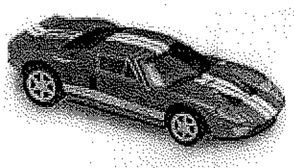
Geometry, 8.1: Ratio and Proportion

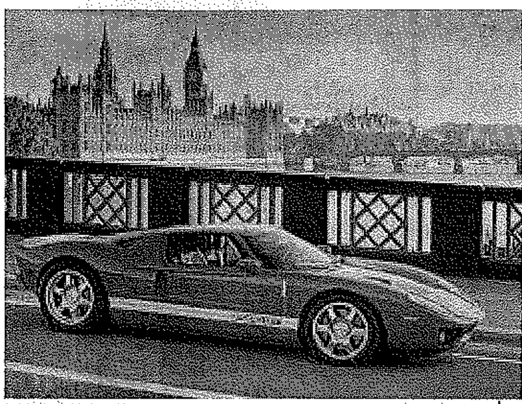
Ratio examples:

Model car: scale factor

$\frac{1}{14}$



14



Model: length = 1 foot
Real car: length = 14 feet

scale factor 14

If side window on model is 4 inches long,
how long is window on real car? $4 \times 14 = \underline{56}''$

Recipe:

Mix: 1 gallon water
The juice from 2 lemons
2 cups sugar

This makes 1 gallon of lemonade. What would you mix if you needed to make 3 gallons of lemonade? Scale factor = 3

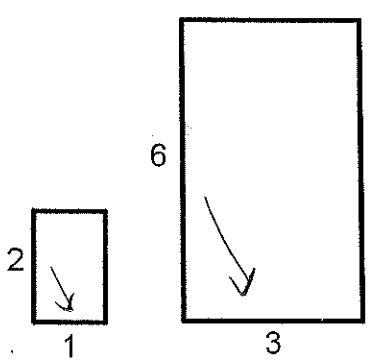
3 gals H_2O
6 lemons
6 cups sugar

Ratio: shows a relation of two numbers. Can be written different ways:

$\frac{2}{1}$ 2:1 2 to 1 $2 \div 1$ 2 cups per gallon

Examples of ratios:

- slope (change in y over change in x)
- map scale (1 inch on map : 100 miles actual distance)
- recipes (2 lemons per gallon water, 2 cups sugar per gallon of water)
- speed (miles per hour)
- shapes of geometric figures (one side twice as long as another side)



$\frac{\text{long side}}{\text{short side}}$

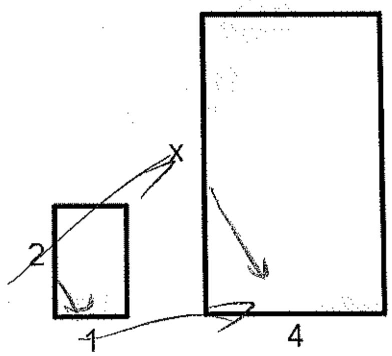
$\frac{2}{1} = 2$ $\frac{6}{3} = 2$

Proportion: An equation stating that two ratios that are equal to each other.

1st term → $\frac{2}{1} = \frac{6}{3}$ ← 3rd term
2nd term → ← 4th term

$\frac{2}{1} = \frac{6}{3}$ 'means' 'extremes'

If you know all but one number of a proportion, you can solve to find the missing value:



$\frac{2}{1} = \frac{x}{4}$ or $\frac{2}{x} = \frac{1}{4}$

$(1)(x) = (2)(4)$ $(1)(x) = (2)(4)$

$x = 8$ $x = 8$

any direction is okay as long as you're consistent.

Solve by **cross-multiplying** (book calls this means-extremes product theorem)
If you have an equation multiplied out, you can turn it back into a proportion...

Find the ratio of x to y if 2x=3y: x to y: $\frac{x}{y} = \frac{3}{2}$ $\boxed{\frac{3}{2}}$

Arithmetic and Geometric mean:

If you have a proportion that looks like this: $\frac{1}{4} = \frac{4}{16}$
with number on bottom of one ratio = number on top of other ratio, then:

- the proportion is called a **mean proportion**
- the common number is called the **geometric mean**

Mean, in general, is a number that describes a group of other numbers. There are 2 kinds of 'means':

Arithmetic mean

- To compute:
  Add numbers, and divide by 2.

- Example:
  Find arithmetic mean of 8 and 18:

$\frac{8+18}{2}$

$\frac{26}{2}$

$\boxed{13}$

Geometric mean

- To compute:
  Make a mean proportion with x as geometric mean, then cross-multiply to solve for x.

- Example:
  Find geometric mean of 8 and 18:

$\frac{8}{x} = \frac{x}{18}$

$x^2 = (8)(18)$

$x^2 = 144$

$x = \pm\sqrt{144}$

$\boxed{x = \pm 12}$

Practice:

#1. Find x: $\frac{3}{2}=\frac{x}{6}$

$2x=(3)(6)$

$\frac{2x}{2}=\frac{18}{2}$

$\boxed{x=9}$

#2. Find b: $\frac{3}{30}=\frac{2}{b}$

$3b=(2)(30)$

$\frac{3b}{3}=\frac{60}{3}$

$\boxed{b=20}$

#3. Find ratio of x to y if 4x=5y

$\frac{x}{y}=\boxed{\frac{5}{4}}$

#4. Find x: $\frac{4}{(x+1)}=\frac{6}{3}$

$6(x+1)=(4)(3)$

$\frac{6(x+1)}{6}=\frac{12}{6}$

$x+1=2$

$\quad -1 \quad -1$

$\boxed{x=1}$

#5. A 20ft steel pole is cut into two parts in the ratio of 3 to 2. How much longer is the longer part than the shorter part?

3:2

3x:2x

20

3x | 2x

3(4) 12 | 2(4) 8

$3x+2x=20$

$\frac{5x}{5}=\frac{20}{5}$

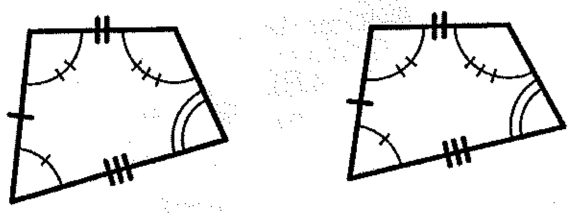
$x=4$

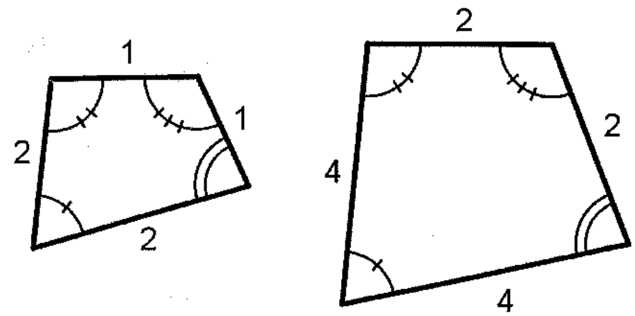
$12 - 8$

$\boxed{\text{4 ft longer}}$

Geometry, 8.2: Similarity

Congruent figures | **Similar figures**





Figures are congruent when:

1. Corresponding sides are congruent.
2. Corresponding angles are congruent.

Figures are similar when:

1. Corresponding sides are proportional
2. Corresponding angles are congruent.

Two ways a figure similar to another figure can be produced:

Dilation = grow, angles same, sides each multiplied by a constant

Reduction = shrink, angle same, sides each divided by a constant

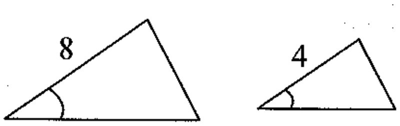
Symbol for 'similar' is ~ example: $\Delta ABC \sim \Delta DEF$

Two figures can be proved similar if: (both must be true)

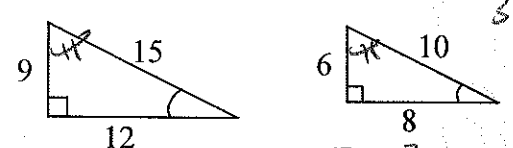
1. The ratios all corresponding sides length are equal.

2. All corresponding angles are congruent.

Practice: Which pairs of polygons ***can be proved*** to be similar?

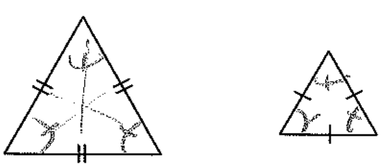


no

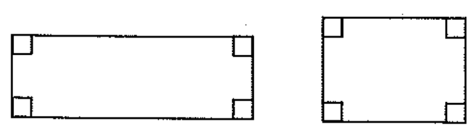


3rd ∠s ≅ by no choice

$\frac{9}{6} = \frac{3}{2}$ $\frac{15}{10} = \frac{3}{2}$ $\frac{12}{8} = \frac{3}{2}$ yes

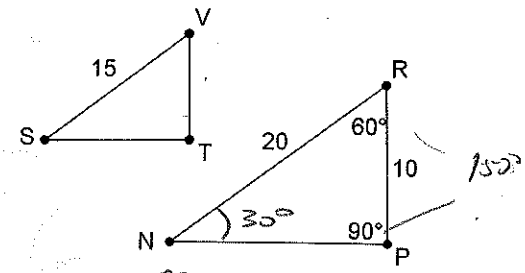


yes (angles are all 60°)



no ∠s ≅, but we don't know if sides are proportional

Practice: Given: $\Delta NPR \sim \Delta STV$ with lengths, angles in diagram
Find: $m\angle T, m\angle S,$ and VT



$m\angle T = 90°$

$m\angle S \cong m\angle N = 30°$

$\frac{20}{15} = \frac{10}{VT}$

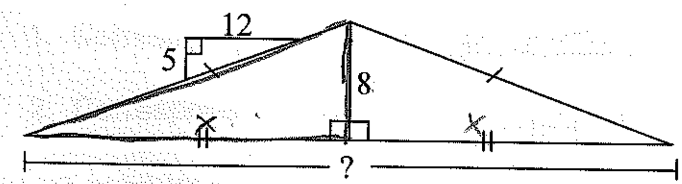
$20VT = 10(15)$

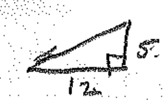
$20VT = 150$

$VT = \frac{150}{20} = \frac{15}{2} = \boxed{7.5}$

180
−150
30

Practice: The roof of a house has a slope of $\frac{5}{12}$. What is the width of the house if the height of the roof is 8 ft?





$\frac{8}{5} = \frac{x}{12}$

$5x = 8(12)$

$5x = 96$

$x = \frac{96}{5} = 19\frac{1}{5}$

width = 2x

$= \boxed{38\frac{2}{5} \text{ ft}}$

$96 \div 5 = 19$ r 1

Extra Information How to find the "**scale factor**" of two similar polygons:

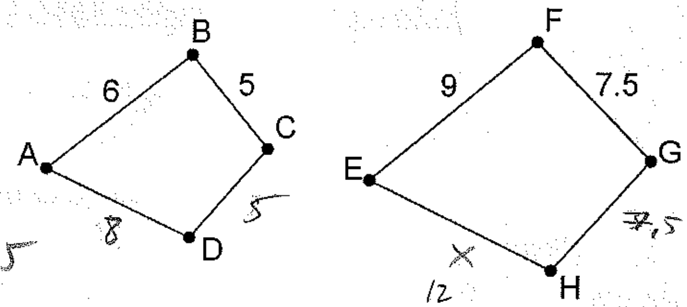
EFGH is a dilation of ABCD. What is the **scale factor**? the ratio of corresponding sides

$\frac{9}{6} = \frac{3}{2} = 1.5$

$\frac{7.5}{5} = 1.5$

$7.5 \div 5 = 1.5$

Scale factor = 1.5



What is the ratio of the perimeters of ABCD and EFGH? What if CD = 5 & AD = 8

$P_{ABCD} = 6+5+5+8$
$= 24$

$\frac{x}{8} = \frac{3}{2}$

$2x = 24$

$x = 12$

$P_{EFGH} = 9+7.5+7.5+12$
$= 36$

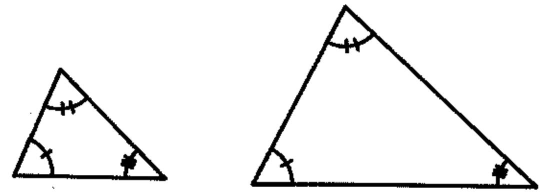
$\frac{36}{24} = \boxed{\frac{3}{2}}$ the scale factor

Geometry, 8.3: Proving Triangles Similar

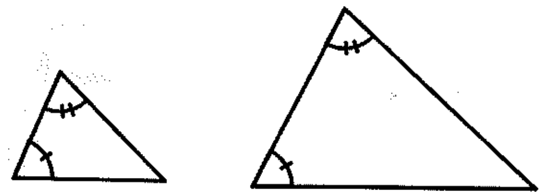
As with proving triangles congruent, we have shortcuts to prove triangles are similar:

Triangles Similar if:

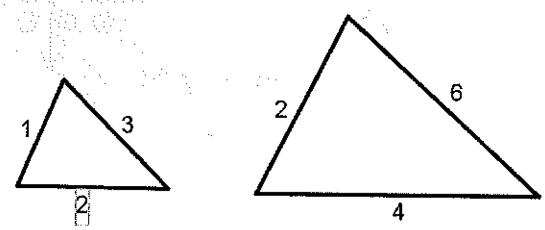
AAA All corresponding angles are congruent.



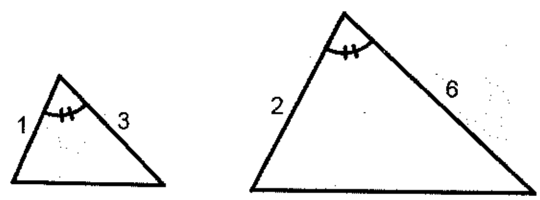
AA Two corresponding angles are congruent.



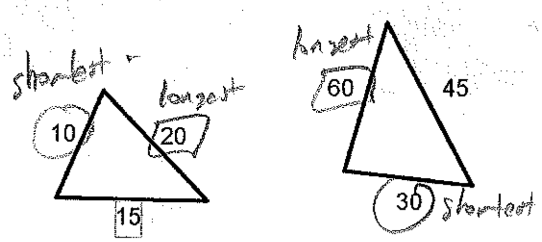
SSS~ The ratios of side lengths are equal for all pairs of corresponding sides.

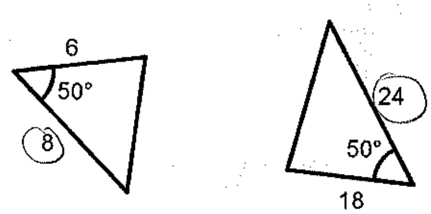


SAS~ The ratios of side lengths are equal for 2 pairs corresponding sides and the angle between is congruent.



Practice: Are the pairs of triangles below similar? If so, by which shortcut?

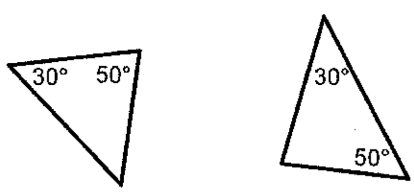




$\frac{30}{10} = 3$ $\frac{60}{20} = 3$

$\frac{45}{15} = 3$

SSS~



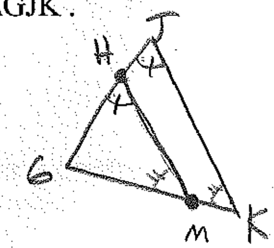
$\frac{24}{8} = 3$ $\frac{18}{6} = 3$

SAS~

AA

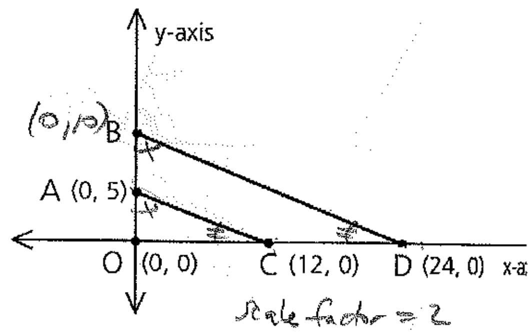
Practice:

Draw a triangle GJK. Then indicate a point H on $\overline{GJ}$ and a point M on $\overline{GK}$ such that $\overline{HM} \parallel \overline{JK}$. Prove that $\triangle GHM \sim \triangle GJK$.



S	R
1. $\triangle GJK$	1. Given
2. $\overline{HM} \parallel \overline{JK}$	2. Given
3. $\angle GHM \cong \angle GJK$	3. ∥ lines ⇒ corr ∠'s ≅
4. $\angle GMH \cong \angle GKJ$	4. ∥ lines ⇒ corr. ∠'s ≅
5. $\triangle GHM \sim \triangle GJK$	5. AA

6 Find the coordinates of B if $\triangle OAC \sim \triangle OBD$. Then write a paragraph proof to show that $\triangle OAC \sim \triangle OBD$. Challenge: Can you find the length of $\overline{BD}$?



Scale factor = 2

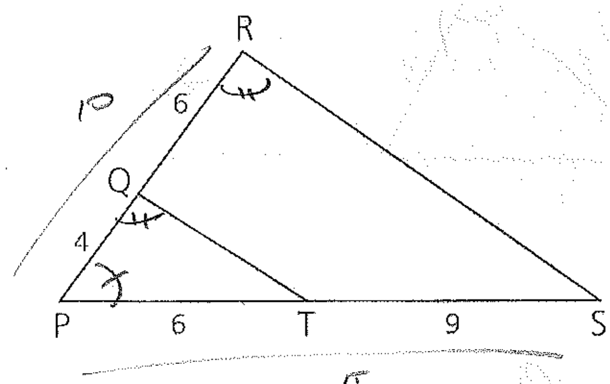
A and C are midpoints so AC is a midline. Midline AC ∥ BD so corr. angles are ≅. △'s similar by AA.

19 Given: Figure as shown

a Is $\triangle PQT \sim \triangle PRS$? Justify your reasoning. yes

b Is $\overline{QT}$ parallel to $\overline{RS}$? Justify your reasoning.

yes,
- angles ≅ for similar △s
- corr ∠'s ≅ ⇒ ∥ lines



$\frac{4}{10} \stackrel{?}{=} \frac{6}{15}$

$4(15) \stackrel{?}{=} 10(6)$

$60 = 60$ ✓

- 2 sides proportional
- angle P between ≅ (reflexive)

yes $\triangle PQT \sim \triangle PRS$

Lesson notes

Geometry, 8.4: Congruence and Proportion in Similar Triangles

Once we proved triangles congruent, we could prove angles or sides congruent by CPCTC

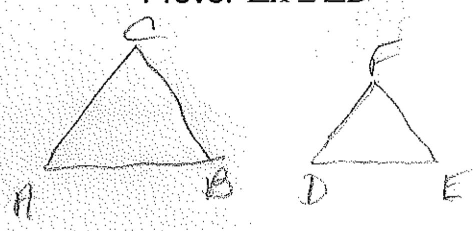
Once we prove triangles are similar we can prove:

1. corresponding sides are proportional ($\sim\Delta$s $\rightarrow$ corr. sides proportional)
2. corresponding angles are congruent ($\sim\Delta$s $\Rightarrow$ corr. angles $\cong$)

If a problem asks you to prove that products of sides are equal use: cross multiply

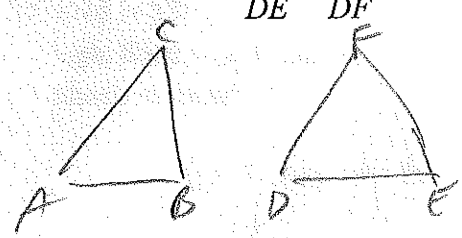
means-extremes products theorem $\left(\frac{a}{b}=\frac{c}{d} \Rightarrow ad=bc\right)$

Example: Given: $\Delta ABC \sim \Delta DEF$
Prove: $\angle A \cong \angle D$



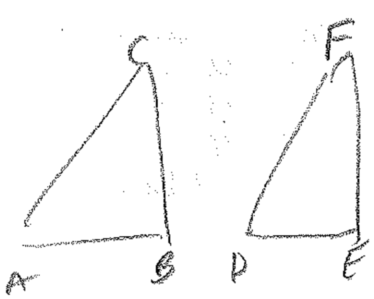
S	R
1. $\Delta ABC \sim \Delta DEF$	1. Given
2. $\angle A \cong \angle D$	2. $\sim\Delta$s $\Rightarrow$ corr. angles $\cong$

Example: Given: $\Delta ABC \sim \Delta DEF$
Prove: $\frac{AB}{DE}=\frac{AC}{DF}$



S	R
1. $\Delta ABC \sim \Delta DEF$	1. Given
2. $\frac{AB}{DE}=\frac{AC}{DF}$	2. $\sim\Delta$s $\Rightarrow$ corr. sides proportional

Example: Given: $\Delta ABC \sim \Delta DEF$
Prove: $AB \cdot DF = AC \cdot DE$

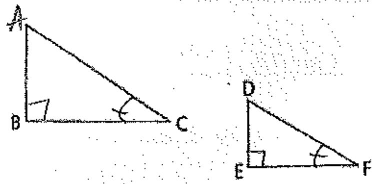


S	R
1. $\Delta ABC \sim \Delta DEF$	1. Given
2. $\frac{AB}{DE}=\frac{AC}{DF}$	2. $\sim\Delta$s $\Rightarrow$ corr. sides proportional
3. $AB \cdot DF = AC \cdot DE$	3. means-extremes products theorem

Practice:

1 Given: $\angle C \cong \angle F$, $\overline{AB} \perp \overline{BC}$, $\overline{DE} \perp \overline{EF}$

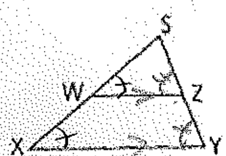
Prove: $\frac{AB}{BC} = \frac{DE}{EF}$



S	R
1. $\angle C \cong \angle F$, $\overline{AB} \perp \overline{BC}$, $\overline{DE} \perp \overline{EF}$	1. Given
2. $\angle B \cong \angle E$	2. rt angles $\cong$
3. $\triangle ABC \sim \triangle DEF$	3. AA
4. $\frac{AB}{DE} = \frac{BC}{EF}$	4. $\sim\triangle$s $\Rightarrow$ corr sides proportional
5. $AB \cdot EF = BC \cdot DE$	5. means-extremes products thm
6. $\frac{AB}{BC} = \frac{DE}{EF}$	6. means-extremes ratio thm.

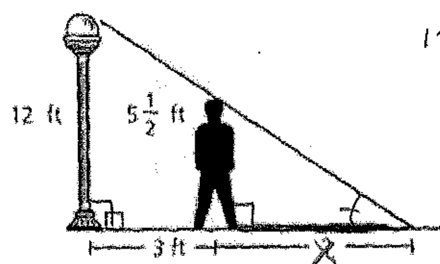
6 Given: $\overleftrightarrow{WZ} \parallel \overleftrightarrow{XY}$

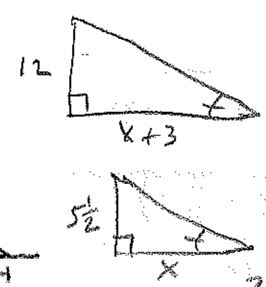
Conclusion: $WS \cdot XY = XS \cdot WZ$



S	R
1. $\overline{WZ} \parallel \overline{XY}$	1. Given
2. $\angle SWZ \cong \angle SXY$	2. $\parallel$ lines $\Rightarrow$ corr. $\angle$'s $\cong$
3. $\angle SZW \cong \angle SYX$	3. $\parallel$ lines $\Rightarrow$ corr. $\angle$'s $\cong$
4. $\triangle SWZ \sim \triangle SXY$	4. AA
5. $\frac{WS}{XS} = \frac{WZ}{XY}$	5. $\sim\triangle$s $\Rightarrow$ corr. sides proportional
6. $WS \cdot XY = WZ \cdot XS$	6. means-extremes products thm.

20 Shad is 3 ft from a lamppost that is 12 ft high. Shad is $5\frac{1}{2}$ ft tall. How long is Shad's shadow?





$\triangle$'s similar by AA

$\frac{x}{x+3} = \frac{5\frac{1}{2}}{12}$

$12x = 5.5(x+3)$

$12x = 5.5x + 5.5(3)$

$6.5x = 16.5$

$12.0 - 5.5 = 6.5$

$5.5 \times 3 = 16.5$

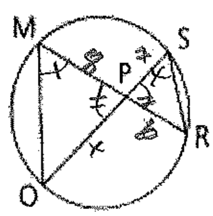
$x = \frac{16.5}{6.5}$

$\boxed{x \approx 2.5 \text{ ft}}$

$16.50 \div 6.5 = 2.54$: 130, 350, 325, 250

17 Given: $\angle M \cong \angle S$, $MP = 8$, $PR = 6$, $SP = 7$

Find: PO



$\triangle$'s are similar by AA

$\frac{8}{7} = \frac{x}{6}$

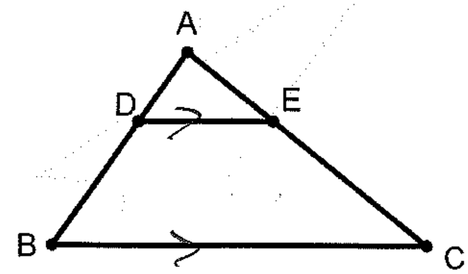
$7x = 48$

$\boxed{x = \frac{48}{7}}$

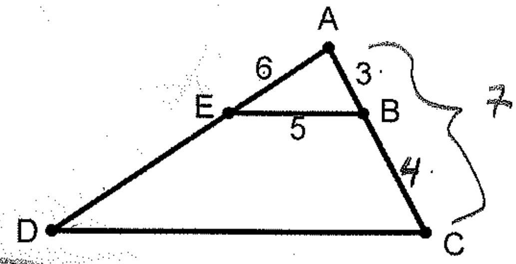
Geometry, 8.5: Three Theorems Involving Proportions

Side-splitter theorem:
If a line is parallel to one side of a triangle and intersects the other two sides, it divides those two sides proportionally.

$\frac{AD}{DB} = \frac{AE}{AC}$ (also $\frac{AD}{AE} = \frac{DB}{EC}$)



Example:
Given: $\overline{BE} \parallel \overline{CD}$, lengths as shown
Find: ED and CD



$\frac{6}{ED} = \frac{3}{4}$

$3ED = 6(4)$

$3ED = 24$

$\boxed{ED = 8}$

$\triangle AEB \sim \triangle ADC$

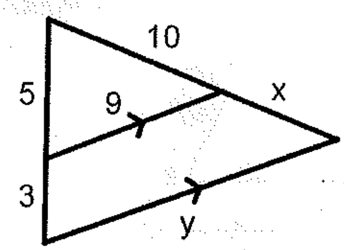
$\frac{3}{5} = \frac{7}{CD}$

$3CD = 7(5)$

$3CD = 35$

$\boxed{CD = \frac{35}{3}}$

Example:
Solve for x and y



$\frac{10}{x} = \frac{5}{3}$

$5x = 10(3)$

$5x = 30$

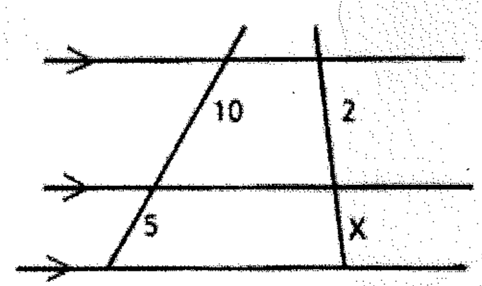
$\boxed{x = 6}$

$\frac{5}{9} = \frac{8}{y}$

$5y = 72$

$\boxed{y = \frac{72}{5}}$

(parallel lines / transversals theorem):
If 3 or more parallel lines are intersected by 2 transversals, the parallel lines divide the transversals proportionally.

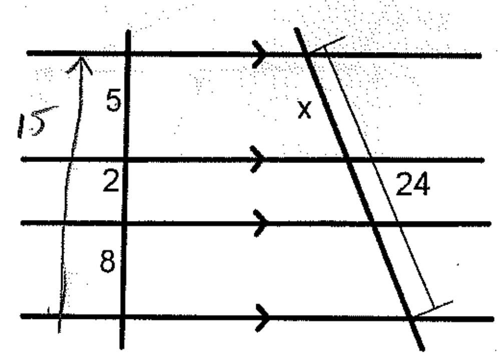


$\frac{10}{5} = \frac{2}{X}$ also $\frac{10}{2} = \frac{5}{X}$

$10X = 10$ $\quad$ $10X = 10$

$X = 1$ $\quad$ $X = 1$

Example:
Find x



$\frac{5}{15} = \frac{x}{24}$

$15x = 5(24)$

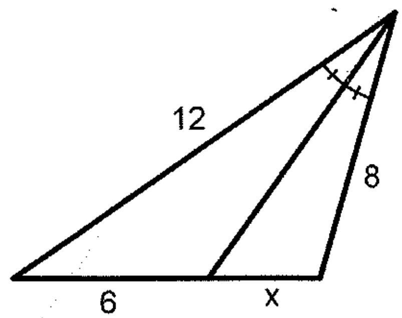
$15x = 120$

$\boxed{x = 8}$

$24 \times 5 = 120$

$120 \div 15 = 8$

(Angle Bisector theorem):
If a ray bisects and angle of a triangle, it divides the opposite side into segments that are proportional to the adjacent sides.



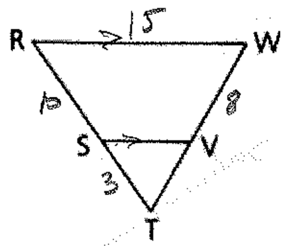
smaller piece → $\frac{x}{6} = \frac{8}{12}$ ← smaller side
larger piece → 6 ← larger side

$12x = 48$

$\boxed{x = 4}$

Practice:

10 Given: $\overleftrightarrow{SV} \parallel \overleftrightarrow{RW}$
$RW = 15,\ RS = 10,$
$ST = 3,\ WV = 8,$
Find: SV and VT



$\frac{10}{3} = \frac{8}{VT}$

$10\,VT = 3(8)$

$10\,VT = 24$

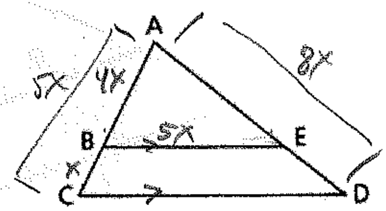
$VT = \frac{24}{10} = \boxed{\frac{12}{5}}$

$\frac{SV}{3} = \frac{15}{13}$ $\boxed{SV = \frac{45}{13}}$

$13SV = 3(15)$

$13SV = 45$

18 Given: $\overleftrightarrow{BE} \parallel \overleftrightarrow{CD}$
$AB = 4x,\ BC = x,$
$AD = 8x,\ BE = 5x,$
Find: AE and CD (in terms of x)



$\frac{AE}{8x} = \frac{4x}{5x}$

$5x\,AE = (8x)(4x)$

$\frac{5x\,AE}{5x} = \frac{32x^2}{5x}$

$\boxed{AE = \frac{32}{5}x}$

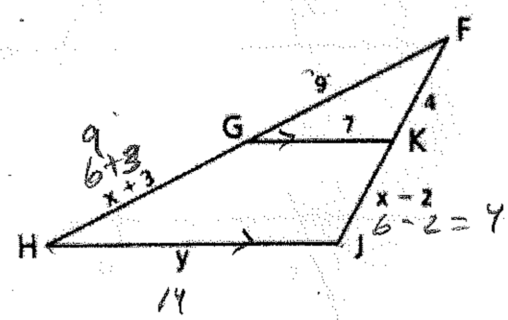
$\frac{5x}{4x} = \frac{CD}{5x}$

$4x\,CD = (5x)(5x)$

$4x\,CD = 25x^2$

$\boxed{CD = \frac{25}{4}x}$

20 Given: $\overleftrightarrow{GK} \parallel \overleftrightarrow{HJ}$
lengths as shown
Find: The perimeter of ΔHJF



$\frac{9}{x+3} = \frac{4}{x-2}$

$4(x+3) = 9(x-2)$

$4x + 12 = 9x - 18$
$\quad +18 \quad\quad +18$

$4x + 30 = 9x$
$-4x \quad\quad -4x$

$\frac{30}{5} = \frac{5x}{5}$

$6 = x$

$\frac{7}{4} = \frac{y}{8}$

$4y = 56$

$y = \frac{56}{4} = 14$

14
18
8
40

perimeter = $\boxed{40}$