HAlg3-4, 2.6 Notes – Rational Functions and Asymptotes

A **Rational Function** is a function in the form of a ratio of polynomials:

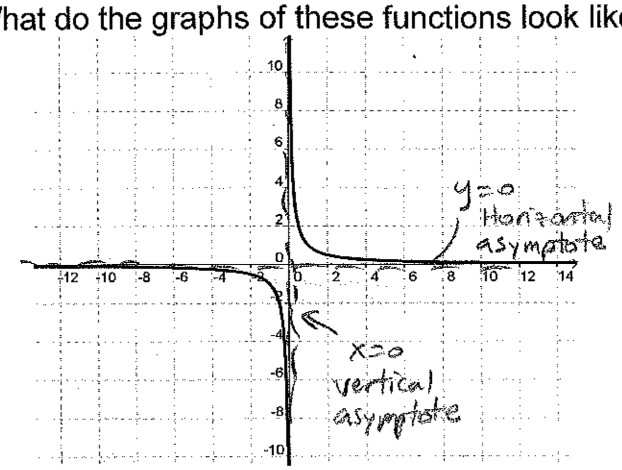
$$f(x)=\frac{N(x)}{D(x)} \qquad \text{examples: } f(x)=\frac{3x^2-2}{4x^3-x^2+2x-1} \qquad g(x)=\frac{1}{x}$$

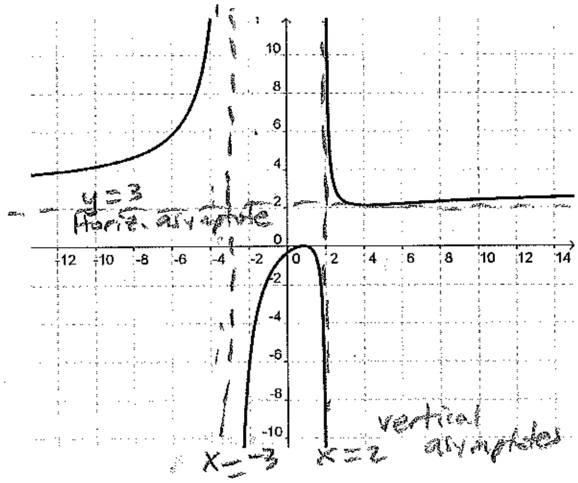
domain of a rational function = all real numbers except where denominator is zero

Find domain of: $f(x)=\dfrac{1}{x}$ ℝ, $x\neq 0$ $\qquad f(x)=\dfrac{3x^2-5x+2}{x^2+x-6}$ ℝ, $x\neq -3$, $x\neq 2$

$(x+3)(x-2)$

What do the graphs of these functions look like?





How to find the Asymptotes of a Rational Function: $f(x)=\dfrac{N(x)}{D(x)}$

Vertical asymptotes: graph of f has vertical asymptotes at the zeros of denominator $D(x)$

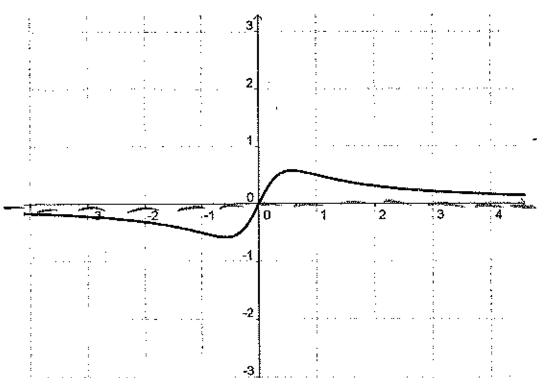
Horizontal asymptotes: graph of f has, at most, one horizontal asymptote, determined by the degrees of the numerator polynomial (n) and the denominator polynomial (m).

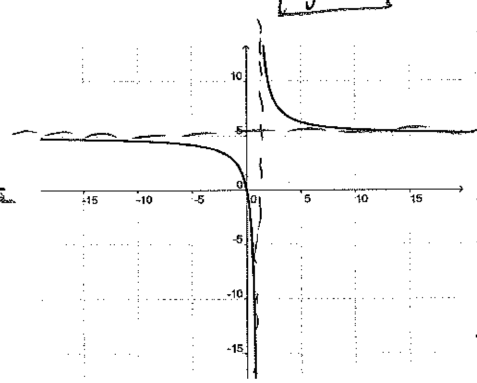
- If n < m, the line $y=0$ (x-axis) is a horizontal asymptote.
- If n = m, the line $y=\dfrac{a_n}{b_m}$ is a horizontal asymptote.
- If n > m, the graph has no horizontal asymptote but may have a slant asymptote.

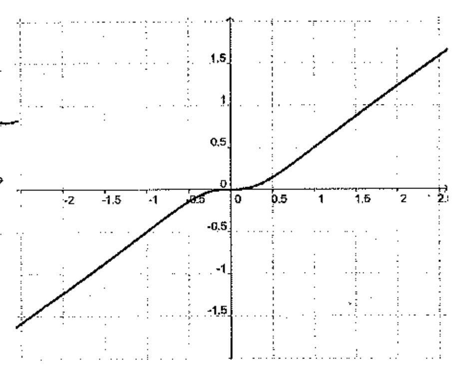
$f(x)=\dfrac{2x}{3x^2+1}$ $n=1$, $m=2$ $\quad$ $n<m$ Horiz. asymptote $\boxed{y=0}$

$f(x)=\dfrac{5x}{x-1}$ $n=1$, $m=1$ $\quad$ $n=m$ H.A. $y=\frac{5}{1}$ $\boxed{y=5}$

$f(x)=\dfrac{2x^3}{3x^2+1}$ $n=3$, $m=2$ $\quad$ $n>m$ $\boxed{\text{No H.A.}}$







vertical asymptote: (denom = 0)
$3x^2+1=0 \quad 3x^2=-1$
$x^2=-\frac{1}{3} \quad x=\pm\sqrt{-1/3}$ $\boxed{\text{no verti. asymptote}}$

V.A.:
$x-1=0$
$\boxed{x=1}$

V.A.
$3x^2+1=0$
$x=\pm\sqrt{-1/3}$
$\boxed{\text{No V.A.}}$

(another, more conceptual, way to find horizontal asymptotes)

Think about what happens when x is very large. The non-x terms become negligible and can be removed. Then simplify the result:

$f(x)=\dfrac{2x}{3x^2+1}$

for large x 1 is negligible

$f(x) \approx \dfrac{2x}{3x^2}$

$= \dfrac{\cancel{x}(2)}{\cancel{x}(3x)}$

$= \dfrac{2}{3x}$

as $x \to \infty$

$f(x) \to 0$

So H.A is

$\boxed{y=0}$

$f(x)=\dfrac{5x}{x-1}$

for large x negligible

$f(x) \approx \dfrac{5x}{x}$

$= 5$

So H.A. is

$\boxed{y=5}$

$f(x)=\dfrac{2x^3}{3x^2+1}$

for large x negligible

$f(x) \approx \dfrac{2x^3}{3x^2}$

$= \dfrac{\cancel{x^2}(2x)}{\cancel{x^2}(3)}$

$= \dfrac{2x}{3}$

as $x \to \infty$

$f(x) \to \infty$

never settles to a number, so

$\boxed{\text{No H.A.}}$

Applications – many real-world problems exhibit 'asymptotic behavior' (approach a value)

A business has a cost function $C = 0.5x + 5000$ where C is cost in dollars and x is number of units produced.
- What is the average cost per unit when the number of units is 1000, and 10,000.
- What is the average cost per unit when a very large number of units is produced?

avg. cost per unit $A(x) = \dfrac{C(x)}{x} = \dfrac{0.5x+5000}{x}$

vertical asymptote; when $x=0$

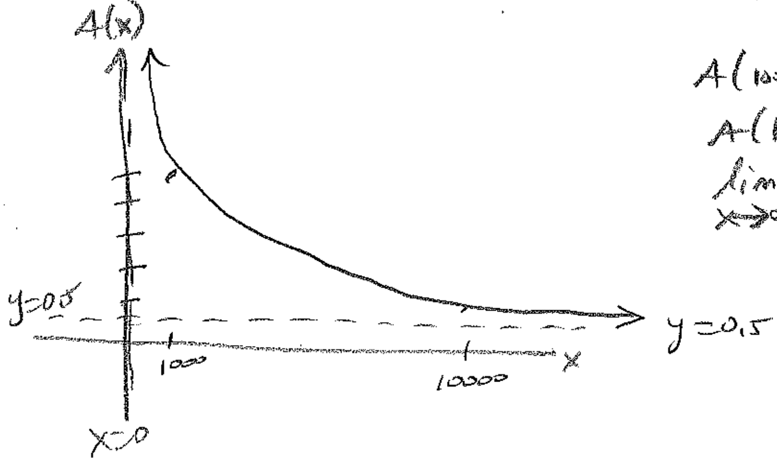
horizontal asymptote: $n=1, m=1$ $n=m$ $\dfrac{(0.5)x+5000}{(1)x}$

So H.A. $y=0.5$

$A(1000) = 5$

$A(10000) = 1$

$\lim_{x\to\infty} A(x) = 0.5$



HAlg3-4, 2.7 Notes – Graphs of Rational Functions

Finding slant asymptotes:

Example: Find horizontal asymptote of $f(x) = \dfrac{x^2 - x}{x+1}$ $n = 2$ $m = 1$

n > m, no horiz. asymptote -or- for large x: $\approx \dfrac{x^2 - x}{x}$
$\approx x - 1$
$\approx \infty$

Slant asymptotes exist when degree of numerator is exactly one greater than degree of denominator. **Find equation of line of slant asymptote by dividing the polynomials**. The quotient (without remainder) is the equation of the slant asymptote.

$$\begin{array}{r|l} & x - 2 \\ x+1 & x^2 - x + 0 \\ & \underline{x^2 + x} \\ & -2x + 0 \\ & \underline{-2x - 2} \\ & 2 \end{array}$$

or by synthetic division:

$$\begin{array}{r|rrr} -1 & 1 & -1 & 0 \\ & & -1 & 2 \\ \hline & 1 & -2 & 2 \end{array}$$

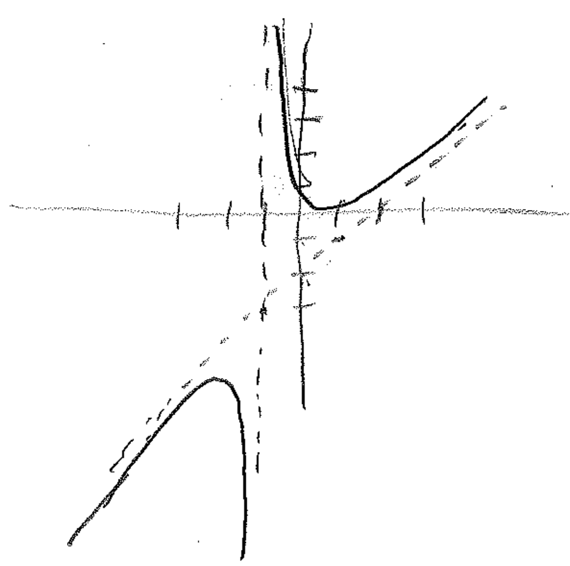
$x - 2$ R 2

$y = x - 2$

$\underbrace{x - 2} + \dfrac{2}{x+1}$ this becomes negligible

slant asymptote: $y = x - 2$

vertical asymptote where denominator = 0: $x = -1$



Sketching rational functions

1) Find f(0) (plug in 0 for x)…this gives y-intercept (if any).

2) Find zeros of numerator polynomial…this gives x-intercepts (if any).

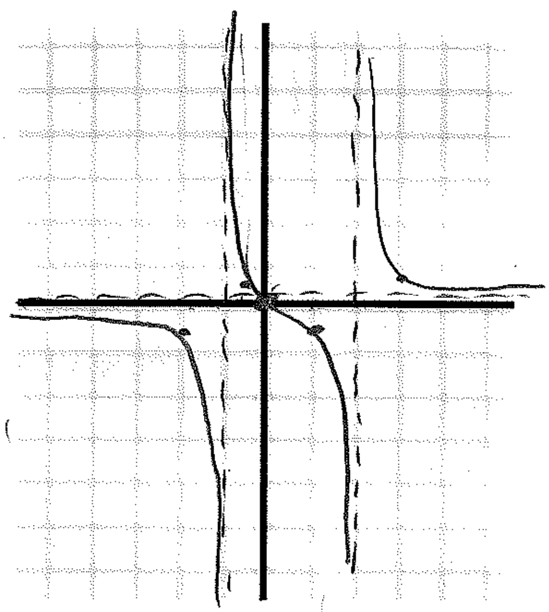
3) Find zeros of denominator polynomial…this gives vertical asymptotes (if any).

4) Use entire rational function to find horizontal or ~~slope~~ slant asymptotes (if any).

5) Plot above on graph and find at least one point in each 'region'.

6) Finish sketch with smooth curve.

Example: sketch $f(x) = \dfrac{x}{x^2 - x - 2}$



1) $f(0) = \dfrac{0}{0^2 - 0 - 2} = \dfrac{0}{-2} = 0$ y-int: $(0, 0)$

2) $x = 0$ x-int: $(0, 0)$

3) $x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

vertical asymptotes at: $x = 2, x = -1$

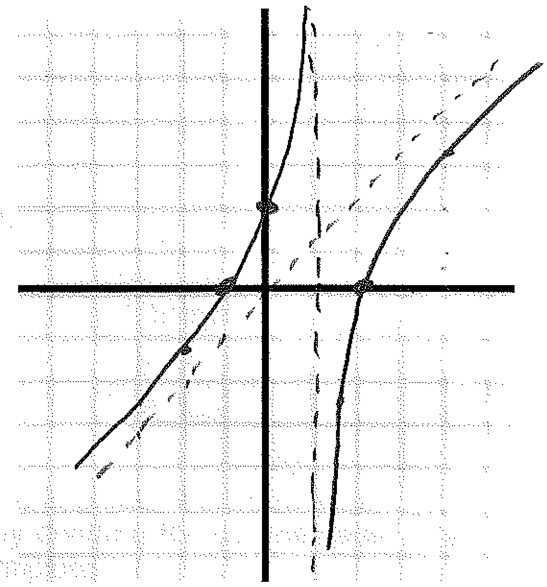
$\dfrac{x}{x^2 - x} = \dfrac{1}{x-1}$

4) $n = 1$, $m = 2$ $n < m$, horiz. asymptote $y = 0$

5)

x	$f(x)$
-2	$\dfrac{-2}{4+2-2} = \dfrac{-2}{4} = -\dfrac{1}{2}$ $\left(-2, -\dfrac{1}{2}\right)$
$-\dfrac{1}{2}$	$\dfrac{-\frac{1}{2}}{\frac{1}{4}+\frac{1}{2}-2} = \dfrac{-\frac{1}{2}}{\frac{1}{4}+\frac{2}{4}-\frac{8}{4}} = \dfrac{-\frac{1}{2}}{-\frac{5}{4}} = -\dfrac{1}{2}\left(-\dfrac{4}{5}\right) = \dfrac{2}{5}$ $\left(-\dfrac{1}{2}, \dfrac{2}{5}\right)$
1	$\dfrac{1}{1-1-2} = \dfrac{1}{-2} = -\dfrac{1}{2}$ $\left(1, -\dfrac{1}{2}\right)$
3	$\dfrac{3}{9-3-2} = \dfrac{3}{4}$ $\left(3, \dfrac{3}{4}\right)$

Example: sketch $f(x)=\dfrac{x^2-x-2}{x-1}$



1) $f(0)=\dfrac{0^2-0-2}{0-1}=\dfrac{-2}{-1}=2$ y-int $(0,2)$

2) $x^2-x-2=0$ x-int: $(2,0)$ and $(-1,0)$

$(x-2)(x+1)=0$

3) $x-1=0$

$x=1$ vertical asymptote

4) horiz. asymptote: for large $\approx \dfrac{x^2-x}{x}$

$\approx x-1$

≈ 0 no horiz.

but n one greater than m, so slant asymptote:

$$\begin{array}{r} x \\ x-1\overline{)\,x^2-x-2} \\ \underline{x^2-x} \\ 0-2 \end{array}$$

$x-\dfrac{2}{x-1}$

$y=x$ slant asymptote

5)

x	$f(x)$
-2	$\dfrac{4+2-2}{-2-1}=\dfrac{4}{-3}=-\dfrac{4}{3}$ $(-2,-\frac{4}{3})$
4	$\dfrac{16-4-2}{4-1}=\dfrac{10}{3}=$ $(4,\frac{10}{3})$
$\frac{3}{2}$	$\dfrac{\frac{9}{4}-\frac{3}{2}-2}{\frac{3}{2}-1}=\dfrac{\frac{9}{4}-\frac{6}{4}-\frac{8}{4}}{\frac{3}{2}-\frac{2}{2}}=\dfrac{-\frac{5}{4}}{\frac{1}{2}}=\left(-\frac{5}{4}\right)(2)=-\frac{5}{2}$ $(\frac{3}{2},-\frac{5}{2})$