HAlg3-4, 3.1 day 1 Notes – Exponential Functions

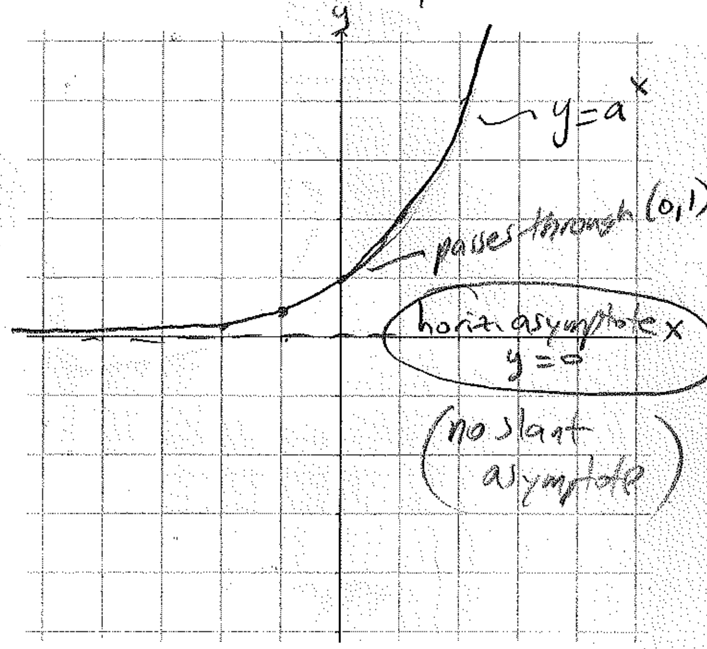
Consider this ancient riddle:

"A courtier once presented a Persian king with a beautiful, hand-made chessboard. The king asked what he would like in return for his gift, and the courtier surprised the king by asking for a single grain of rice to be placed on the first square, and then on each successive square, for double the amount of rice from the previous to be placed (e.g., two grains on the second square, four grains on the third, etc.) Thinking this a paltry sum, the king readily agreed and asked for the rice to be brought."

What do you think happened when the rice was brought in? Did the king pay a fair price for his new chessboard?

squares	rice
start	$1 = 1(2)^0$
1	$2 = [1](2) = (1)2^1$
2	$4 = [1(2)](2) = (1)2^2$
3	$8 = [1(2)(2)](2) = (1)2^3$
4	$16 = [1(2)(2)(2)](2) = (1)2^4$
5	32
6	64
7	128
.	.
.	.
63	? ~ 9,223,372,036,854,789000 79 quintillion
x	$f(x) = (1)(2)^x$

$2^{-1} = \frac{1}{2}$, $2^{-2} = \frac{1}{2^2} = \frac{1}{4}$



Exponential equations: $f(x) = a^x$

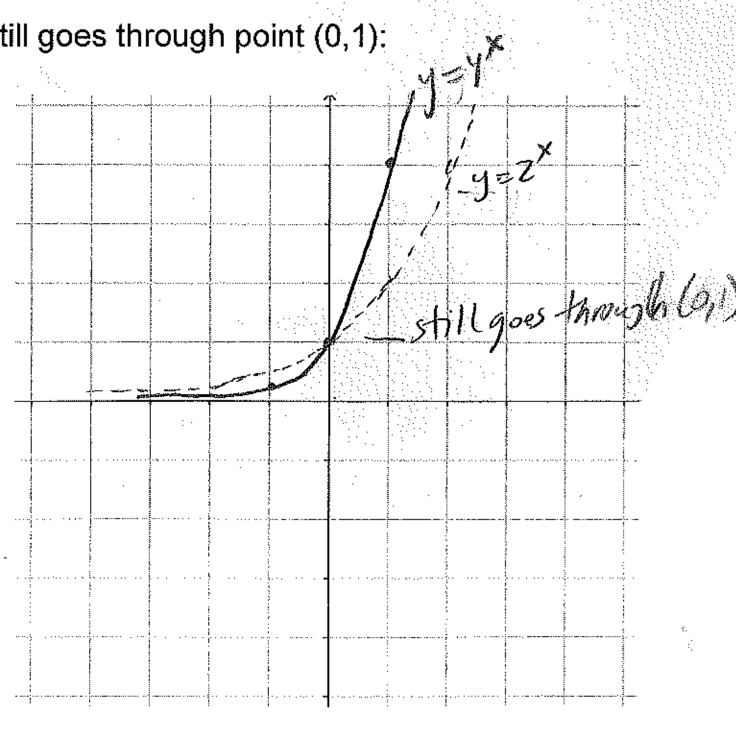
a = the 'base' of the exponential function

Graph of an exponential equation: $f(x) = 2^x$

Changing base changes steepness of curve, but still goes through point (0,1):

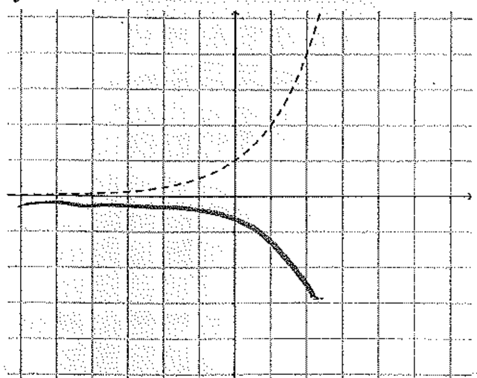
$f(x) = 4^x$

x	2^x	4^x
0	1	1
1	2	4
2	4	16
-1	$\frac{1}{2}$	$\frac{1}{4}$

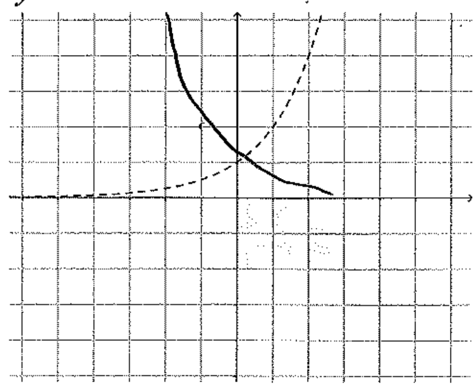


Try graphing the following and comparing to $y = 2^x$. How does each change in equation affect the graph?

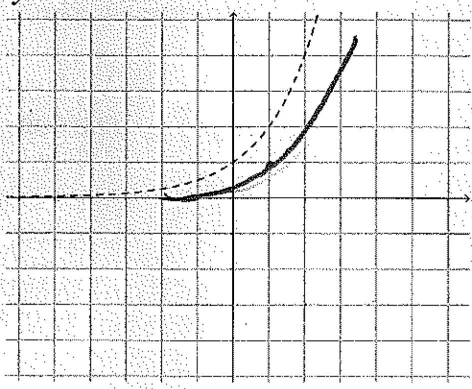
$y = -2^x$



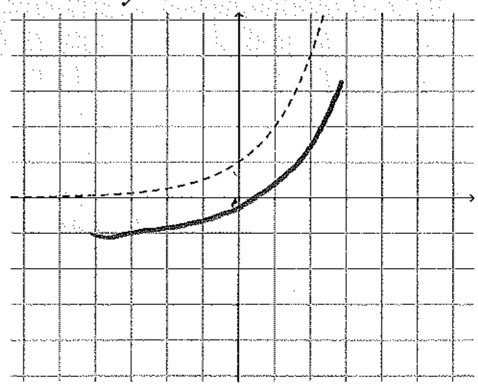
$y = 2^{(-x)}$



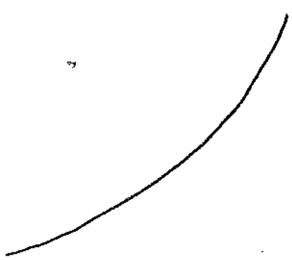
$y = 2^{(x-1)}$

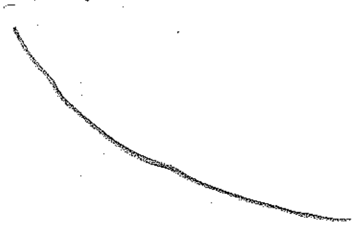


$y = 2^{(x)} - 1$



all same shifting, reflecting rules from ch1 apply





'exponential growth'

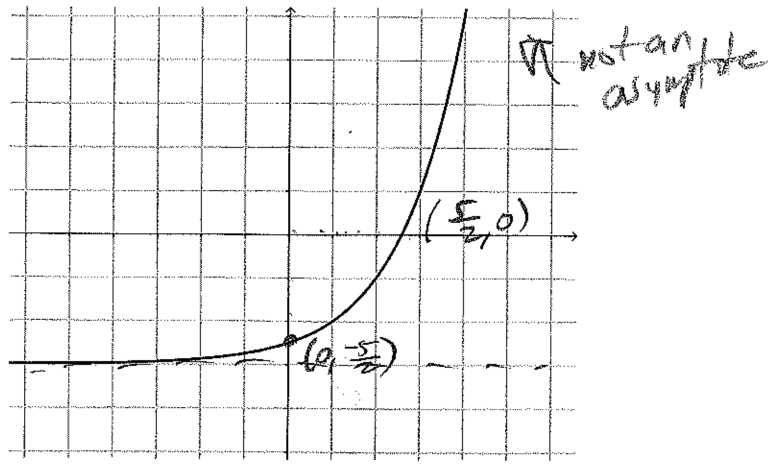
- bacteria population
- nuclear chain reactions
- financial 'pyramid schemes'
- rumors

'exponential decay'

- radioactive decay (halflife)
- temperature equilization

Finding x and y intercepts and domain, range:

$y = 2^{(x-1)} - 3$



y-int: when $x = 0$

$y = 2^{(0-1)} - 3$

$y = 2^{-1} - 3$

$y = \frac{1}{2} - 3$ $\quad (0, \frac{-5}{2})$

$y = \frac{-5}{2}$

x-int: when $y = 0$ $\quad 0 = 2^{(x-1)} - 3$

(next chapter we learn how algebraically — for now, must graph)

$(\frac{5}{2}, 0)$

horiz. asymptote: $y = -3$

domain: $(-\infty, \infty)$

range: $(-3, \infty)$

oblems like the HW (partners try — share answers)

#12. $2^{2x+6} = 2^{2x} \cdot 2^{6} = 64 \cdot 2^{2x} = 64\left((2)^{x}\right)^{2}$

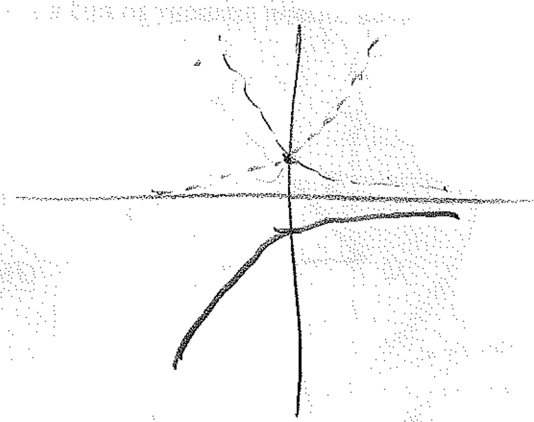
$= 64\left((2)^{2}\right)^{x}$

$= 64(4)^{x}$

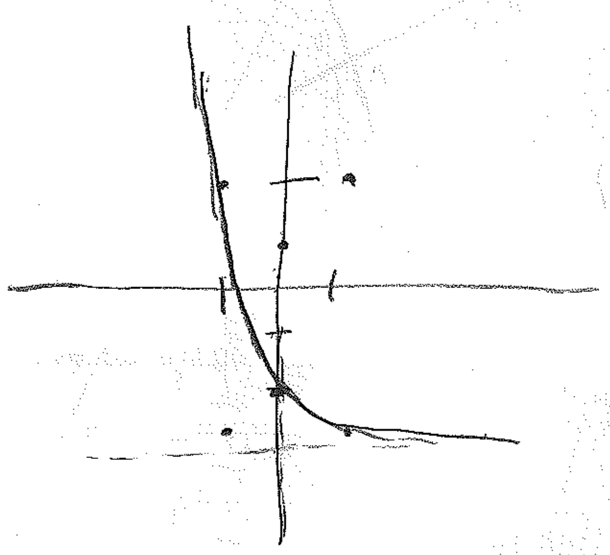
$g(x) = h(x)$

#18. $f(x) = -2^{-x}$ ← flip horiz.

flip vertical



#33. $g(x) = 5^{-x} - 3$



(a) $y = -3$

(b) y-int when x=0

$g(0) = 5^{-0} - 3$

$g(0) = 1 - 3$

$g(0) = -2$ $\boxed{(0,-2)}$

x-int when y=0

$0 = 5^{-x} - 3$

$(-0.6826, 0)$

(c) decreasing

HAlg3-4, 3.1 day 2 Notes – Compound Interest and Natural Base e

Some financial terms...

Principal – The initial (starting) amount of an investment or a loan.

Interest rate – An extra amount that is added to the principal each year. For an investment, it is extra money in the account at the end of the year. For a loan, it is extra money you pay each year. Interest rate is usually specified as a percentage annually. (e.g. 5% annual interest rate.)

Compounding – After a period of time (the compounding period) the amount is adjusted and interest is added to the principal. A new compounding period starts, and the end amount becomes the new start amount for the next compounding period.

Example: If you invest $100 in a bank account with a 3% annual interest rate, and the account compounds annually (once at the end of each year):

$r = .03$ (3%)

t, in years	Amount at start of year	Interest earned that year (3%)	Amount at end of year
1	$100.00 P	× $3.00 rP	$103.00 $= P + rP = P(1+r)$
2	$103.00	$3.09	$106.09 $= [P(1+r)](1+r) = P(1+r)^2$
3	$106.09	$3.18	$109.27 $= [P(1+r)^2](1+r) = P(1+r)^3$

1.03

an exponential function

Compounding annually: $A = P(1+r)^t$ gives the amount, A, after t years.

What if we compound each month instead of only at the end of the year?

$$A = P\left(1+\frac{r}{12}\right)^{12t}$$

← we compound 12 times each year

we only get 1/12 of the annual interest each month

For compounding 'n' times per year:

* $$\boxed{A = P\left(1+\frac{r}{n}\right)^{nt}}$$

compound annually:	n = 1
compound quarterly:	n = 4
compound monthly:	n=12
compound daily:	n = 365

Example: Invest $5000 in an account with annual interest rate of 5% for 10 years. What is the amount in the account at the end of 10 years if the account compounds:

students try it

(a) annually $(n=1)$ $A = 5000\left(1+\frac{.05}{1}\right)^{1(10)} = \$8144.47$

(b) monthly $(n=12)$ $A = 5000\left(1+\frac{.05}{12}\right)^{12(10)} = \$8235.05$

(c) daily $(n=365)$ $A = 5000\left(1+\frac{.05}{365}\right)^{365(10)} = \$8243.32$

(d) every hour $(n = 365 \cdot 24$, $n = 8760)$ $A = 5000\left(1+\frac{.05}{8760}\right)^{8760(10)} = \$8243.59$

hat if compounded every 'instant'...compounded 'continuously'?

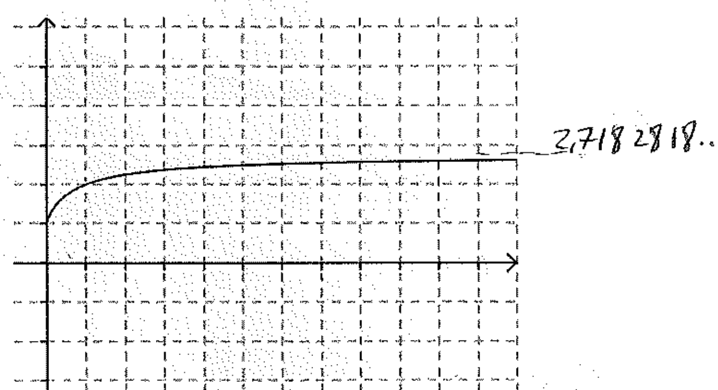
$$A = P\left(1+\frac{r}{n}\right)^{nt} \qquad \text{define } m = \frac{n}{r} \qquad \text{then, } \frac{r}{n} = \frac{1}{m} \quad \text{and } n = rm$$

substituting these into the equation:

$$A = P\left(1+\frac{1}{m}\right)^{rmt} \qquad \text{and} \qquad A = P\left[\left(1+\frac{1}{m}\right)^{m}\right]^{rt}$$

When $n \to \infty, m \to \infty$, so what does the expression in the square brackets do as we compound continuously (as $m \to \infty$)?

m	$\left(1+\frac{1}{m}\right)^{m}$
1	2.0000
10	2.5937...
100	2.7048...
1000	2.7169...
10000	2.7181...



As m increases, the expression in the brackets approaches a number. That number is called 'e'

$e = 2.718281828459....$ (Super important number in math)

e is called the 'natural base' and is an irrational number, like π.

We can then rewrite our compounding equation for the 'continuous compounding' case:

For continuous compounding: * $\boxed{A = Pe^{rt}}$

Earlier example: $5000 invested in a 5% account for 10 years compounded:

(a) annually: A = $8144.47

(b) monthly: A = $8235.05

(c) daily: A = $8243.32

(d) hourly: A = $8243.59

(e) continuously: $A = Pe^{rt} = 5000\,e^{.05(10)} = \$8243.61$

Exponential curves are used to model other things as well. Another example:

Radioactive Decay: Let y represent the mass of a quantity of a radioactive element whose half-life is 25 years. After t years, the mass (in grams) is $y = 10\left(\frac{1}{2}\right)^{\frac{t}{25}}$

initial mass

(a) What is the initial mass (when t=0)?

$y = 10\left(\frac{1}{2}\right)^{\frac{0}{25}}$

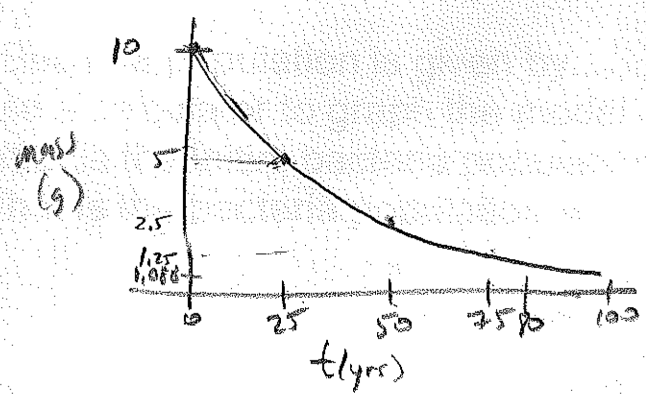
$y = 10\left(\frac{1}{2}\right)^{0}$

$y = 10(1)$

$\boxed{y = 10 \text{ grams}}$

(b) How much of the initial mass is present after 80 years?

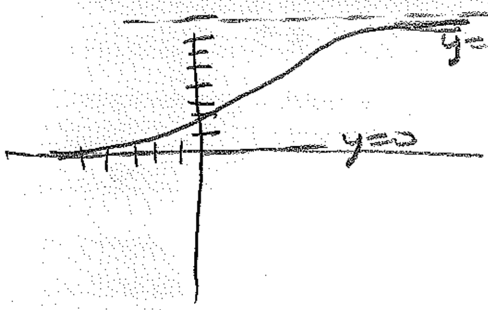
$y = 10\left(\frac{1}{2}\right)^{\frac{80}{25}} = 1.088$ grams



Rows 1, 4:

#55. Use a graphing utility to (a) graph the function and (b) find any asymptotes numerically by creating a table of values for the function.

$$f(x) = \frac{8}{1+e^{-0.5x}}$$



X	f(x)
−5	.6069
−50	1e−10
5	7.39
50	8

y = 0

y = 8

Rows 2, 5:

#73. Bacteria population is given by.......

$$P(t) = 100e^{0.2197t}$$

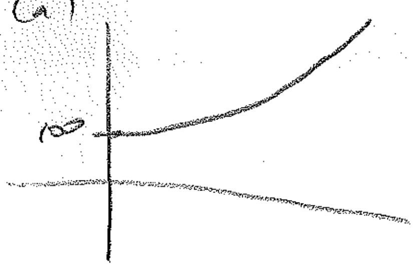
(*t is time in hours*)

(a) Use a calculator to graph the model.

(b) What is the initial bacteria population?

(c) What is the bacteria population at 5 hours and 10 hours?

(a)



(b)

$P(0) = 100e^{0.2197(0)} = 100$

(c) $P(5) = 100e^{0.2197(5)} = 299.966$

$P(10) = 100e^{.2197(10)} = 899.797$

Rows 3, 6:

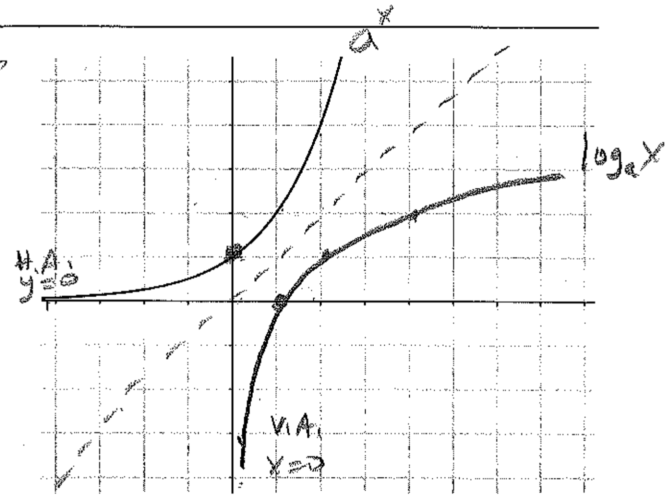
#63. Complete the table to determine the balance A for P=$2500 invested at a rate r = 8% for t = 10 years.

n	1	~~15~~ 12	~~30~~ 365	Continuous
A	5397.3	5549.1	5563.4	5563.85

$A = P(1+\frac{r}{n})^{nt}$ $\qquad$ $A = Pe^{rt}$

HAlg3-4, 3.2 Notes – Logarithms

Does the exponential function have an inverse? Yes



$f(x)=a^x \Longleftrightarrow f^{-1}(x)=\log_a x$

exponential function

logarithmic function
"log base a of x"

Logarithm and exponential functions (of same base) are inverses – they 'undo' each other:

might skip this

$3^x = 8$

$\log_3(3^x)=\log_3 8$

$x=\log_3 8$

$\log_4 x = 2$

$4^{(\log_4 x)} = 4^{(2)} = 16$

$x = 16$

"a logarithm is an exponent"...

$a^y = x \quad \leftrightarrow \quad y = \log_a x$

"Exponential form" "Logarithmic form"

if a=10 (base is 10):

$10^2 = 100 \quad \leftrightarrow \quad \log_{10} 100 = 2$

$10^3 = 1000 \quad \leftrightarrow \quad \log_{10} 1000 = 3$

$\log_{10} 100 = 2$ raised to $\quad 10^2 = 100$

if a=2 (base is 2):

$2^2 = 4 \quad \leftrightarrow \quad \log_2 4 = 2$

$2^3 = 8 \quad \leftrightarrow \quad \log_2 8 = 3 \qquad 2^3 = 8$

raise base to an exponent, get a number $\leftrightarrow$ log of that number gives the exponent

Examples:

$\log_2 32 = x \; \boxed{5}$ $\qquad \log_3 27 = \boxed{3}$ $\qquad \log_{10}\left(\frac{1}{100}\right) = \boxed{-2}$ $\qquad \log_4 2 = \boxed{1/2}$ (square root)

$2^x = 32 \qquad 3^x = 27 \qquad 10^x = \frac{1}{100} \qquad 4^x = 2 = 1/2$

$f(x) = \log_{10} x$ 'common logarithmic function'

$f(x) = \log_e x$ 'natural logarithmic function'

$\log_e x = \ln x$ (calculators have buttons for common log, and natural log)

If not marked, $\log x = \log_{10} x$

assume 10 unless marked

p6 start

A few properties of logarithmic functions:

$\log_a 1 = 0$	$\ln 1 = 0$	$a^0=1 \quad e^0=1$
$\log_a a = 1$	$\ln e = 1$	$a^1=a \quad e^1=e$
$\log_a(a^x) = x$	$\ln(e^x) = x$	inverse functions
$a^{(\log_a x)} = x$	$e^{(\ln x)} = x$	undo each other

if $\log_a x = \log_a y$, *then* $x = y$ *if* $\ln x = \ln y$, *then* $x = y$ 1:1 property

matching

Examples: Find x...

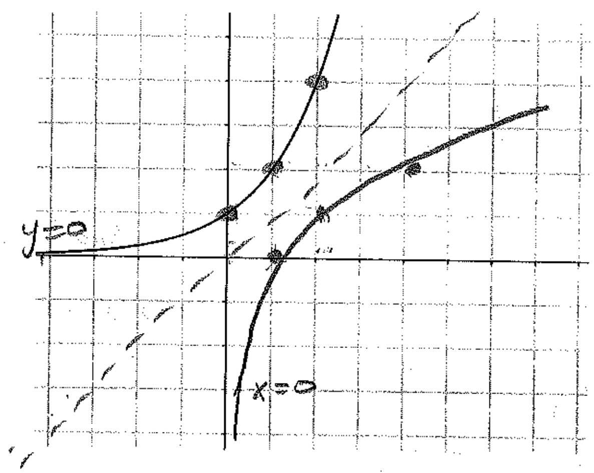
$\log_5 1 = x$
$5^x = 1$
$\boxed{x=0}$

$\log_3 3 = x$
$3^x = 3$
$\boxed{x=1}$

$\log_5 x = \log_5 8$
$\boxed{x=8}$

Graphs of logarithmic functions:

	$f(x) = 2^x$	$g(x) = \log_2 x$
Domain:	$(-\infty, \infty)$	$(0, \infty)$
Range:	$(0, \infty)$	$(-\infty, \infty)$
x-intercepts:	none	$(1,0)$
y-intercepts:	$(0,1)$	none
asymptotes:	$y=0$	$x=0$
increasing/ decreasing	increasing	increasing



Two ways to graph: **1) Make an x/y table and plot points.**
2) Graph exponential function and reflect over y=x line.

All shifting rules apply – affects domain, intercepts and asymptotes.

Example: Comparing $y = \log_{10} x$ and $y = \log_{10}(x-2)+3$

shift rt 2, up 3

Domain: $(2, \infty)$

Range: $(-\infty, \infty)$

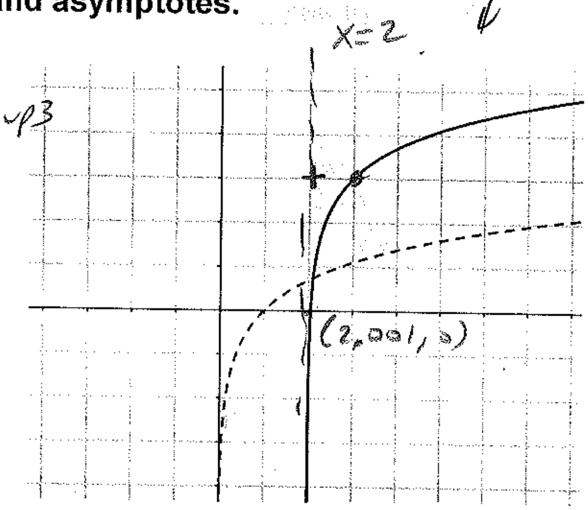
x-intercepts: $(2.001, 0)$

y-intercepts: none

asymptotes: $x=2$

x-int when y=0
$\log_{10}(x-2)+3=0$
$\log_{10}(x-2)=-3$
$10^{-3}=x-2$
$.001 = x-2$
$x=2.001$
$(2.001, 0)$

graph is actually $\log_2$ not $\log_{10}$



Honors Algebra 3-4
Exponent / Logarithms Worksheet 1

Name Key
Period______

Write each equation in exponential form:

#1. $\log_4 64 = 3$ $4^3 = 64$

#2. $\log_3 81 = 4$ $3^4 = 81$

#3. $\log_7 \frac{1}{49} = -2$ $7^{-2} = \frac{1}{49}$

#4. $\log_{10} \frac{1}{1000} = -3$ $10^{-3} = \frac{1}{1000}$

#5. $\log_{32} 4 = \frac{2}{5}$ $32^{2/5} = 4$

#6. $\log_{16} 8 = \frac{3}{4}$ $16^{3/4} = 8$

#7. $\ln 1 = 0$ $e^0 = 1$

#8. $\ln 4 = 1.386....$ $e^{1.386\ldots} = 4$

Write each equation in logarithmic form:

#9. $5^3 = 125$ $\log_5 125 = 3$

#10. $8^2 = 64$ $\log_8 64 = 2$

#11. $81^{\frac{1}{4}} = 3$ $\log_{81} 3 = \frac{1}{4}$

#12. $9^{\frac{3}{2}} = 27$ $\log_9 27 = \frac{3}{2}$

#13. $6^{-2} = \frac{1}{36}$ $\log_6 \frac{1}{36} = -2$

#14. $10^{-3} = 0.001$ $\log_{10} 0.001 = -3$

#15. $e^3 = 20.0855...$ $\ln 20.0855\ldots = 3$

#16. $e^x = 4$ $\ln 4 = x$

Evaluate each expression without using a calculator:

#17. $\log_2 16$ $2^x = 16$ $\boxed{4}$

#18. $\log_{27} 9$ $27^x = 9$ $27^{2/3} = (27^{1/3})^2 = 9$ $\boxed{2/3}$

#19. $\log_{16}\left(\frac{1}{4}\right)$ $16^x = \frac{1}{4}$ $16^{-1/2} = \frac{1}{16^{1/2}}$ $\boxed{-\frac{1}{2}}$

#20. $\log_2\left(\frac{1}{8}\right)$ $2^x = \frac{1}{8}$ $\boxed{-3}$

#21. $\log_{10} 0.01$ $10^x = .01$ $\boxed{-2}$

#22. $\log_{10} 1000$ $10^x = 1000$ $\boxed{3}$

Solve each equation for x:

#23. $\log_7 x = \log_7 9$ $\boxed{x = 9}$

#24. $\log_5 5 = x$ $5^x = 5$ $\boxed{x = 1}$

#25. $\ln e^8 = x$ $\boxed{x = 8}$

#26. $\log_5 x = 2$ $5^2 = x$ $\boxed{x = 25}$

#27. $5^x = 125$ $\boxed{x = 3}$

#28. $e^x = 42$
$\ln(e^x) = \ln 42$
$x = \ln 42 \approx 3.7377$

Honors Algebra 3-4 Name Key
Exponent / Logarithms Worksheet 2 Period______

Evaluate using your calculator and the change of base formula (round to nearest 3 decimal places):

#1. $\log_3 7 = \frac{\ln 7}{\ln 3} = 1.771$

#2. $\log_7 4 = \frac{\ln 4}{\ln 7} = 0.712$

#3. $\log_{\left(\frac{1}{2}\right)} 4 = \frac{\ln 4}{\ln .5} = -2$

#4. $\log_{\left(\frac{1}{8}\right)} 64 = \frac{\ln 64}{\ln 1/8} = -2$

#5. $\log_9 (0.8) = \frac{\ln .8}{\ln 9} = -0.102$

#6. $\log_{\left(\frac{1}{3}\right)} (0.015) = \frac{\ln 0.015}{\ln 1/3} = 3.823$

#7. $\log_{15} 1460 = \frac{\ln 1460}{\ln 15} = 2.691$

#8. $\log_{20} 135 = \frac{\ln 135}{\ln 20} = 1.637$

Rewrite the logarithm as a multiple (fraction) of (a) a common logarithm (b) a natural logarithm.

#9. $\log_5 x = \frac{\log_{10} x}{\log_{10} 5} = \frac{\ln x}{\ln 5}$

#10. $\log_3 x = \frac{\log_{10} x}{\log_{10} 3} = \frac{\ln x}{\ln 3}$

#11. $\log_x \left(\frac{3}{10}\right) = \frac{\log_{10} 3/10}{\log_{10} x} = \frac{\ln 3/10}{\ln x}$

#12. $\log_x \left(\frac{3}{4}\right) = \frac{\log_{10} 3/4}{\log_{10} x} = \frac{\ln 3/4}{\ln x}$

#13. $\log_{2.6} x = \frac{\log_{10} x}{\log_{10} 2.6} = \frac{\ln x}{\ln 2.6}$

#14. $\log_{\left(\frac{1}{3}\right)} x = \frac{\log_{10} x}{\log_{10} 1/3} = \frac{\ln x}{\ln 1/3}$

Use the properties of logarithms to write the expression as a sum, difference, and/or constant multiple of logarithms (assume all variables are positive).

#15. $\log_{10} 5x = \log_{10} 5 + \log_{10} x$

#16. $\log_{10}\left(\frac{y}{2}\right) = \log_{10} y - \log_{10} 2$

#17. $\log_6 z^{-3} = -3 \log_6 z$

#18. $\ln \sqrt[3]{t} = \ln t^{1/3} = \frac{1}{3} \ln t$

#19. $\ln \frac{xy}{z} = \ln x + \ln y - \ln z$

#20. $\ln\left(\frac{x^2-1}{x^3}\right)$, $x>1$
$\ln(x^2-1) - \ln x^3$
$\ln(x^2-1) - 3\ln x$
$\ln[(x-1)(x+1)] - 3\ln x$
$\boxed{\ln(x-1) + \ln(x+1) - 3\ln x}$

Write the expression as the logarithm of a single quantity.

#21. $\ln y + \ln s = \ln(ys)$

#22. $\log_5 8 - \log_5 t$
$= \log_5\left(\frac{8}{t}\right)$

#23. $3\ln x + 2\ln y - 4\ln z$
$\ln x^3 + \ln y^2 - \ln z^4 = \ln\left(\frac{x^3 y^2}{z^4}\right)$

#24. $\frac{5}{2}\log_7 (z-4) = \log_7 (z-4)^{5/2}$

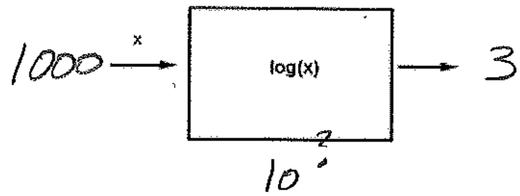
#25. $4\left[\ln z + \ln(z+5)\right] - 2\ln(z-5)$
$4\ln z + 4\ln(z+5) - \ln(z-5)^2$
$\ln z^4 + \ln(z+5)^4 - \ln(z-5)^2 = \boxed{\ln\left(\frac{z^4(z+5)^4}{(z-5)^2}\right)}$

#26. $\frac{3}{2}\ln 5t^6 - \frac{3}{4}\ln t^4$
$\ln(5t^6)^{3/2} - \ln(t^4)^{3/4}$
$\ln 5^{3/2} t^9 - \ln t^3$
$\ln 5^{3/2} t^9 - \ln t^3$
$\ln (5^3)^{1/2} \frac{t^9}{t^3}$
$\ln \sqrt{125}\, t^6$
$\boxed{\ln 5\sqrt{5}\, t^6}$

HAlg3-4, 3.3 day 1 Notes – Properties of Logarithms

The logarithmic function works the same way as other functions, accepts an input, provides an output:

$$f(x) = \log_{10}(x)$$
$$f(1000) = \log_{10}(1000)$$
$$f(1000) = 3$$



2 ways to evaluate this function...

By hand

$\log_{10}(1000) = x$

$10^x = 1000$

$x = 3$

Calculator

LOG(1000)

If our calculators can only find log base 10 or base e, how do we find $\log_2 8$?

Change of base formula: $\log_a x = \dfrac{\log_b x}{\log_b a}$ changes from base a to base b.

Base b can be anything, but we usually use b=10 or b=e because we have calculator keys for logarithms in those bases.

Examples:	using 'log' key:	using 'ln' key:
$\log_2 8 =$	$\dfrac{\log_{10} 8}{\log_{10} 2} = \dfrac{.90308...}{.3010...} = 3$	$\dfrac{\ln 8}{\ln 2} = \dfrac{2.079...}{.6931...} = 3$
$\log_4 30 =$	$\dfrac{\log_{10} 30}{\log_{10} 4} = 2.4534...$	$\dfrac{\ln 30}{\ln 4} = 2.4534...$
$\log_5 18 =$	$\dfrac{\log_{10} 18}{\log_{10} 5} = 1.7958...$	$\dfrac{\ln 18}{\ln 5} = 1.7958...$

More properties of logarithms:

$$\log_a(uv) = \log_a(u) + \log_a(v) \qquad \ln(uv) = \ln(u) + \ln(v)$$
$$\log_a\left(\frac{u}{v}\right) = \log_a(u) - \log_a(v) \qquad \ln\left(\frac{u}{v}\right) = \ln(u) - \ln(v)$$
$$\log_a u^n = n\log_a u \qquad \ln u^n = n\ln u$$

(Note: 'similar' to exponent rules: $x^u x^v = x^{(u+v)}$ and $\dfrac{x^u}{x^v} = x^{(u-v)}$)

Examples using properties:

$\log_3(2xy) = \log_3 2 + \log_3 x + \log_3 y$

$\log_5(2x^3y^2) = \log_5 2 + \log_5 x^3 + \log_5 y^2$
$= \log_5 2 + 3\log_5 x + 2\log_5 y$

$\log_4(4y) = \log_4 4 + \log_4 y$
$= 1 + \log_4 y$

$\ln\left(\frac{14x^3}{2h^2w^4}\right) = \ln(14x^3) - \ln(2h^2w^4)$
$= \ln 14 + \ln x^3 - [\ln 2 + \ln h^2 + \ln w^4]$
$= \ln 14 + 3\ln x - \ln 2 - 2\ln h - 4\ln w$

$\log_5(h) + \log_5(w) = \log_5(hw)$

$\log_5(2x) - 2\log_5(y) = \log_5(2x) - \log_5(y^2)$
$= \log_5\left(\frac{2x}{y^2}\right)$

$2\log_3 x - 3\log_3 y + \frac{1}{2}\log_3 z =$
$\log_3(x^2) - \log_3(y^3) + \log_3(z^{1/2})$
$\log_3\left(\frac{x^2 z^{1/2}}{y^3}\right)$
$\log_3\left(\frac{x^2\sqrt{z}}{y^3}\right)$

$\frac{1}{3}(2\ln x - 4\ln y - \ln(z+2)) =$
$\frac{1}{3}[\ln x^2 - \ln y^4 - \ln(z+2)]$
$\frac{1}{3}\ln\left(\frac{x^2}{y^4(z+2)}\right)$
$\ln\left(\frac{x^2}{y^4(z+2)}\right)^{1/3}$
$\ln\left(\sqrt[3]{\frac{x^2}{y^4(z+2)}}\right)$

HAlg3-4, 3.3 day 2 Notes – Rewriting log expressions, Applications of logarithms

Sometimes, you can find the exact values of logarithmic expressions without a calculator or rewrite logarithmic express using log properties:

teacher $\log_6 \sqrt[3]{6}$ $\quad \log_6 6^{1/3} \quad \frac{1}{3}\log_6 6 \quad (6^x=6, x=1)$ $\quad \frac{1}{3}(1) \quad \boxed{\frac{1}{3}}$

students $\log_5 \frac{1}{125}$ $\quad \log_5 1 - \log_5 125 \quad 5^x=1 \quad 0 \quad 5^x=125 \quad -3 \quad \boxed{-3}$ or $\log_4(64) \quad \log_4 4^3 = 3$

students $\log_4(-16)$ $\quad 4^x=-16$ (not possible)

teacher $\log_4 2+\log_4 32$ $\quad 4^x=2 \quad 4^x=32 \rightarrow$ but $2^5=32$ rewrite 4^x as $(2^2)^x$, $2^{2x}=32=2^5$ $\quad 2x=5 \quad x=\frac{5}{2}$

$\frac{1}{2}+\frac{5}{2}=\frac{6}{2} \quad \boxed{3}$

Practice:

teacher $\log_5 \frac{1}{15}$ $\quad \log_5 1 - \log_5 15$

$0 - \log_5(3\cdot5) \quad \boxed{-1-\log_5 3}$

$0 - \log_5 5 - \log_5 3$

$0 - 1 - \log_5 3$

students $\log_2(4^2 \cdot 3^4)$ $\quad \log_2 4^2 + \log_2 3^4$

$2\log_2 4 + 4\log_2 3$

$2(2)+4\log_2 3 \quad \boxed{4+4\log_2 3}$

students $\log_{10}\left(\frac{3}{300}\right)$ $\quad \log_{10} 3 - \log_{10} 100 \quad 10^x=100$

$\boxed{\log_{10} 3 - 2}$

students $\ln \frac{6}{e^2}$ $\quad \ln 6 - \ln e^2$

$\ln 6 - 2\ln e$

$\ln 6 - 2$

Applications of logarithms: Situations where a variable changes rapidly at first, then varies more slowly. Example:

Students participating in a psychological experiment attended several lectures. After the last lecture, and every month for the next year, the students were tested to see how much of the material they remembered. The average scores for the group were given by the memory model:

$f(t) = 90 - 15\log_{10}(t+1) \quad 0 \le t \le 12$ where t is time in months.

(a) What was the average score on the original exam (t=0)? $\quad 90-15\log_{10}(1) \quad 10^x=1 \quad x=0 \quad 90-0 \quad \boxed{90}$

(b) What was the average score after 6 months? $\quad 90-15\log_{10}(7) = \boxed{77.3}$

(c) What was the average score after 12 months? $\quad 90-15\log_{10}(13) = \boxed{73.3}$

(d) How long did it take for the average score to decrease to 75?

$90-15\log_{10}(t+1)=75$

$15\log_{10}(t+1)=15$

$\log_{10}(t+1)=1$

$10^1=t+1$

$10=t+1$

$-1 \quad -1$

$9=t$

$\boxed{9 \text{ months}}$

New (make sure slider match)

HAlg3-4, 3.4 day 1 Notes – Solving Exponential and Logarithmic Equations

Reminder: How do we solve for x in these cases?

$2^x = 3$

Compare: $\frac{2x}{2}=\frac{3}{2}$ $\quad x^2=3$ $\quad \sqrt{x^2}=\sqrt{3}$

need to 'undo' 2^x

$\log_2(2^x)=\log_2(3)$

$x=\log_2(3)$

$\log_2 x = 3$

$2^{(\log_2 x)}=2^3$

$x=2^3$

$x=8$

easier: write in exp. form:

$\log_2 x = 3$

$2^3=x$

$8=x$

Strategies for solving exponential and logarithmic equations:

Use inverse functions:

- First, isolate the term with x.
- Use log properties to combine to one term.
- Substitute a variable to get a quadratic.

Use the 1:1 property for logarithms or exponents:

- Find a common base on both sides.

#1. Solve for x: $\ln x = -1$ (use inverse function)

$e^{\ln x}=e^{-1}$

$x=e^{-1}$

$\boxed{x=0.3678\ldots}$

** (check answer)

#2. Solve for x: $\log_{10} x = -\frac{1}{2}$ (use inverse function)

$10^{-1/2}=x$

$x=\frac{1}{10^{1/2}}$

$\boxed{x=\frac{1}{\sqrt{10}}}$

or $10^{(\log_{10} x)}=10^{-1/2}$

$x=10^{-1/2}$

** check answer

#3. Solve for x: $5e^{x+2}-8=14$ (isolate term w/x, then use inverse)

$5e^{x+2}=22$

$e^{x+2}=\frac{22}{5}$

$\ln(e^{x+2})=\ln\left(\frac{22}{5}\right)$

$x+2=\ln\left(\frac{22}{5}\right)$

$\boxed{x=-2+\ln\left(\frac{22}{5}\right)}$

$x=-0.51839\ldots$

** check answer

#4. Solve for x: $\log_{10} x - \log_{10}(x-3) = 1$

(log properties, then inverse)

$\log_{10}\left(\frac{x}{x-3}\right) = 1$

$10^1 = \frac{x}{x-3}$

$\frac{x}{x-3} = \frac{10}{1}$

$x = 10(x-3)$

$x = 10x - 30$

$-9x = -30$

$x = \frac{30}{9} = \boxed{\frac{10}{3}}$

**check answer

#5. Solve for x: $\ln\sqrt{x+2} = \ln x$

(use 1:1 property)

$\sqrt{x+2} = x$

$(\sqrt{x+2})^2 = x^2$

$x+2 = x^2$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$\boxed{x = 2}$ $\cancel{x = -1}$ extraneous

**check answer

#6. Solve for x: $\left(\frac{3}{4}\right)^x = \frac{27}{64}$

(get same base, use 1:1 property)

$\left(\frac{3}{4}\right)^x = \left(\frac{3^3}{4^3}\right)$

$\left(\frac{3}{4}\right)^x = \left(\frac{3}{4}\right)^3$

$\boxed{x = 3}$

**check answer

#7. Solve for x: $e^{2x} - e^x - 20 = 0$

(substitute a variable to get a quadratic)

$u = e^x$

$u^2 - u - 20 = 0$

$(u-5)(u+4) = 0$

$(e^x - 5)(e^x + 4) = 0$

**check answer

$e^x - 5 = 0$ $e^x + 4 = 0$

$e^x = 5$ $e^x = -4$

$\ln(e^x) = \ln(5)$ $\ln(e^x) = \ln(-4)$

$\boxed{x = \ln 5}$ $x = \cancel{\ln(-4)}$ -4 not in domain of ln

#8. Solve for x: $\ln(x-2) + \ln(2x-3) = 2\ln x$

(log properties, then 1:1)

$\ln[(x-2)(2x-3)] = \ln(x^2)$

**check answer

$(x-2)(2x-3) = x^2$

$2x^2 - 7x + 6 = x^2$

$x^2 - 7x + 6 = 0$

$(x-6)(x-1) = 0$

$\boxed{x = 6}$ $\cancel{x = 1}$ ext.

$\ln(1-2)\ldots$

$\ln(-1)$

not in domain

HAlg3-4, 3.4 day 2 Notes – Solving Exponential and Logarithmic Equations

Summary of solving strategies:

1) Combine all exponential or logarithmic expressions into 1 term on 1 side, then use inverse.

$$\log_4 x - \log_4(x-1) = \frac{1}{2}$$

$\log_4\left(\frac{x}{x-1}\right) = \frac{1}{2}$

$4^{\log_4\left(\frac{x}{x-1}\right)} = 4^{1/2} = \sqrt{4}$

$\frac{x}{x-1} = 2$

$\frac{x}{x-1} = 2$

$x = 2(x-1)$

$x = 2x - 2$

$\boxed{2 = x}$

2) Combine all exponential or logarithmic expressions into 1 term on each side, then use 1:1.

$$\ln x - \ln 5 = 0$$

$\ln x = \ln 5$

$\boxed{x = 5}$

How would we handle this one? $\ln x = x^2 - 2$

$e^{\ln x} = e^{(x^2-2)}$

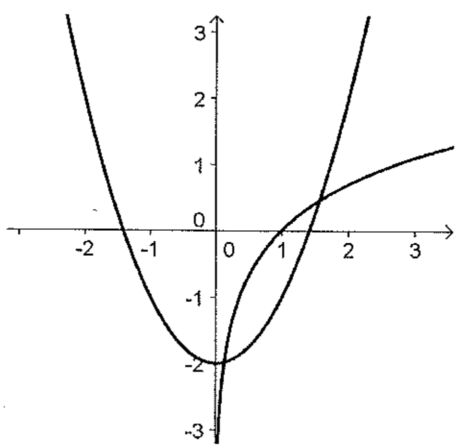
$x = e^{(x^2-2)}$??

$x = \cancel{e^{x^2}e^{-2}}$

Regular strategies don't work in cases like this, so we resort to graphing calculator. Two ways to do this:

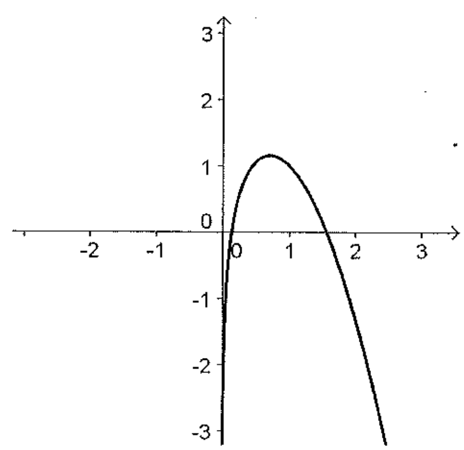
1) Enter each side of the equation in as a separate equation and find the intersection points.

(Use calculator 'intersect' feature)
2nd (trace) for CALC menu, intersect
--try window x: -5 to 5, y: -5 to 5



2) Move everything to one side to make a single equation = 0, then find zeros.

(Use calculator 'zero' feature)
2nd (trace) for CALC menu, zero
--try window x: -1 to 3, y: -2 to 2



Can also use graphing to quickly check for extraneous solutions:

Solve for x: $\log_{10} x + \log_{10}\left(x^2 - 8\right) = \log_{10} 8x$

$\log_{10} x \cdot (x^2 - 8) = \log_{10} 8x$

$x(x^2 - 8) = 8x$

$x^3 - 8x = 8x$

$x^3 - 16x = 0$

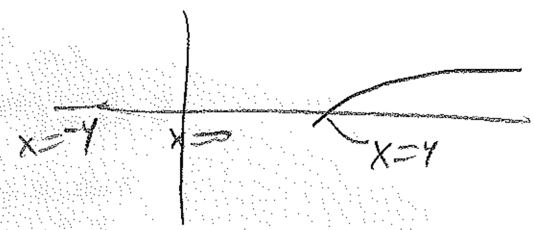
$x(x^2 - 16) = 0$

$x(x+4)(x-4) = 0$

$x = 0, \ x = -4, \ x = 4$

~~$x=0$~~, ~~$x=-4$~~ extraneous

graph $\log x + \log(x^2 - 8) - \log 8x = 0$



Example: How long would it take for an investment to double if the interest was compounded continuously at 8%? $r = .08$

$A = Pe^{rt}$

starting amount $= P$
at time t, amount doubles to $2P$

@ t_0

$A = Pe^{.08(0)}$
$A = Pe^{0}$
$A = P(1)$
$A = P$

$A = Pe^{rt}$
$2P = Pe^{.08t}$
$2 = e^{.08t}$
$e^{.08t} = 2$
$\ln(e^{.08t}) = \ln 2$
$.08t = \ln 2$
$t = \frac{\ln 2}{.08}$
$t = \boxed{8.66 \text{ years}}$

t	A
0	P
t	$2P$

Example: You have $50,000 to invest. You need to have $350,000 to retire in 30 years. At what continuously compounded interest rate would you need to invest to reach your goal?

$P = \$50{,}000$
$A = \$350{,}000$ when $t = 30$
$r = ?$

$A = Pe^{rt}$
$350000 = 50000\, e^{r(30)}$
$35 = 5e^{30r}$
$7 = e^{30r}$
$e^{30r} = 7$
$\ln(e^{30r}) = \ln 7$
$30r = \ln 7$
$r = \frac{\ln 7}{30}$
$r = .0649$

$\boxed{\approx 6.5\%}$

Aredore (slides match?)

HAlg3-4, 3.5 day 1 Notes – Exponential and Logarithmic Models

Five most common math models using exponential or logarithmic functions:

1) Exponential growth model: $y = ae^{bx}$ $(b > 0)$

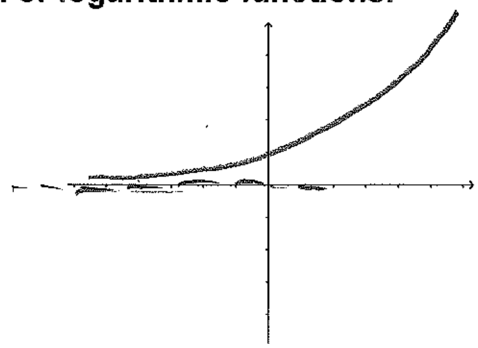
- populations (unrestrained)
- interest

$A = Pe^{rt}$

$A = P\left(1+\frac{r}{n}\right)^{nt}$

growth $= (>1)^x$

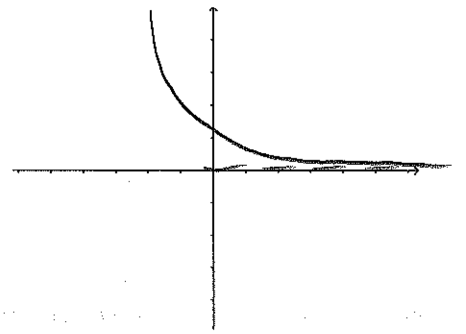
decay $= (<1)^x$



2) Exponential decay model: $y = ae^{-bx}$ $(b > 0)$

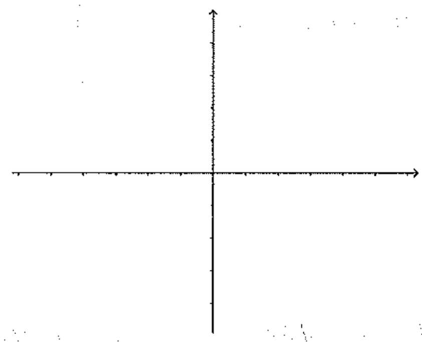
- radioactive decay

$Q(t) = Q_0\left(\frac{1}{2}\right)^{\frac{t}{halflife}}$



3) Gaussian model: $y = ae^{-(x-b)^2/c}$ 'bellcurve'

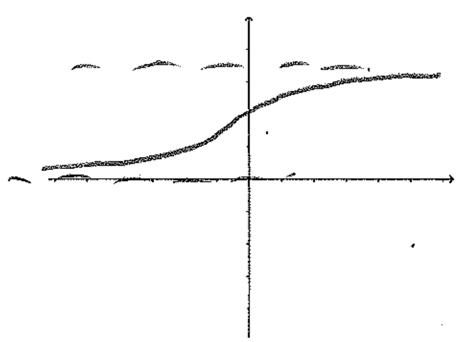
- IQ
- test scores



4) Logistic growth model: $y = \dfrac{a}{1+be^{-rx}}$

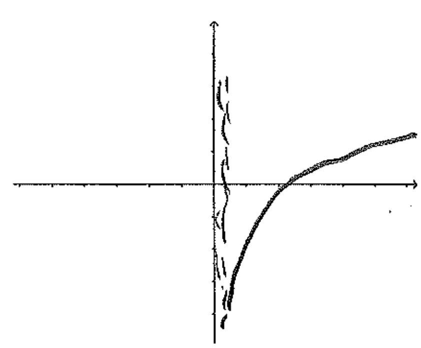
(sigmoidal curve)

- spread of virus
- populations in constrained environment



5) Logarithmic models: $y = a + b\ln x$ $y = a + b\log_{10} x$

- Sound intensity
- earthquake magnitudes



Foredore (slides match?)

To create a model, pick the general equation most closely matching data and solve (using data points) to find constants for specific model.

Similar to 'find an equation of a line with a given slope through a point':
Find the equation of a line with slope of -2 through point (1,4). What is the y value of the point on the line if x=5?

$y = mx + b$
$y = -2x + b$
$(4) = -2(1) + b$
$4 = -2 + b$
$6 = b$

x	y
1	4
5	-4

complete model:
$y = -2x + 6$
use the model
$y = -2(5) + 6$
$y = -10 + 6$
$y = -4$

Compound Interest (example of exponential growth)

Separate models for compounding n times per year, and compounding continuously:

n times per year: $A = P\left(1 + \frac{r}{n}\right)^{nt}$

continuous compounding: $A = Pe^{rt}$

Example, solving to find constants: Find a model equation for the amount in an investment account for time t in years. The investment is $5000, interest is compounded continuously, and investments in this account double every 8 years.

$A = Pe^{rt}$
$5000 = Pe^{r(0)}$
$5000 = P(1)$
$P = 5000$

t	A
0	5000
8	10000

$A = 5000e^{rt}$
$10000 = 5000e^{r(8)}$
$2 = e^{r8}$
$\ln 2 = \ln(e^{r8})$
$\ln 2 = r8$
$r = \frac{\ln 2}{8} = .086643$

$A = 5000e^{.086643t}$

New definition: Effective Yield (EY) (or 'effective interest rate')
Effective yield = equivalent annual interest rate (compounded annually) that would give the same interest as the actual investment returns.

Effective Yield = (total interest earned in a year) / (amount invested)
For continuous compounding, Effective Yield = $e^r - 1$

Example: If $100 is invested in an account with an annual interest rate of 6% compounded monthly, what is the effective interest rate (effective yield)?

$A = P\left(1 + \frac{r}{n}\right)^{nt}$

$A = 100\left(1 + \frac{.06}{12}\right)^{12t}$

at 1yr, t=1

$A = 100\left(1 + \frac{.06}{12}\right)^{12(1)}$

$A = 106.16778$

6% annual interest rate
but actually earned $6.16778 interest
So effective interest rate is:

$\frac{6.16778}{100} = .0616778$

$\approx \boxed{6.17\%}$

Amanda (videos middle?)

groups *

Example: Find all missing values to complete the table for an investment which compounds continuously:

** do all teacher example 1st (both pages)

	Initial Investment	Annual % rate	Time to double	Amount after 10 years
1	\$1000	12%	5.77 yrs	\$ 3320.12
2	\$750	9.22	7.5 years	\$1889.88
3	\$600	9.22	7.5 yrs	\$1505
4	8986.58	8%	8.7 yrs	\$20,000

(1)

$A = Pe^{rt}$

$A = 1000e^{.12t}$

$2000 = 1000e^{.12t}$

$2 = e^{.12t}$

$\ln(2) = .12t$

$t = \frac{\ln 2}{.12} = 5.77$

$A = 1000e^{.12(10)}$

$= 3320.12$

t	A
0	1000
?	2000

(2) example

$A = Pe^{rt}$

$A = 750e^{rt}$

$1500 = 750e^{r(7.5)}$

$2 = e^{r(7.5)}$

$\ln 2 = r(7.5)$

$r = \frac{\ln 2}{7.5} = .0924$

$A = 750e^{.0924(10)}$

$= 1889.88$

t	A
0	750
7.5	1500

(3)

$A = Pe^{rt}$

$A = 600e^{rt}$

$1505 = 600e^{r(10)}$

$\frac{1505}{600} = e^{r(10)}$

$\ln\left(\frac{1505}{600}\right) = r(10)$

$r = \frac{\ln\left(\frac{1505}{600}\right)}{10}$

$r = .092$

$1200 = 600e^{.092t}$

$2 = e^{.092t}$

$\ln 2 = .092t$

$t = \frac{\ln 2}{.092} \approx 7.5$

t	A
0	600
10	1505
?	1200

(4)

$A = Pe^{rt}$

$A = Pe^{.08t}$

$\frac{20000}{e^{.08(10)}} = \frac{Pe^{.08(10)}}{e^{.08(10)}}$

$P = \frac{20000}{e^{.08(10)}} = 8986.58$

$2P = Pe^{.08t}$

$2 = e^{.08t}$

$\ln 2 = .08t$

$t = \frac{\ln 2}{.08} = 8.66$

t	A
0	P
10	20000
?	2P

Exponential Growth and Decay

$$Q(t) = Q_0 e^{kt} \quad (k > 0, growth \quad k < 0, decay)$$

Example: Find all missing values to complete the table for the decay of a radioactive substance:

	Half-life (years)	Initial Quantity	Amount After 1000 years	Equation model
1	1620	2.3 g	1.5 g	$Q = 2.3 e^{-4.279 \times 10^{-4} t}$
2	5730	3 g	2.66 g	$Q = 3 e^{-1.2097 \times 10^{-4} t}$
3	1943	10 g	7 g	$Q = 10 e^{-3.5667 \times 10^{-4} t}$

(3)

$$Q = Q_0 e^{kt}$$
$$Q = 10 e^{kt}$$
$$7 = 10 e^{k(1000)}$$
$$\frac{7}{10} = e^{k1000}$$
$$\ln\left(\frac{7}{10}\right) = k1000$$
$$k = \frac{\ln\left(\frac{7}{10}\right)}{1000} = -3.5667 \times 10^{-4}$$

*

t	Q
0	10
1000	7

$$\frac{1}{2} = e^{-3.5667 \times 10^{-4} t}$$
$$\ln\left(\frac{1}{2}\right) = -3.5667 \times 10^{-4} t$$
$$t = \frac{\ln\left(\frac{1}{2}\right)}{-3.5667 \times 10^{-4}} = 1943$$

book also uses this model: $Q(t) = Q_0 \left(\frac{1}{2}\right)^{\frac{t}{halflife}}$

(1)

$$Q = Q_0 e^{kt}$$
$$\frac{1}{2} Q_0 = Q_0 e^{k1620}$$
$$\frac{1}{2} = e^{k1620}$$
$$\ln\left(\frac{1}{2}\right) = k1620$$
$$k = \frac{\ln\left(\frac{1}{2}\right)}{1620}$$
$$k = -4.279 \times 10^{-4}$$
$$Q = Q_0 e^{-4.279 \times 10^{-4} t}$$
$$1.5 = Q_0 e^{-4.279 \times 10^{-4}(1000)}$$
$$Q_0 = \frac{1.5}{e^{-4.279 \times 10^{-4}(1000)}} = 2.3$$

t	Q
0	Q_0
1620	$\frac{1}{2} Q_0$
1000	1.5

$\boxed{\text{* show how to store}}$

(2)

$$Q = Q_0 e^{kt}$$
$$Q = 3 e^{kt}$$
$$1.5 = 3 e^{k(5730)}$$
$$\frac{1}{2} = e^{k(5730)}$$
$$\ln\left(\frac{1}{2}\right) = k5730$$
$$k = \frac{\ln\left(\frac{1}{2}\right)}{5730} = -1.2097 \times 10^{-4}$$
$$Q = 3 e^{-1.2097 \times 10^{-4}(1000)}$$
$$= 2.66$$

t	Q
0	3
5730	1.5

HAlg3-4, 3.5 day 2 Notes – Exponential and Logarithmic Models

partners pair do these

More examples of exponential and logarithmic models:

#1. **Population** The population of a city is given by the model: $P = 240{,}360e^{.012t}$ where t=0 represents the starting year, 2000. According to this model, when will the population reach 275,000?

d1 row1

$275{,}000 = 240{,}360e^{.012t}$

$e^{.012t} = \frac{275{,}000}{240{,}360}$

$.012t = \ln\frac{275000}{240{,}360}$

$t = \frac{\ln\frac{275000}{240{,}360}}{.012} = 11.219$

$\boxed{2011}$

t	P
0	240,360
t	275000

#2. **Bacteria Growth**. The number of bacteria, N, in a culture is given by the model: $N = 250e^{kt}$ where t is the time (in hours.) If N=280 when t=10, estimate the time required for the population to double in size.

d2 row2

$280 = 250e^{k(10)}$

$\frac{280}{250} = e^{10k}$

$10k = \ln\frac{280}{250}$

$k = \frac{\ln\frac{280}{250}}{10} = .0113328...$

$2 = e^{.0113328t}$

$\ln 2 = .011332..t$

$t_{double} = \frac{\ln 2}{k} = 61.16$ hrs.

$\boxed{\approx 61.2 \text{ hrs}}$

t	N
0	250
10	280
t	500

#3. **Depreciation**. A computer that costs \$4600 new has a 'book value' of \$3000 after 2 years.

t	V
0	4600
2	3000

(a) Find the linear model for the value of the computer over time: $V = mt + b$

$3000 = m(2) + 4600$

$-1600 = m2$

$-800 = m$

$\boxed{V = -800t + 4600}$

d3 row3

(b) Find the exponential model for the value over time: $V = ae^{kt}$

$3000 = 4600e^{k(2)}$

$\frac{3000}{4600} = e^{k(2)}$

$\ln(\frac{30}{46}) = k(2)$

$k = \frac{\ln(30/46)}{2}$

$k = -.2137$

$\boxed{V(t) = 4600e^{-.2137t}}$

row6

(c) Plot both using a graphing calculator. Which model depreciates faster in the first year?

$\boxed{\text{exponential model}}$

(d) Use each model to find the book values of the computer at 1 year and at 3 years.

lin: $V(1) = -800(1) + 4600 = \$3800$

$V(3) = -800(3) + 4600 = \$2200$

(use calc, table feature)

exp: $V(1) = 4600e^{(-.2137)(1)} = \$3774.90$

$V(3) = 4600e^{(-.2137)(3)} = \$2422.9$

#4. **Sales and Advertising**. The sales, S (in thousands of units), of a product after x hundred dollars is spent on advertising is: $S = 10\left(1 - e^{kx}\right)$. When \$500 is spent on advertising, 2500 units are sold.

(a) Complete the model by solving for k.

$2.5 = 10(1 - e^{k5})$ $\quad e^{k5} = .75$

$.25 = 1 - e^{k5}$ $\quad k5 = \ln .75$

$-.75 = -e^{k5}$ $\quad k = \frac{\ln .75}{5} = -0.0575364$

d2

(b) Estimate the number of units that will be sold if \$700 is spent on advertising.

row 4

$S = 10(1 - e^{(-.0575364)7})$

$S = 3.315$ thousand units

so $\boxed{3315}$

#5. **Intensity of Sound**. The level of sound, β (in decibels), with an intensity I is: $\beta(I) = 10\log\frac{I}{I_0}$ where I_0 is an intensity of 10^{-12} watt per square meter (faintest sound that can be heard by a human.)

Determine the level of sound, β, if:

(a) $I = 10^{-3.5}$ watt per square meter (jet 4 miles from takeoff)

$\beta = 10 \log \frac{10^{-3.5}}{10^{-12}} = 10 \log 10^{(-3.5+12)} = 10 \log 10^{8.5} = 10(8.5) = \boxed{85 \text{ dB}}$

(b) $I = 10^{-3}$ watt per square meter (diesel truck at 25 feet)

d2

$\beta = 10 \log \frac{10^{-3}}{10^{-12}} = 10 \log \frac{10^{12}}{10^{3}} = 10 \log 10^{9} = 10(9) = \boxed{90 \text{ dB}}$

row 5

(c) $I = 10^{-1.5}$ watt per square meter (auto horn at 3 feet)

$\beta = 10 \log \frac{10^{-1.5}}{10^{-12}} = 10 \log \frac{10^{12}}{10^{1.5}} = 10 \log 10^{10.5} = 10(10.5) = \boxed{105 \text{ dB}}$