HAlg3-4, 6.1 day 1 Notes – Law of Sines

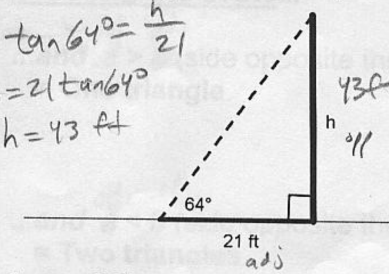
We know how to do right triangle problems like this...

Find the height of the telephone pole:

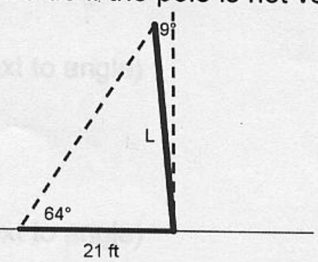
$\tan 64^\circ = \frac{h}{21}$

$h = 21\tan 64^\circ$

$h = 43$ ft

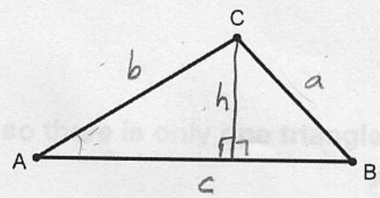


But what do we do if the pole is not vertical?



Law of Sines

brief proof...



$\sin A = \frac{h}{b}$ $\quad \sin B = \frac{h}{a}$

$h = b\sin A$ $\quad h = a\sin B$

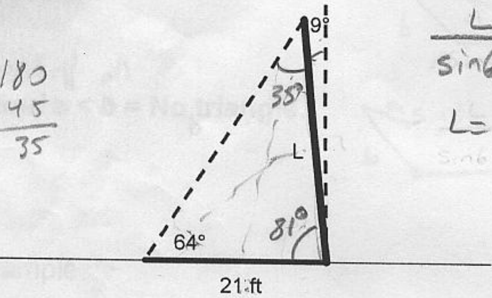
$b\sin A = a\sin B$

$\frac{b\sin A}{\sin B} = a \rightarrow \frac{b}{\sin B} = \frac{a}{\sin A}$

Law of Sines: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

ASA case:

$180 - 145 = 35$



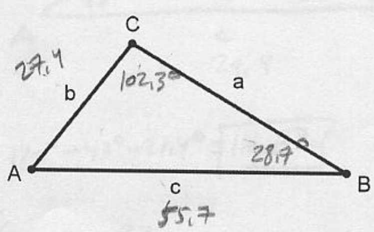
$\frac{L}{\sin 64^\circ} = \frac{21}{\sin 35^\circ}$

$L = 21\frac{\sin 64^\circ}{\sin 35^\circ} = 32.9$ ft

student!

AAS case:

If C=102.3°, B=28.7°, and b=27.4 ft, find the remaining angle and sides.



$\frac{c}{\sin 102.3^\circ} = \frac{27.4}{\sin 28.7^\circ}$

$c = 27.4\frac{\sin 102.3^\circ}{\sin 28.7^\circ} = 55.7$ ft

$\angle A = 180 - 102.3 - 28.7 = 49^\circ$

$\frac{a}{\sin 49^\circ} = \frac{27.4}{\sin 28.7^\circ}$

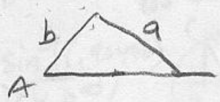
$a = 27.4\frac{\sin 49^\circ}{\sin 28.7^\circ} = 43.1$ ft

ASS case (the ambiguous case): There are 3 possible situations: 0, 1 or 2 triangles. How do you know which case you have? It depends upon the angle and the side lengths...

If the angle is acute...

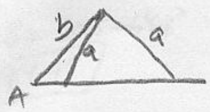
opp adj

***...and** a* **>** b (side opposite the angle is greater than side next to angle)
= One triangle.



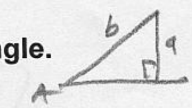
opp adj

***...and** a* **<** b (side opposite the angle is smaller than side next to angle)
= Two triangles.

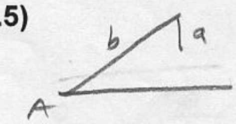


Special cases:

- 90° case - you'll find the angle is 90 degrees, so there is only one triangle.



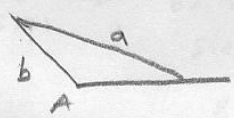
- No triangle case – you won't be able to calculate angle (e.g. sinA=1.5)



If the angle is obtuse...

opp adj

***...and** a* **>** b **= One triangle.**

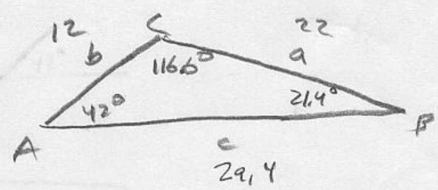


opp adj

***...and** a* **<** b **= No triangle.**

Examples:

a=22, b=12, A=42°
Find remaining sides and angles.



acute, a > b, 1 triangle

$$\frac{22}{\sin 42^\circ} = \frac{12}{\sin B}$$

$$22 \sin B = 12 \sin 42^\circ$$

$$\sin B = \frac{12 \sin 42^\circ}{22} = .36498$$

$$B = \sin^{-1}(.36498) = \boxed{21.4^\circ}$$

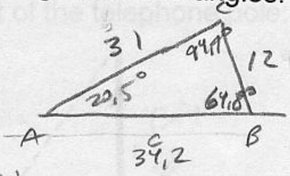
$$C = 180^\circ - 42^\circ - 21.4^\circ = \boxed{116.6^\circ}$$

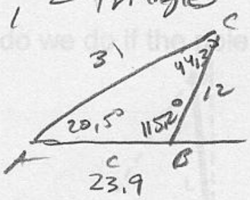
$$\frac{c}{\sin 116.6^\circ} = \frac{22}{\sin 42^\circ}$$

$$c = 22 \frac{\sin 116.6^\circ}{\sin 42^\circ} = \boxed{29.4}$$

a=12, b=31, A=20.5°
Find remaining sides and angles.

acute, a<b , 2 triangles





$\dfrac{12}{\sin 20.5°} = \dfrac{31}{\sin B}$

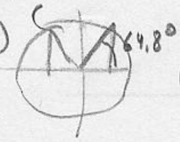
$12 \sin B = 31 \sin 20.5°$

$\sin B = \dfrac{31 \sin 20.5°}{12} = .90470$

$B = \sin^{-1}(.90470)$

$B = \boxed{64.8°}$

115.2°
180 − 64.8°



$B = \boxed{115.2°}$

$C = 180 - 20.5 - 64.8°$

$C = \boxed{94.7°}$

$C = 180 - 20.5° - 115.2°$

$C = \boxed{44.3°}$

$\dfrac{c}{\sin 94.7°} = \dfrac{12}{\sin 20.5°}$

$c = 12 \dfrac{\sin 94.7°}{\sin 20.5°} = \boxed{34.2}$

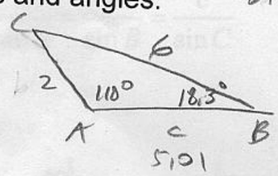
$\dfrac{c}{\sin 44.3°} = \dfrac{12}{\sin 20.5°}$

$c = 12 \dfrac{\sin 44.3°}{\sin 20.5°} = \boxed{23.9}$

try it

a=6, b=2, A=110°
Find remaining sides and angles.

obtuse, a>b , 1 triangle



$\dfrac{6}{\sin 110°} = \dfrac{2}{\sin B}$

$6 \sin B = 2 \sin 110°$

$\sin B = \dfrac{2 \sin 110°}{6} = .31323$

$B = \sin^{-1}(.31323)$

$B = \boxed{18.3°}$

$C = 180 - 110 - 18.3 = \boxed{51.7°}$

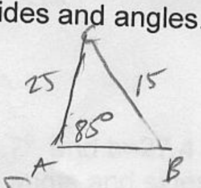
$\dfrac{c}{\sin 51.7°} = \dfrac{6}{\sin 110°}$

$c = 6 \dfrac{\sin 51.7°}{\sin 110°}$

$c = \boxed{5.01}$

a=15, b=25, A=85°
Find remaining sides and angles.

acute, a<b , 2 triangles



$\dfrac{15}{\sin 85°} = \dfrac{25}{\sin B}$

$15 \sin B = 25 \sin 85°$

$\sin B = \dfrac{25 \sin 85°}{15} = 1.66$

no sol'n

$\boxed{\text{No triangle}}$

HAlg3-4, 6.1 day 2 Notes – Law of Sines

Area of an Oblique Triangle

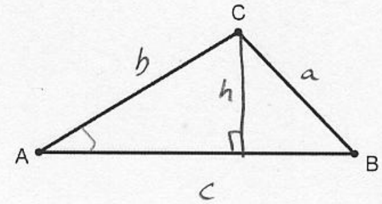
brief proof...

$\sin A = \frac{h}{b}$

$h = b\sin A$

area $= \frac{1}{2}$ base $\cdot$ height $= \frac{1}{2} c\,(b\sin A)$

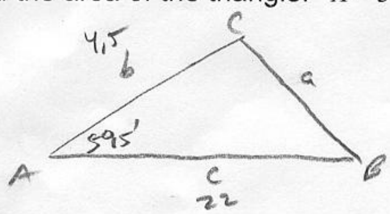
area $= \frac{1}{2} bc\sin A$



$$area = \frac{1}{2}bc\sin A = \frac{1}{2}ab\sin C = \frac{1}{2}ac\sin B$$

← use when you have 2 sides and angle between.

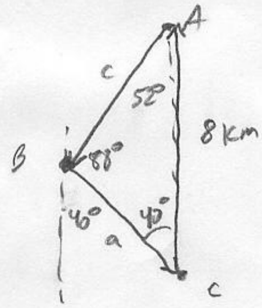
Example: Find the area of the triangle: $A = 5°15', b = 4.5, c = 22$



$A = \frac{1}{2}(4.5)(22)\sin 5°15'$

$\boxed{A = 4.529 \ u^2}$

Example: The course for a boat race starts at point A and proceeds in the direction S52°W to point B, then in the direction S40°E to point C, and finally back to point A. The point C lies 8 kilometers directly south of point A. Approximate the total distance of the race course.

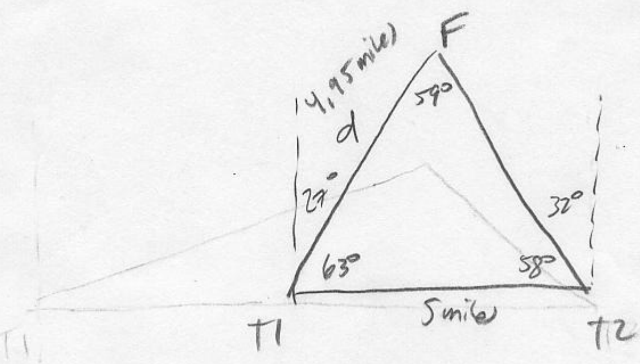


$\frac{8}{\sin 88°} = \frac{a}{\sin 52°}$ $\quad a = \frac{8\sin 52°}{\sin 88°} = 6.308$ km

$\frac{8}{\sin 88°} = \frac{c}{\sin 40°}$ $\quad c = \frac{8\sin 40°}{\sin 88°} = 5.1454$ km

dist $= 6.308 + 5.145 + 8 = \boxed{19.453 \text{ km}}$

Example: Two fire ranger towers lie on the east-west line and are 5 miles apart. There is a fire with a bearing of N27°E from tower 1 and N32°W from tower 2. How far is the fire from tower 1?



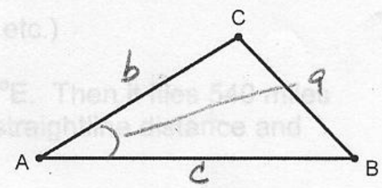
$\frac{5}{\sin 59°} = \frac{d}{\sin 58°}$

$d = \frac{5\sin 58°}{\sin 59°} = 4.95$ miles

HAlg3-4, 6.2 Notes – Law of Cosines

Law of Sines works for AAS, ASA, and ASS cases. What about SAS and SSS?

Law of Cosines:



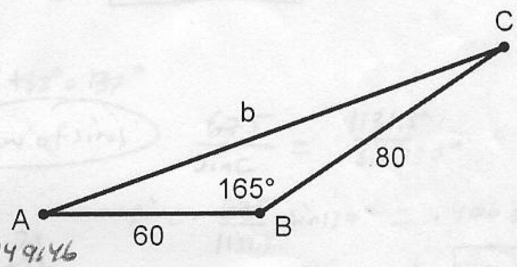
$a^2 = b^2 + c^2 - 2bc\cos A \qquad \cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

$b^2 = a^2 + c^2 - 2ac\cos B \qquad \cos B = \dfrac{a^2 + c^2 - b^2}{2ac}$

$c^2 = a^2 + b^2 - 2ab\cos C \qquad \cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

Like Pythagorean thm with extra term

Example (SAS case):
Find remaining sides and angles.



$b^2 = a^2 + c^2 - 2ac\cos B$

$b^2 = 80^2 + 60^2 - 2(80)(60)\cos 165°$

$b = \boxed{138.83}$

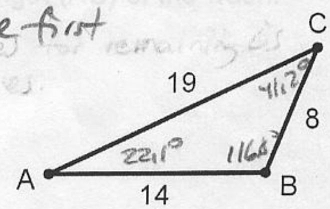
Use law of sines

$\dfrac{138.83}{\sin 165°} = \dfrac{80}{\sin A} \qquad \sin A = 80\,\dfrac{\sin 165°}{138.83} = .149146$

$A = \sin^{-1}(.149146) = \boxed{8.6°}$

$C = 180° - 165° - 8.6° = \boxed{6.4°}$

Example (SSS case): Find angle opposite biggest side first
Find the angles.



$\cos B = \dfrac{a^2 + c^2 - b^2}{2ac} = \dfrac{8^2 + 14^2 - 19^2}{2(8)(14)} = -.45089\ldots$

$B = 116.8°$

law of sines

$\dfrac{19}{\sin 116.8°} = \dfrac{14}{\sin C} \qquad \sin C = .65769 \qquad C = 41.2 \qquad A = 180 - 41.2 - 116.8 = 22.1$

Heron's Area Formula:

$$\text{area of triangle} = \sqrt{s(s-a)(s-b)(s-c)}, \quad \text{where } s = \frac{(a+b+c)}{2}$$

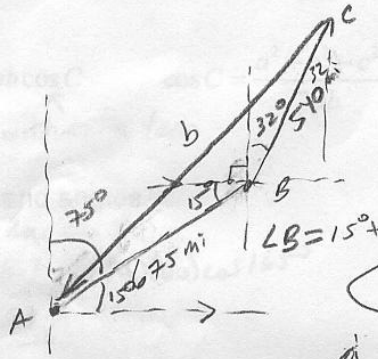
Example: Find the area of a triangle with sides of 3.5, 10.2, and 9.

$$s = \frac{a+b+c}{2} = \frac{3.5+10.2+9}{2} = 11.35$$

$$A = \sqrt{11.35(11.35-3.5)(11.35-10.2)(11.35-9)} = \boxed{15.517}\ u^2$$

(Note: remember to look for parallel lines, alternate interior angles, etc.)

Example: A plane flies 675 miles from A to B with a bearing of N75°E. Then it flies 540 miles from B to C with a bearing of N32°E. Draw a diagram and find the straightline distance and bearing from C to A.



law of cosines

$$b^2 = a^2 + c^2 - 2ac\cos B$$

$$b^2 = 675^2 + 540^2 - 2(675)(540)\cos 137°$$

$$b = \boxed{1131.5 \text{ miles}}$$

$$\angle B = 15° + 90° + 32° = 137°$$

law of sines:

$$\frac{675}{\sin C} = \frac{1131.5}{\sin 137°}$$

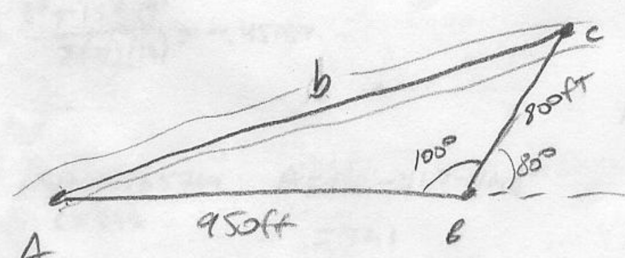
$$\sin C = \frac{675}{1131.5}\sin 137° = .406848$$

$$C = \sin^{-1}(.406848) = \boxed{24.°}$$

$$\begin{array}{r} 24 \\ +32 \\ \hline 56° \end{array}$$

$\boxed{S56°W}$

Example; To approximate the length of a wash, a surveyor walks 950 ft from point A to point B, then turns 80° and walks 800 feet to point C. Approximate the length (AC) of the wash.



$$b^2 = a^2 + c^2 - 2ac\cos B$$

$$b^2 = 950^2 + 800^2 - 2(950)(800)\cos 100°$$

$$AC = b = \boxed{1344 \text{ ft}}$$

HAlg3-4, 6.5 day 1 Notes – Trigonometric Forms of Complex Numbers

We've defined a complex number, having a real and imaginary part which can be plotted on the complex plane:

$$z = -3 + 4i$$

Two ways to specify a complex number:

$r^2 = -3^2 + 4^2$

$r = \sqrt{9+16}$

$r = \sqrt{25}$

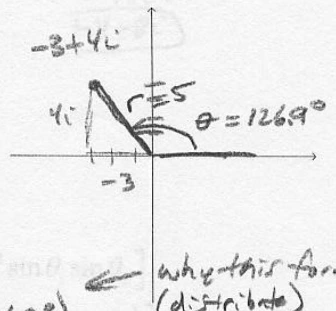
$r = 5$

$\tan\theta = \frac{(y)}{(x)}$

$\tan\theta = \frac{4}{-3}$

$\theta = \tan^{-1}\left(\frac{-4}{3}\right) = -53.1°$ (wrong quadrant) other side of circle $(+180°)$

$\theta = 126.9°$



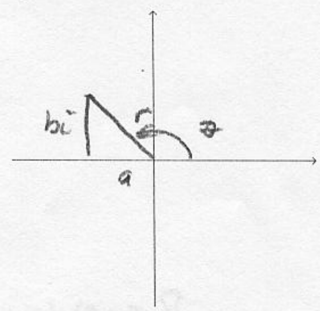
$-3 + 4i$ —or— $5(\cos 126.9° + i \sin 126.9°)$

why this form? (distribute)

$5\cos 126.9° + 5\sin 126.9°\, i$

$-3 \quad + 4i$

More generally:



Standard (or rectangular) Form	**Trigonometric (or polar) Form**
$z = a + bi$	$z = (r\cos\theta) + i(r\sin\theta)$
$r = \sqrt{a^2+b^2}$	$z = r(\cos\theta + i\sin\theta)$
$\tan\theta = \frac{b}{a}$ (careful about quadrant)	$a = r\cos\theta$
a – 'real component'	$b = r\sin\theta$
b – 'imaginary component'	r – 'modulus'
	θ – 'argument'

Example: Write $-2+2i$ in trig form

$r = \sqrt{a^2+b^2} = \sqrt{(-2)^2+(2)^2} = \sqrt{4+4}$

$r = \sqrt{8} = 2\sqrt{2}$

$\tan\theta = \frac{b}{a} = \frac{2}{-2} = -1$

$\theta = \frac{3\pi}{4}$ or 135

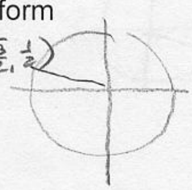
trig form: $r(\cos\theta + i\sin\theta)$

$\boxed{2\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)}$

Write $4\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$ in a+bi form

$4\cos\frac{5\pi}{6} + 4\sin\frac{5\pi}{6}\, i$

$\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$



$4\left(-\frac{\sqrt{3}}{2}\right) + 4\left(\frac{1}{2}\right)i$

$\boxed{-2\sqrt{3} + 2i}$

Absolute Value of a complex number: defined to be distance from origin (r)

$z = a + bi$

$|z| = |a+bi|$

$|z| = \sqrt{a^2+b^2}$

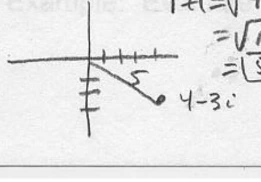
Example: Find the absolute values:

(a) $z = 4 - 3i$

$|z| = \sqrt{4^2+3^2}$

$= \sqrt{16+9}$

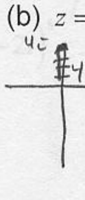
$= \boxed{5}$



(b) $z = 4i$

$|z| = \sqrt{4^2+0^2}$

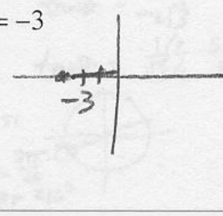
$|z| = \boxed{4}$



(c) $z = -3$

$|z| = \sqrt{3^2+0^2}$

$= \boxed{3}$



Multiplication and Division of Complex Numbers

Standard form: Like we learned last semester... Example: $(-2+2i)(3-i)$

(divide by multiplying top and bottom by complex conjugate of denominator)

$-6+2i+6i-2i^2$

$-6+8i+2$

$\boxed{-4+8i}$

Trigonometric form:

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1),$$
$$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

$$\begin{aligned} z_1 z_2 = \; & r_1(\cos\theta_1 + i\sin\theta_1)\cdot r_2(\cos\theta_2 + i\sin\theta_2) \\ & r_1 r_2\left[\cos\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 + i^2\sin\theta_1\sin\theta_2\right] \\ & r_1 r_2\left[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)\right] \\ & r_1 r_2\left[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right] \end{aligned}$$

$$\boxed{z_1 z_2 = r_1 r_2\left[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right]}$$

$$\boxed{\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\right]}$$

Example: $z_1 = 8(\cos 120° + i\sin 120°)$, $z_2 = 6(\cos 150° + i\sin 150°)$

Find $z_1 z_2 = r_1 r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$

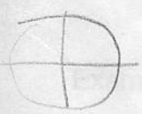
$= (8)(6)[\cos(120°+150°) + i\sin(120°+150°)]$

$= 48[\cos 270° + i\sin 270°]$

$= 48[0 + i(-1)]$

$= 0 - 48i$

$= \boxed{-48i}$



Find $\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)]$

$= \frac{8}{6}[\cos(120°-150°) + i\sin(120°-150°)]$

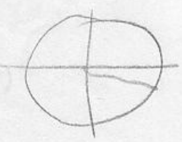
$= \frac{4}{3}[\cos(-30°) + i\sin(-30°)]$

$= \frac{4}{3}\left[\frac{\sqrt{3}}{2} + i\left(-\frac{1}{2}\right)\right]$

$= \boxed{\frac{2\sqrt{3}}{3} - \frac{2}{3}i}$

DeMoivre's Theorem: used to find powers of complex numbers

$$z = r(\cos\theta + i\sin\theta)$$
$$z^2 = rr(\cos(\theta+\theta) + i\sin(\theta+\theta)) = r^2(\cos 2\theta + i\sin 2\theta)$$
$$z^3 = r^3(\cos 3\theta + i\sin 3\theta)$$



$$z^n = \left[r(\cos\theta + i\sin\theta)\right]^n$$
$$z^n = r^n(\cos n\theta + i\sin n\theta)$$

Example: Evaluate $\left(-2\sqrt{3} - 2i\right)^5$

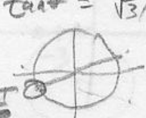
1st convert to trig form:

$r = \sqrt{(-2\sqrt{3})^2 + (-2)^2}$ $\tan\theta = \frac{b}{a}$

$= \sqrt{4\cdot 3 + 4}$ $\tan\theta = \frac{-2}{-2\sqrt{3}}$

$= \sqrt{16}$ $\tan\theta = \frac{-1/2}{-\sqrt{3}/2}$ $\frac{\sin}{\cos}$

$r = 4$



$\theta = \frac{7\pi}{6}$ or $210°$

$\left[4(\cos 210° + i\sin 210°)\right]^5$

$= 4^5(\cos(5\cdot 210°) + i\sin(5\cdot 210°))$

$= 1024(\cos 1050° + i\sin(1050°))$

$= 1024(\cos 330° + i\sin 330°)$

$= 1024\left(\frac{\sqrt{3}}{2} + i\left(-\frac{1}{2}\right)\right)$

$= \boxed{512\sqrt{3} - 512i}$

HAlg3-4, 6.5 day2 Notes – Roots of Complex Numbers

What are the square roots of 4? $2, -2$ $(2)(2)=4$ $(-2)(-2)=4$

What are the cube roots of 8? $2, -1+\sqrt{3}i, -1-\sqrt{3}i$

$(2)(2)(2)$

8 ✓

$(-1+\sqrt{3}i)(-1+\sqrt{3}i)(-1+\sqrt{3}i)$

$(1-\sqrt{3}i-\sqrt{3}i+3i^2)(-1+\sqrt{3}i)$

$(-2-2\sqrt{3}i)(-1+\sqrt{3}i)$

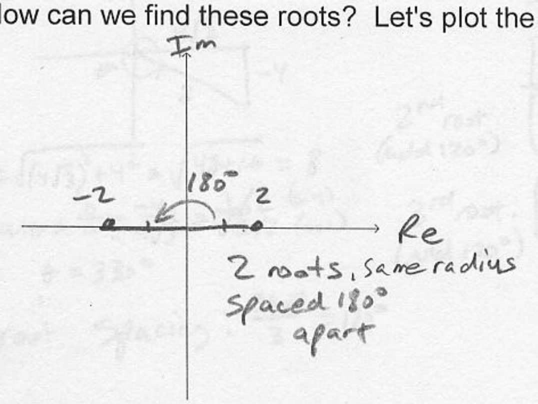
$2-2\sqrt{3}i+2\sqrt{3}i-2\cdot 3i^2$

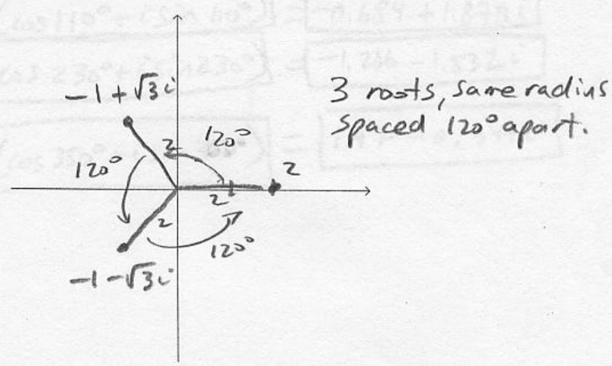
$2-6i^2$

$2+6$

8 ✓

How can we find these roots? Let's plot the roots found above on the complex plane:





For a complex number, z, the nth roots of z, $\sqrt[n]{z}=\sqrt[n]{r}(\cos\theta+i\sin\theta)$ (convert to trig form first)

- There will be n complex roots.
- The first root will be $\sqrt[n]{r}\left(\cos\frac{\theta}{n}+i\sin\frac{\theta}{n}\right)$ (this is just DeMoivre's theorem)
- The other roots will be evenly spaced around a circle, angle between roots = $\frac{360°}{n}$ or $\frac{2\pi}{n}$

$n=4$

Find the four 4th roots of 16 $= 16(\cos 0° + i\sin 0°)$

Convert to trig form

1st root: $\sqrt[n]{r}\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right)$

$\sqrt[4]{16}\left(\cos\frac{0°}{4} + i\sin\frac{0°}{4}\right)$

$\boxed{2(\cos 0° + i\sin 0°)} = 2 + 0i = \boxed{2}$

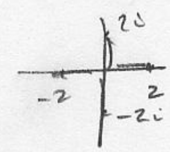
root spacing $= \frac{360°}{n} = \frac{360°}{4} = 90°$ apart

2nd root (add 90°): $\boxed{2(\cos 90° + i\sin 90°)} = 0 + 2i = \boxed{2i}$

3rd root (add 90°): $\boxed{2(\cos 180° + i\sin 180°)} = -2 + 0i = \boxed{-2}$

4th root (add 90°): $\boxed{2(\cos 270° + i\sin 270°)} = 0 - 2i = \boxed{-2i}$

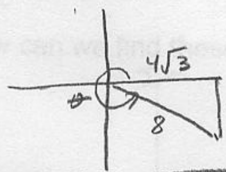
roots:



$n=3$

Find the three 3rd roots of $4\sqrt{3} - 4i$ $= 8(\cos 330° + i\sin 330°)$

Convert to trig form:



$r = \sqrt{(4\sqrt{3})^2 + 4^2} = \sqrt{48+16} = 8$

$\tan\theta = \frac{b}{a} = \frac{-4}{4\sqrt{3}} = \frac{-1/2}{\sqrt{3}/2}$ (sin) (cos)

$\theta = 330°$

root spacing: $\frac{360°}{3} = 120°$

1st root: $\sqrt[n]{r}\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right)$

$\sqrt[3]{8}\left(\cos\frac{330°}{3} + i\sin\frac{330°}{3}\right)$

$\boxed{2(\cos 110° + i\sin 110°)} = \boxed{-0.684 + 1.879i}$

2nd root (add 120°): $\boxed{2(\cos 230° + i\sin 230°)} = \boxed{-1.286 - 1.532i}$

3rd root (add 120°): $\boxed{2(\cos 350° + i\sin 350°)} = \boxed{1.97 - 0.347i}$

roots (110°)

One more example from yesterday: For the expression $(3+i)(1+i)$

a) give the trig form of the complex numbers

b) perform the indicated operation using trig form

c) perform the indicated operation using standard form

a) $3+i$:

$r = \sqrt{3^2+1^2} = \sqrt{10}$

$\tan\theta = \frac{b}{a} = \frac{1}{3}$

$\theta = \tan^{-1}\left(\frac{1}{3}\right) = 18.4°$

$\boxed{\sqrt{10}(\cos 18.4° + i\sin 18.4°)}$

$1+i$

$r = \sqrt{1^2+1^2} = \sqrt{2}$

$\tan\theta = \frac{1}{1} = 1$

$\theta = 45°$

$\boxed{\sqrt{2}(\cos 45° + i\sin 45°)}$

b) $z_1 = \sqrt{10}(\cos 18.4° + i\sin 18.4°)$

$z_2 = \sqrt{2}(\cos 45° + i\sin 45°)$

$z_1 z_2 = r_1 r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))$

$= \sqrt{10}\sqrt{2}(\cos(18.4°+45°) + i\sin(18.4°+45°))$

$= \sqrt{20}(\cos 63.4° + i\sin 63.4°)$

$= \boxed{2 + 4i}$

c) $(3+i)(1+i)$

$3 + 3i + i + i^2$

$3 + 4i - 1$

$\boxed{2+4i}$