Precalculus)

HAlg3-4, 2.1 (day1) Notes – Polynomial Functions, Quadratic Functions

* Sketch this $y = 2(x-1)^2+3$

$(y-3) = 2(x-1)^2$

Polynomial Functions

'many terms'

classified by 'degree'

Examples

Degree 0: $f(x) = a_0$ constant function $f(x) = 3$

Degree 1: $f(x) = a_1x + a_0$ (line) $f(x) = -2x+4$

$mx+b$

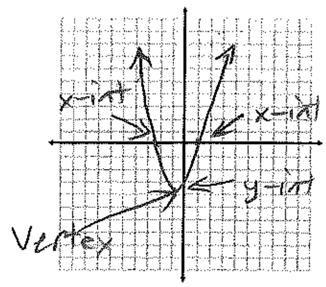
Degree 2: $f(x) = a_2x^2 + a_1x + a_0$ (quadratic) $f(x) = 3x^2-2x+1$

Degree n: $f(x) = a_nx^n + a_{n-1}x^{n-1} + \dots + a_2x^2 + a_1x + a_0$ $f(x) = 3x^{10} - 2x^9 + \dots + 3x^2 - 2x + 4$

Quadratic functions: Shape is: parabola

$f(x) = a_2x^2 + a_1x + a_0$

$f(x) = ax^2 + bx + c$



$f(x) = 2x^2 + 3x - 4$

The point where the curve changes direction is called the vertex

X, Y intercepts are where the other variable is zero

How does changing a, b, and c affect the graph of $f(x) = ax^2 + bx + c$?

Changing a causes: changes width (bigger = skinnier), negative flips vertically

Changing b causes: moves vertex around (how?)

Changing c causes: + shifts up, − shifts down

Difficult to sketch a quadratic function given in form $f(x) = ax^2 + bx + c$

Easier to sketch if converted to **standard form of a quadratic function:**

$f(x) = a(x-h)^2 + k$ (h,k) is the vertex

a affects direction and vertical scale or 'stretch'

$(y-k) = a(x-h)^2$ (like $(x-h)^2 + (y-k)^2 = r^2$)

To convert a quadratic to standard form, complete the square:

teacher [on board]

$f(x) = 2x^2 + 8x + 7$

$f(x) = (2x^2 + 8x \quad) + 7$

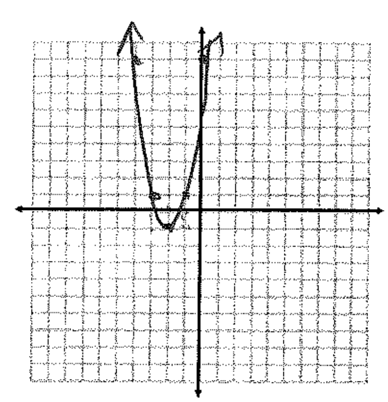
$f(x) = 2(x^2 + 4x + 4) + 7 - 8$

$f(x) = 2(x+2)^2 - 1$

$f(x) = a(x-h)^2 + k$

$(h,k) = (-2,-1)$

$a = 2$



$y = 2x^2$ curve starting at vertex

Practice: Convert the quadratic to standard form, and sketch.

teacher

#1. $f(x) = 3x^2 + 6x + 1$

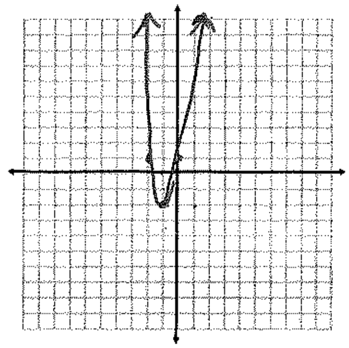
$f(x) = (3x^2 + 6x \quad) + 1$

$f(x) = 3(x^2 + 2x + 1) + 1 - 3$

$f(x) = 3(x+1)^2 - 2$

$(h, k) = (-1, -2)$

$a = 3$



student)

#2. $f(x) = -x^2 - 2x + 1$

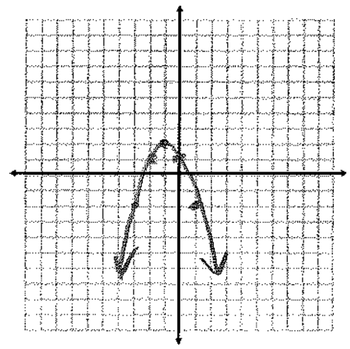
$f(x) = (-x^2 - 2x \quad) + 1$

$f(x) = -(x^2 + 2x + 1) + 1 + 1$

$f(x) = -(x+1)^2 + 2$

$(h, k) = (-1, 2)$

$a = -1$



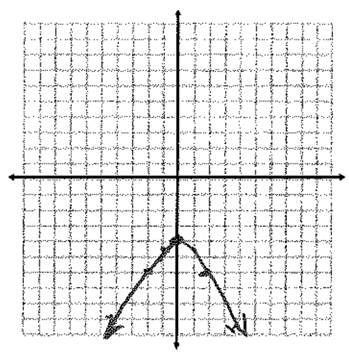
teacher ← no 'x' term, just rewrite:

#3. $f(x) = -\frac{1}{2}x^2 - 4$

$f(x) = -\frac{1}{2}(x-0)^2 - 4$

$(h, k) = (0, -4)$

$a = \frac{-1}{2}$



student)

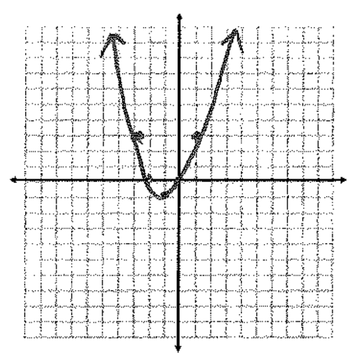
#4. $f(x) = x^2 + 2x$

$f(x) = (x^2 + 2x + 1) - 1$

$f(x) = (x+1)^2 - 1$

$(h, k) = (-1, -1)$

$a = 1$



Finding x- and y-intercepts (set other variable to zero)...can help in sketching:

$f(x) = x^2 + 2x$

y-int when $x = 0$

$f(0) = 0^2 + 2(0) = 0$

$(0, 0)$

x-int when $y = 0$ $(f(x) = 0)$

$0 = x^2 + 2x$

$x^2 + 2x = 0$

$x(x+2) = 0$

$x = 0 \quad x = -2$

$(0, 0) \quad (-2, 0)$

Precalculus/

HAlg3-4, 2.1 (day2) Notes – Polynomial Functions, Quadratic Functions

Given vertex and a point, find the equation: $f(x) = a(x-h)^2 + k$

$(y-k) = a(x-h)^2$

Example: Find equation of quadratic function with vertex at (1,2) passing through point (3,-6).

1) replace h, k in general equation $\quad f(x) = a(x-1)^2 + 2$
2) replace x, f(x) using point on curve $\quad -6 = a(3-1)^2 + 2$
3) Solve for a $\quad -6 = 4a + 2$

$-8 = 4a$

$-2 = a$

4) rewrite general equation $\quad \boxed{f(x) = -2(x-1)^2 + 2}$

Practice: Find the equation of a quadratic function with vertex at (-2, -2) passing through point (-1, 0).

$f(x) = a(x+2)^2 - 2$

$0 = a(-1+2)^2 - 2$

$0 = a(1)^2 - 2$

$0 = a - 2$

$a = 2$

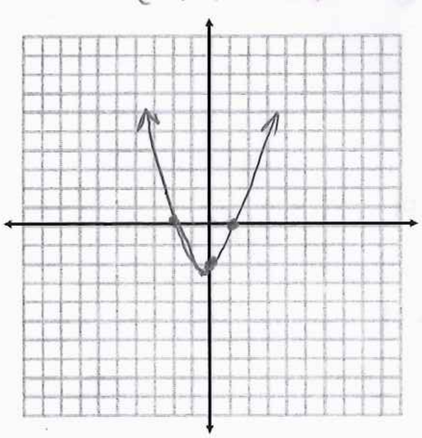
$\boxed{f(x) = 2(x+2)^2 - 2}$

Given equation, find the x- and y-intercepts:

$f(x) = x^2 + x - 2$

y-int
$x = 0$
$f(0) = (0)^2 + (0) - 2$
$= -2$
$(0, -2)$

x-int
$y = 0$
$x^2 + x - 2 = 0$
$(x+2)(x-1) = 0$
$x = -2 \quad x = 1$
$(-2, 0) \quad (1, 0)$

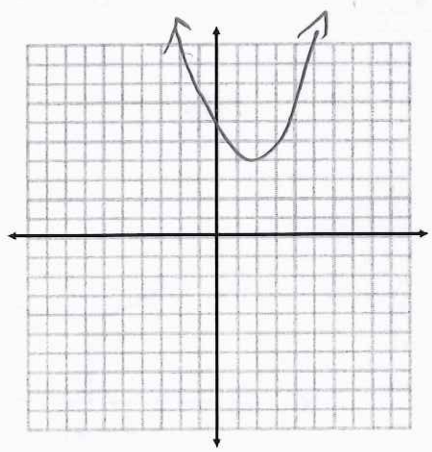


$f(x) = x^2 - x + 7$

Can't factor

$x = \dfrac{+1 \pm \sqrt{-1^2 - 4(1)(7)}}{2(1)} = \dfrac{1 \pm \sqrt{-27}}{2}$

no solution



Finding the vertex of a quadratic, real-world applications:

vertex: Performing our 'completing the square' procedure with general form $f(x) = ax^2 + bx + c$:

$$f(x) = a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right) \qquad \text{therefore:}$$

$h = -\dfrac{b}{2a}$ = x coordinate of vertex, x value where curve max or min occurs.

$k = c - \dfrac{b^2}{4a}$ = y coordinate of vertex, the min or max value of function.

Sometimes, easier to use formulas to quickly get the vertex (h, k), rather than convert equation to standard form. Finding the vertex is often useful in real-world applications.

Example: A textile manufacturer has daily production costs of:

$C(x) = 10{,}000 - 110x + 0.45x^2$ where C is total cost in dollars and x is number of units produced.

How many units should be produced each day to yield a minimum cost?

$C(x) = 0.45x^2 - 110x + 10000$

$a = 0.45$

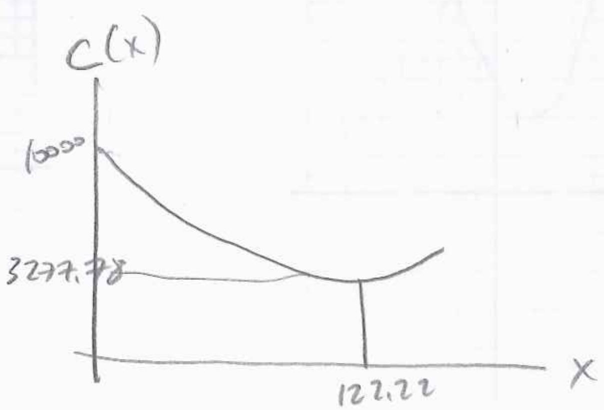
$b = -110$

$c = 10000$

$h = \dfrac{-b}{2a} = \dfrac{+110}{2(0.45)} = 122.22$

$k = c - \dfrac{b^2}{4a} = 10000 - \dfrac{(-110)^2}{4(0.45)} = 3277.78$

Graph: something happening at $(122.22, 3277.78)$



deriving the Quadratic Formula

$$ax^2+bx+c=0$$

$$ax^2+bx=-c$$

$$x^2+\frac{b}{a}\cancel{x}=\frac{-c}{a}$$

$$\left(x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2\right)=\frac{-c}{a}+\left(\frac{b}{2a}\right)^2$$

$$\left(x+\frac{b}{2a}\right)^2=\frac{-c}{a}+\frac{b^2}{4a^2}$$

$$\left(x+\frac{b}{2a}\right)^2=\frac{-4ac}{4a^2}+\frac{b^2}{4a^2}$$

$$\left(x+\frac{b}{2a}\right)^2=\frac{-4ac+b^2}{4a^2}$$

$$x+\frac{b}{2a}=\pm\sqrt{\frac{-4ac+b^2}{4a^2}}=\pm\frac{\sqrt{-4ac+b^2}}{2a}$$

$$x=\frac{-b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$$

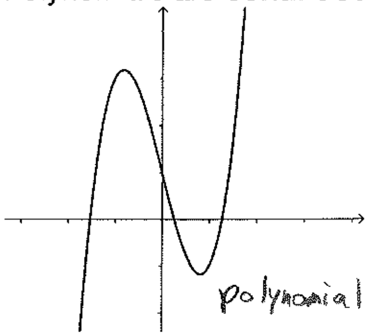
$$\boxed{x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}}$$

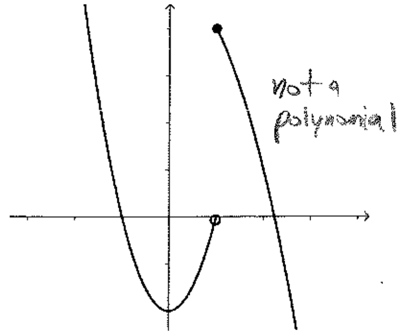
Precalculus/

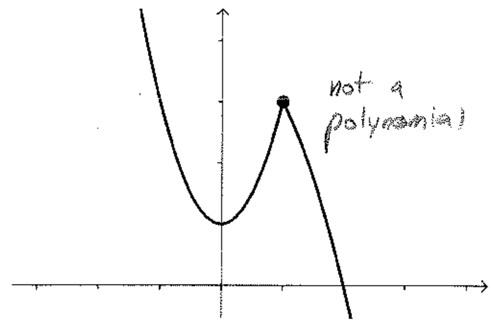
HAlg3-4, 2.2 (day1) Notes – Polynomial Functions of Higher Degree

Graphs of polynomial functions of higher degree: (Degree = highest variable exponent.)

Polynomials are **continuous** and **smooth**:







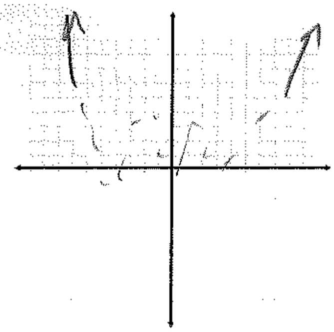
All polynomials eventually rise or fall to infinity on left and right side. To sketch left and right side behavior, use the **Leading Coefficient Test**:

Even degree polynomials:

- Left and right side in same direction
- Positive (+) leading coefficient, both rise (like x^2)
- Negative (-) leading coefficient, both fall (like $-x^2$)

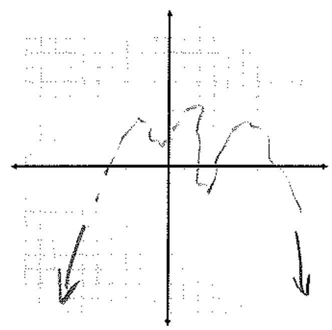
\+ even

$f(x) = 3x^4 - 2x^3 + x^2 - x - 1$



\- even

$f(x) = -4x^6 - 2x^5 + x^2 - 2x - 2$

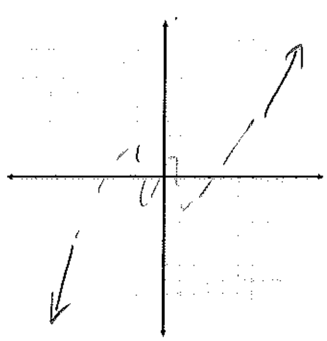


Odd degree polynomials:

- Left and right side in opposite directions
- Positive (+) leading coefficient, left falls, right rises ('positive slope' like x^3)
- Negative (-) leading coefficient, left rises, right falls ('negative slope' like $-x^3$)

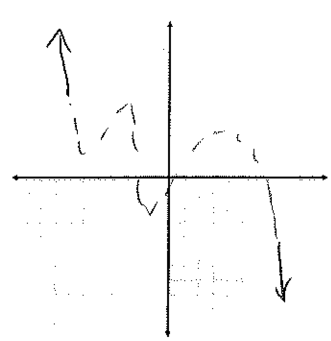
\+ odd

$f(x) = 2x^5 - 2x^4 + x^3 - x - 1$



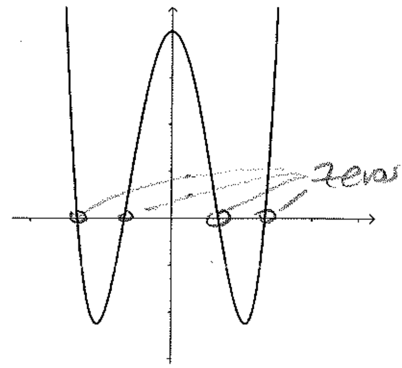
\- odd

$f(x) = -x^3 + x^2 - 3x + 2$



Zeros of a polynomial:

A zero is an x value where f(x) is zero:



<u>The following are all equivalent statements</u>:

- (a,0) is an *x-intercept* of the graph of f.
- x=a is a *zero* of the function f.
- x=a is a *solution* of the polynomial equation f(x)=0.
- (x – a) is a *factor* of the polynomial f(x).

Zeros help in sketching the area between the left and right hand side behavior.

Finding zeros of a polynomial algebraically: Set f(x)=0 and solve for x. (factoring often used)

Examples: Find zeros algebraically:

$f(x) = x^3 - x^2 - 2x = 0$

$x(x^2 - x - 2) = 0$

$x(x-2)(x+1) = 0$

zeros: 0, 2, −1

$f(x) = x^4 - 4x^2 = 0$

$x^2(x^2 - 4) = 0$

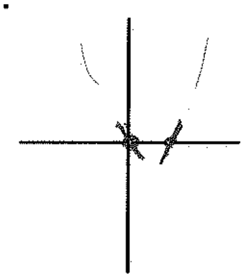
$x^2(x+2)(x-2) = 0$

zeros: 0, −2, 2

Multiplicity of a zero: how many times that factor appears.
Multiplicity of a zero affects how the curve behaves near that zero.
The higher the multiplicity, the 'flatter' the curve is near the x-axis, and:

<u>Multiplicity = 1</u> – curve just crosses x-axis at the zero: $f(x) = x(x-2)$ $\quad x^2 - 2x$

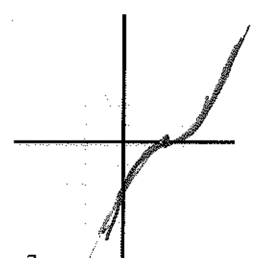
zeros: 0 (multiplicity = 1)
2 (multiplicity = 1)



<u>Multiplicity is odd (>1)</u> – curve has 'inflection' at the zero: $f(x) = (x-2)^3$ $\quad x^3 - 6x^2 + 12x - 8$

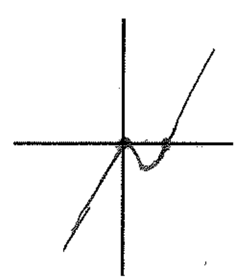
Crosses and changes curvature

zeros: 2 (multiplicity = 3)



<u>Multiplicity is even</u> – curve 'tangent' to x-axis at the zero (just touches): $f(x) = x^2(x-2)$ $\quad x^3 - 2x^2$

zeros: 0 (multiplicity = 2)
2 (multiplicity = 1)



Procedure to sketch a higher-degree polynomial:

1) **Use Leading Coefficient Test to find left, right behavior. Add lines on left and right.**

2) **Find zeros and multiplicity of each zero. Sketch behavior of curve near zeros.**
- Set f(x)=0 and solve for x.
- Factor if possible, quadratic equation if not factorable.

3) **Plot a few more points if needed to see behavior in between.**

4) **Connect lines up to sketch.**

Example: Sketch $f(x) = x^3 - x^2 - x + 1$

$(x^3 - x^2) - (x-1)$

$x^2(x-1) - (x-1)$

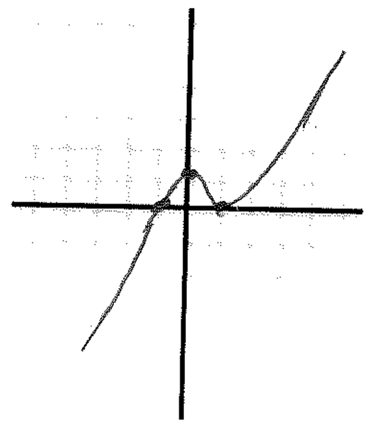
$(x-1)(x^2-1)$

$(x-1)(x+1)(x-1)$

$(x+1)(x-1)^2$

Zeros: -1 (mult = 1)

1 (mult = 2

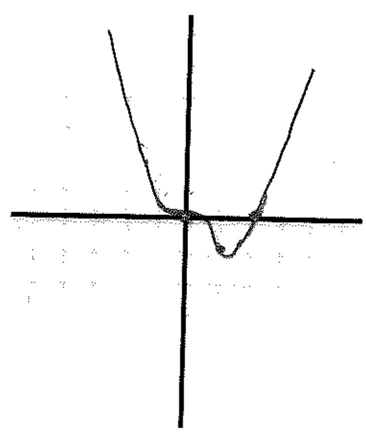


Example: Sketch $f(x) = x^6 - 2x^5$

$x^5(x-2)$

Zeros: 0 (mult = 5)

2 (mult = 1)



Precalculus)

HAlg3-4, 2.2 (day2) Notes – Polynomial Functions of Higher Degree

talk about multiplier

talk about distribution

Finding a polynomial function, given its zeros:
Just write as factors, then FOIL:

Example: Find a polynomial function that has zeros: 3, 1, and -2.

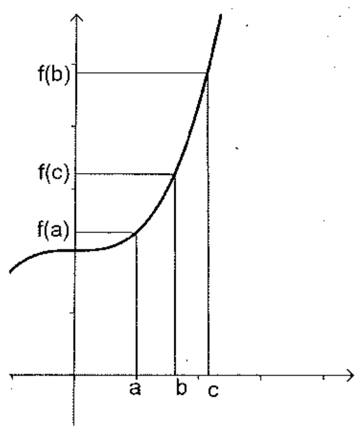
$f(x) = A(x-3)(x-1)(x+2)$
$= (x^2-4x+3)(x+2)$
$= x^3-4x^2+3x+2x^2-8x+6$
$f(x) = x^3-2x^2-5x+6$

Finding relative minimum and maximum points: (extrema)
Two ways: 1) try points
2) graph with a calculator and use trace feature. ← easiest

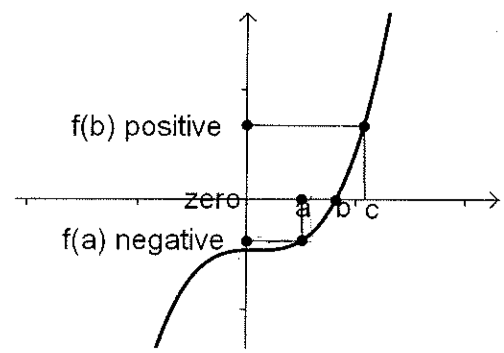
Example: Find relative minima and maxima of $f(x) = x^4 - 3x^2 + 1$

minima: $(-1.2, -1.2)$, $(1.2, -1.2)$
Maximum: $(0, 1)$

Intermediate Value Theorem:



Can use to approximate zeros:



If you can find where the function changes from positive to negative or negative to positive, there is a zero in between those x values. You can use your calculator's TABLE feature for this.

Book example: Find 3 intervals of length 1 in which the polynomial $f(x) = 12x^3 - 32x^2 + 3x + 5$ is guaranteed to have a zero.

- Enter the equation in the first y= slot (for y1)
- Press 2nd-Window to get the TBLSET menu. Set TblStart= -10 and $\Delta Tbl = 1$.
- Instead of pressing 'graph', press 2nd-graph to get the TABLE function.

This displays an X, Y table. Use the up/down arrows to scroll through the table.

The y-values start negative, but switch to positive somewhere between x= -1 and x=0, and then switch again between x=0 and x=1, then switch again between x=2 and x=3.

By the intermediate value theorem, there are three zeros: somewhere between -1 and 0, between 0 and 1 and between 2 and 3.

Starter/ $f(x)=x^3+9x^2+8x-60$ zero at $x=2$
$=(x-2)(\quad)$

Precalculus/

HAlg3-4, 2.3 (day1) Notes – Real Zeros of Polynomial Functions

There are a number of procedures that allow us to factor larger polynomials or find zeros of polynomials. We can do these things graphically with modern tools, but techniques were developed in the past to speed the process. Some of these techniques are still worth studying.

Dividing Polynomials:

Remember long division? $100 \div 3$

$$3\overline{)100} = 33 \text{ R } 1 \quad \boxed{33\tfrac{1}{3}}$$

33 ← quotient; divisor; dividend; ← remainder

$$3\overline{)99} = 33 \quad \boxed{0} \qquad 99 = 3(33) \text{ factors}$$

You can also divide polynomials: $\dfrac{x^3+4x^2-3x-12}{x^2-3}$

$$x^2-3\overline{)x^3+4x^2-3x-12}$$
$$-(x^3 \quad -3x)$$
$$4x^2-12$$
$$4x^2-12$$
$$0$$

$\boxed{x+4}$

If you have missing terms, in the numerator you need placeholders (0 terms): $\dfrac{x^5+12}{x^2+1}$

$$x^2+1\overline{)x^5+0x^4+0x^3+0x^2+0x+12} \quad \text{quotient } x^3-x$$
$$x^5 \quad +x^3$$
$$-x^3+0x^2+0x$$
$$-x^3 \quad -x$$
$$x+12$$

$\boxed{x^3-x+\dfrac{x+12}{x^2+1}}$

Synthetic Division:

A way to divide faster, if divisor is x – a.

Procedure:

- Put the 'a' outside the bracket.
- List the coefficients of each term, highest to lowest, left to right inside the bracket (insert zero terms where terms are missing.)
- Bring down the first term.
- Multiply bottom number by outside number and place result up and to right (diagonal multiply).
- Add this to number above and put result below.
- Repeat until the end.

The numbers on the bottom are the coefficients of the quotient. The last number is the remainder.

If the remainder is zero, then a is (x-a) is a factor of the polynomial (a is a zero.)
You can use synthetic division to quickly test if a number is a zero/factor of a polynomial.

Examples:

$\dfrac{3x^3-16x^2-72}{x-6}$ $a=6$

6	3	−16	0	−72
		18	12	72
	3	2	12	0

$3x^2+2x+12$ remainder 0

$\dfrac{5x^2-17x-12}{x-4}$ $a=4$

4	5	−17	−12
		20	12
	5	3	0

$5x+3$, remainder 0

Two theorems help us use the results of polynomial division:

The Remainder Theorem:

If a polynomial f(x) is divided by x – k, the remainder r = f(k).

You can use synthetic division to quickly evaluate a function. Divide by x – k, and the remainder is what you would get by computing f(k):

Example: $f(x)=3x^3+8x^2+5x-7$ Evaluate $f(-2)$

$$\begin{array}{r|rrrr} -2 & 3 & 8 & 5 & -7 \\ & & -6 & -4 & -2 \\ \hline & 3 & 2 & 1 & \boxed{-9} \end{array}$$

$f(-2)=-9$

The Factor Theorem:

If f(k) = 0, remainder = 0, so (x – k) is a factor of the f(x).

(Use synthetic division to quickly compute f(k))

Example: Is $(x-2)$ a factor of $f(x)=2x^4+7x^3-4x^2-27x-18$?

$$\begin{array}{r|rrrrr} 2 & 2 & 7 & -4 & -27 & -18 \\ & & 4 & 22 & 36 & 18 \\ \hline & 2 & 11 & 18 & 9 & \boxed{0} \end{array}$$

remainder is zero, yes x-2 is a factor

$2x^3+11x^2+18x+9$

$(x-2)(2x^3+11x^2+18x+9)$

Practice examples:

Use synthetic division to evaluate $f(4)$ for $f(x)=x^3-x^2-14x+11$

$$\begin{array}{r|rrrr} 4 & 1 & -1 & -14 & +11 \\ & & 4 & 12 & -8 \\ \hline & 1 & 3 & -2 & 3 \end{array}$$

$f(4)=3$

Show that $(x-2)$ *and* $(x+3)$ are factors of $f(x)=2x^4+7x^3-4x^2-27x-18$ and finish factoring.

$$\begin{array}{r|rrrrr} 2 & 2 & 7 & -4 & -27 & -18 \\ & & 4 & 22 & 36 & 18 \\ \hline & 2 & 11 & 18 & 9 & 0 \end{array}$$

yes

$(x-2)(2x^3+11x^2+18x+9)$

$$\begin{array}{r|rrrr} -3 & 2 & 11 & 18 & 9 \\ & & -6 & -15 & -9 \\ \hline & 2 & 5 & 3 & 0 \end{array}$$

yes

$(x-2)(x+3)\underbrace{(2x^2+5x+3)}$ $=$ $\boxed{(x-2)(x+3)(x+1)(2x+3)}$

$\dfrac{(2x+2)(2x+3)}{2 \qquad 1}$

x	+
6	5
(2)(3)	2+3

Precalculus)

HAlg3-4, 2.3 (day2) Notes – Real Zeros of Polynomial Functions

More procedures and theorems that help in finding zeros (factors) of polynomials…

2 **Descartes' Rule of Signs:** Determines the number of positive and negative real zeros of a polynomial.

\# of positive real zeros = # of sign changes in f(x) or is less than that number by a multiple of 2. (Find number of signs and keep subtracting 2 until you go past zero – see example.)

\# of negative real zeros = # of sign changes in f(-x) or is less than that number by a multiple of 2. (see example.)

Example: Use Descartes' Rule of Signs to determine possible number of real positive and negative zeros of the polynomial $f(x) = 4x^3 - 3x^2 + 2x - 1$

$+ \ - \ + \ -$
$\Delta \ \Delta \ \Delta$

\# positive real zeros: 3, 1

$f(-x) = 4(-x)^3 - 3(-x)^2 + 2(-x) - 1$
$f(-x) = \underbrace{-4x^3 - 3x^2 - 2x - 1}_{\text{no changes}}$

\# negative real zeros: none

1 **Rational Zero Test:** Gives us a list of possible rational ~~roots~~ zeros of a polynomial. Limited to functions with only integer coefficients and lists only the possible rational zeros (there may still be irrational zeros this procedure doesn't list.)

Procedure: look at the 'constant' term (last term) and coefficient of first term (lead coefficient).

Possible rational zeros are: $\dfrac{\pm \textit{factors of last term}}{\pm \textit{factors of first term}}$

Example: Find possible rational zeros of $f(x) = -3x^3 + 20x^2 - 36x + 16$

Factors of last (constant) term: (16) 1, 2, 4, 8, 16
Factors of first (lead) term: (−3) 1, 3

Possible rational zeros: $\dfrac{\pm1, \pm2, \pm4, \pm8, \pm16}{\pm1, \pm3} = \pm1, \pm2, \pm4, \pm8, \pm16, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{4}{3}, \pm\frac{8}{3}, \pm\frac{16}{3}$

3 **Bounds of Real Zeros by synthetic division:** Select a possible zero, c, and do synthetic division of f(x) for x – c.

- If c > 0, and each number in last row is either positive or zero, then c is an **upper bound** for the real zeros of f(x).
- If c < 0, and the signs of the numbers in the last row alternate (zero entries count as positive or negative) then c is a **lower bound** for the real zeros of f(x).

Examples: For $f(x) = 2x^3 - 3x^2 - 12x + 8$

show x=4 is upper bound

4	2	−3	−12	8
		8	20	32
	2	5	8	40

all positive
so 4 is upper bound on real zeros
(all real zeros are ≤ 4)

show x= -3 is lower bound

−3	2	−3	−12	8
		−6	27	−45
	2	−9	15	−37

sign alternate
so −3 is lower bound on real zeros
(all real zeros are ≥ −3)

Procedure for finding real zeros of (factoring) a polynomial:

1) Use Descartes' Rule of Signs to narrow down the number of positive and negative real zeros.

2) Use the Rational Zero Test to give a list of possible rational zeros.

3) Find bounds - use synthetic division on some possible zeros to see if you can eliminate some from the list.

4) From the list of possible zeros, find one that works
   Possible strategies:
   - Check with synthetic division (remainder = 0 means it is a factor – Remainder theorem).
   - Plug in a possible zero, if f(k) = 0, k is a zero (by the Factor theorem.)

5) When you have a quadratic left, use traditional factoring techniques to factor completely.

Example: Fully factor $f(x)=6x^3-4x^2+3x-2$

1) Descartes Rule: $f(x)=6x^3-4x^2+3x-2$ Δ Δ Δ

+ zeros: 3 or 1

$f(-x)=6(-x)^3-4(-x)^2+3(-x)-2$

$f(-x)=\underbrace{-6x^3-4x^2-3x-2}_{\text{no changes}}$

− zeros: none

2) Rational Zero Test: last term $=-2$, factors: 1, 2
first term $=6$, factors: 1, 2, 3, 6

possible: $\dfrac{\pm1, \pm2}{\pm1, \pm2, \pm3, \pm6}$

(no negatives)

$1, 2, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{6}$

3) Bounds: try 1

$$\begin{array}{r|rrrr} 1 & 6 & -4 & 3 & -2 \\ & & 6 & 2 & 5 \\ \hline & 6 & 2 & 5 & 3 \end{array}$$

$\neq 0$, 1 is not a zero

but all +, 1 = upper bound (throw out 2)

try $\frac{2}{3}$

$$\begin{array}{r|rrrr} \frac{2}{3} & 6 & -4 & 3 & -2 \\ & & 4 & 0 & 2 \\ \hline & 6 & 0 & 3 & 0 \end{array}$$

$\frac{2}{3}$ is a zero

$(6x^2 \quad +3)(x-\frac{2}{3})$

4) $(6x^2+3)(x-\frac{2}{3})$

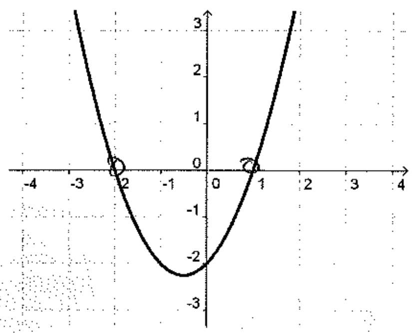
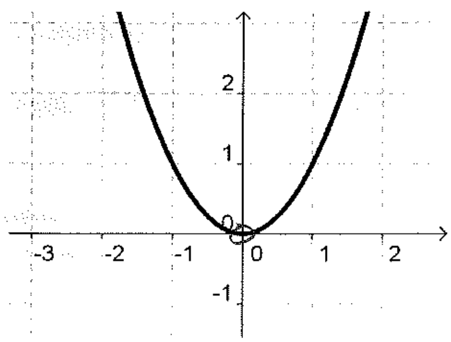
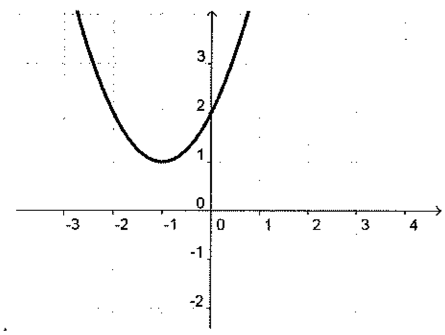
$\boxed{3(2x^2+1)(x-\frac{2}{3})}$

Precalculus)

HAlg3-4, 2.4 Notes – Complex Numbers

Find the solutions:

$x^2+x-2=0$	$x^2=0$	$x^2+2x+2=0$
$x=\dfrac{-1\pm\sqrt{1^2-4(1)(-2)}}{2(1)}$	$x=\dfrac{0\pm\sqrt{0-4(1)(0)}}{2(1)}$	$x=\dfrac{-2\pm\sqrt{2^2-4(1)(2)}}{2(1)}$
$x=\dfrac{-1\pm\sqrt{9}}{2}$	$x=\dfrac{0\pm\sqrt{0}}{2}$	$x=\dfrac{-\cancel{2}\pm\sqrt{-4}}{2}$
$x=\dfrac{-1\pm 3}{2}$	$x=\dfrac{0\pm 0}{2}$	$x=no\ solution$
$x=1,-2$	$x=0$	

Some quadratic functions have no real solutions (negative square root in quadratic formula):

Mathematicians expanded the number system to include new numbers so that equations such as this do have solutions (but they're not 'real' solutions.)

Definition of imaginary unit, i: $i=\sqrt{-1}$

Mathematicians:
- why 2, 1, or 0 zeros?
- why no solutions to $x^2+1=0$?

need a new kind of number

And...

$$i^2=\left(\sqrt{-1}\right)^2=-1$$
$$i^3=i^2\bullet i=(-1)i=-i$$
$$i^4=i^2\bullet i^2=(-1)(-1)=1$$
$$i^5=i^4\bullet i=(1)i=i$$

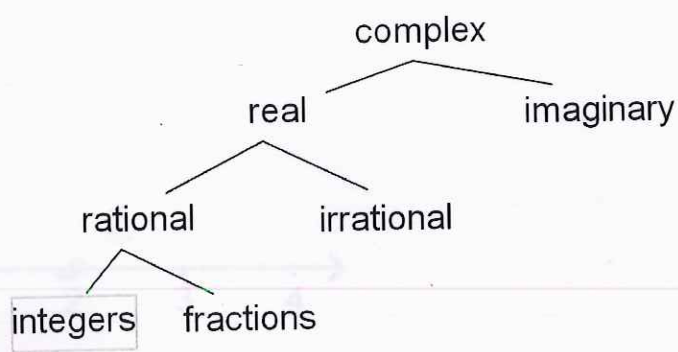
In above example, factoring out the square-root of -1 lets us define solutions:

$$x=\frac{-2\pm\sqrt{-4}}{2}$$
$$x=\frac{-2\pm\sqrt{4}\sqrt{-1}}{2}$$
$$x=\frac{-2\pm 2i}{2}$$

$x=-1\pm i$

2 solutions: $-1+i$ and $-1-i$

Extended number system:



Standard form of complex numbers: $a+bi$

$a+bi$

real part

imaginary part

Examples:

If a = 0, number is pure imaginary: $3i$

If b = 0, number is pure real: -4

examples: $2-3i$

$-\frac{1}{2}+0.7i$

Properties of Complex Numbers...

<u>Equality</u>: Two complex numbers are equal, if their real parts match and their imaginary parts match:

$2+3i = 2+3i$

$3-4i \neq 3+2i$

does not match

<u>Adding/substracting</u>: Add (subtract) real and imaginary parts separately:

$(2+3i)+(4+i) = 6+4i$

$(6-3i)-(2+2i) = 4-5i$

<u>Multiplying</u>: Treat real and imaginary part as separate terms and multiply using FOIL:

$(3i)(2+4i)$

$6i+12i^2 \quad (i^2=-1)$

$6i-12$

$\boxed{-12+6i}$

$(2+3i)(2-4i)$

$4-8i+6i-12i^2$

$4-2i-12(-1)$

$\boxed{16-2i}$

<u>Complex Conjugates</u>: Same real part, opposite imaginary part:

$3+2i$

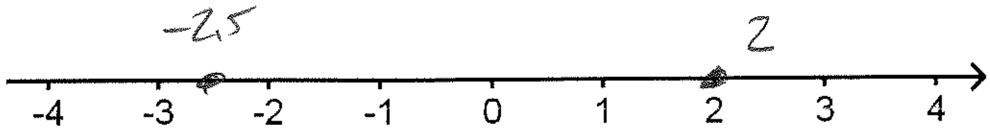
$3-2i \leftarrow$ complex conjugate

<u>Dividing</u>: Similar to 'rationalizing' a fraction with radical on bottom, you multiply top and bottom by the complex conjugate of the denominator:

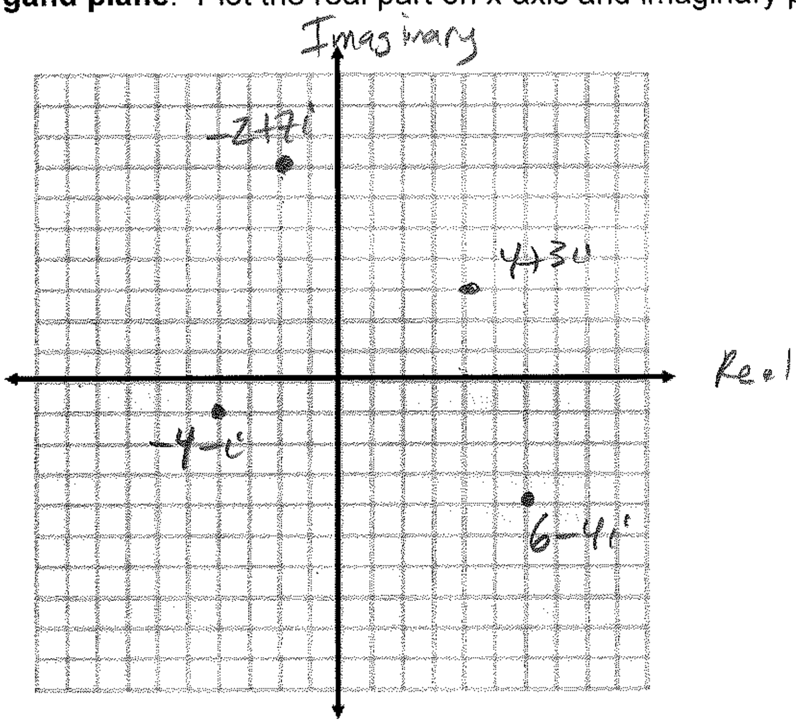
$$\frac{(3+2i)}{(2+4i)}\frac{(2-4i)}{(2-4i)} = \frac{6-12i+4i-8i^2}{4-8i+8i-16i^2} = \frac{6-8i+8}{4+16} = \boxed{\frac{14-8i}{20}}$$

Plotting on complex plane:

Real numbers we can plot on a number line:



The 'number line' for complex numbers is a 2-dimensional plane called the **complex plane**, or **Argand plane**. Plot the real part on x-axis and imaginary part on y-axis.



Precalculus

HAlg3-4, 2.5 Notes – Fundamental Theorem of Algebra, Complex zeros

What are the zeros of this polynomial: $f(x)=x^2+2x+3=0$

$$x=\frac{-2\pm\sqrt{4-4(1)(3)}}{2(1)}=\frac{-2\pm\sqrt{4-12}}{2}=\frac{-2\pm\sqrt{-8}}{2}=\frac{-2\pm\sqrt{4}\sqrt{2}\sqrt{-1}}{2}=\frac{-2\pm2\sqrt{2}i}{2}$$

$$x=-1\pm\sqrt{2}i$$

$$x=-1+\sqrt{2}i,\ -1-\sqrt{2}i$$

You <u>can</u> have complex numbers as zeros.

Fundamental Theorem of Algebra (Gauss, 1799)
"If f(x) is a polynomial of degree n, where n>0, f has at least one zero in the complex number system."

From this, it's possible to derive the **Linear Factorization Theorem**:
"If f(x) is a polynomial of degree n, where n>0, f has precisely n linear factors:

$$f(x)=a_n(x-c_1)(x-c_2)...(x-c_n)$$

where $c_1, c_2, ..., c_n$ are complex numbers."

(Degree n = number of factors = number of zeros.)
Examples…

Solve $x^3+6x-7=0$

Rational zero test:
last, -7, factors: $1, 7$
1st, 1: factors: 1
possible rational zeros: $\frac{\pm1,\pm7}{\pm1}=\pm1,\pm7$

try 1
$$\begin{array}{r|rrrr} 1 & 1 & 0 & 6 & -7 \\ & & 1 & 1 & 7 \\ \hline & 1 & 1 & 7 & 0 \checkmark \end{array}$$

$(x-1)(x^2+x+7)$

quadratic formula:
$$\frac{-1\pm\sqrt{1-4(7)}}{2}$$
$$\frac{-1\pm\sqrt{-27}}{2}=\frac{-1\pm\sqrt{9}\sqrt{3}\sqrt{-1}}{2}$$
$$\frac{-1\pm3\sqrt{3}i}{2}=-\frac{1}{2}\pm\frac{3\sqrt{3}}{2}i$$

$\boxed{\text{3 zeros: } 1,\ -\frac{1}{2}+\frac{3\sqrt{3}}{2}i,\ -\frac{1}{2}-\frac{3\sqrt{3}}{2}i}$

Find all zeros and factors of $f(x)=2x^3-5x^2+12x-5$

Rational zero test: last: -5, factors: $1, 5$
1st 2, factors: $1, 2$
possible rational zeros: $\frac{\pm1,\pm5}{\pm1,\pm2}=\pm1,\pm5,\pm\frac{1}{2},\pm\frac{5}{2}$

try 1
$$\begin{array}{r|rrrr} 1 & 2 & -5 & 12 & -5 \\ & & 2 & -3 & 9 \\ \hline & 2 & -3 & 9 & 4 \times \end{array}$$

try $\frac{1}{2}$
$$\begin{array}{r|rrrr} \frac{1}{2} & 2 & -5 & 12 & -5 \\ & & 1 & -2 & 5 \\ \hline & 2 & -4 & 10 & 0 \checkmark \end{array}$$

$(x-\frac{1}{2})(2x^2-4x+10)$

quadratic formula:
$$\frac{4\pm\sqrt{16-4(2)(10)}}{2(2)}=\frac{4\pm\sqrt{16-80}}{4}$$
$$\frac{4\pm\sqrt{-64}}{4}=\frac{4\pm8i}{4}=1\pm2i$$

$\boxed{\text{Zeros: } \frac{1}{2},\ 1+2i,\ 1-2i}$

$\boxed{\text{factors: } (x-\frac{1}{2})(x-1-2i)(x-1+2i)}$

Complex zeros occur in conjugate pairs (if original polynomial has only real coefficients).

Example: Find a 4^{th} degree polynomial with real coefficients that has 0, 1, and $\underbrace{i \text{ as zeros}}$.

conjugate pair
i and $-i$

$f(x) = x(x-1)(x-i)(x+i)$
$f(x) = (x^2-x)(x^2+1)$
$f(x) = x^4+x^2-x^3-1$
$\boxed{f(x) = x^4-x^3+x^2-1}$

Some terminology associated with zeros, factoring and polynomials:

Linear/quadratic factors

From above example, factoring $f(x)=x^3+6x-7$, when we determined that 1 was a zero, we wrote:

$$f(x)=(x-1)(x^2+x+7)$$

$(x-1)$ is a <u>linear factor</u>
(x^2+x+7) is a <u>quadratic factor</u>

'Write as a product of linear factors' = keep factoring until all factors are linear factors (real or complex)

Other examples of 'linear factors': $(x-2+i)$, $(x-2)^2 \leftarrow$ multiplicity 2, but still a linear factor
could write as: $(x-2)(x-2)$

Irreducible over reals / Irreducible over rationals

Factoring $f(x)=x^4+6x^2-27$, yields: $f(x)=(x^2+9)(x^2-3)$

How much farther to factor?

$x^2+9=0$
$x^2=-9$
$x=\pm\sqrt{-9}$
$x=\pm\sqrt{9}\sqrt{-1}$
$x=\pm 3i$
$(x+3i)(x-3i)$

$x^2-3=0$
$x^2=3$
$x=\pm\sqrt{3}$
$(x+\sqrt{3})(x-\sqrt{3})$

$f(x)=(x^2+9)(x^2+3)$ --- 'irreducible over the rationals' (can't factor any more unless we go beyond rational numbers to allow radicals)

$f(x)=\left(x^2+9\right)\left(x+\sqrt{3}\right)\left(x-\sqrt{3}\right)$ --- 'irreducible over the reals' (can't factor any more unless we go beyond reals to allow complex)

$f(x)=(x+3i)(x-3i)\left(x+\sqrt{3}\right)\left(x-\sqrt{3}\right)$ --- 'fully factored' or 'written as a product of linear factors'

Final example: Find all zeros of $f(x) = x^4 - 4x^3 + 12x^2 + 4x - 13$ given that $(2+3i)$ is a factor.

divide using synthetic division:

$$\begin{array}{r|rrrrr} 2+3i & 1 & -4 & 12 & 4 & -13 \\ & & 2+3i & -13 & -2-3i & 13 \\ \hline & 1 & -2+3i & -1 & 2-3i & 0 \end{array}$$

conjugate also a factor

$(2+3i)(-2+3i)$
$-4-9$
-13

$(2+3i)(2-3i)$
$4+9$
13

$$\begin{array}{r|rrrr} 2-3i & 1 & -2+3i & -1 & 2-3i \\ & & 2-3i & 0 & -2+3i \\ \hline & 1 & 0 & -1 & 0 \end{array}$$

$x^2 - 1$

$x^2 - 1 = 0$

$x^2 = 1$

$x = \pm\sqrt{1} = \pm 1$

Zeros: $+1, -1, 2+3i, 2-3i$

factors: $(x-1)(x+1)(x-2-3i)(x-2+3i)$