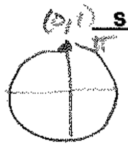
Precalculus – Lesson Notes: Chapter 5 Analytical Trigonometry

5.1 day 1: Using Trigonometric Identities

Equation vs. Identity

An equation is true only for **some** values of the variable:



$$\sin\theta = 1$$

$\boxed{\theta = \pi}$

An identify is true for **all** values of the variable:

$$\sin\theta = \frac{1}{\csc\theta}$$

******Start Memorizing These...***

Reciprocal identities:

$$\sin x = \frac{1}{\csc x} \qquad \cos x = \frac{1}{\sec x} \qquad \tan x = \frac{1}{\cot x}$$

$$\csc x = \frac{1}{\sin x} \qquad \sec x = \frac{1}{\cos x} \qquad \cot x = \frac{1}{\tan x}$$

Quotient identities:

$$\tan x = \frac{\sin x}{\cos x}$$

$$\cot x = \frac{\cos x}{\sin x}$$

Pythagorean identities:

$$\sin^2 x + \cos^2 x = 1 \qquad 1 + \tan^2 x = \sec^2 x \qquad 1 + \cot^2 x = \csc^2 x$$

$$\sin^2 x = 1 - \cos^2 x \qquad \sec^2 x - 1 = \tan^2 x \qquad \csc^2 x - 1 = \cot^2 x$$

$$\cos^2 x = 1 - \sin^2 x \qquad \sec^2 x - \tan^2 x = 1 \qquad \csc^2 x - \cot^2 x = 1$$

Deriving these from:

$$\sin^2 x + \cos^2 x = 1$$

$$\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$(\tan^2 x + 1 = \sec^2 x)$$

$$\frac{\sin^2 x}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x} = \frac{1}{\sin^2 x}$$

$$(1 + \cot^2 x = \csc^2 x)$$

Cofunction identities:

$$\sin\left(\frac{\pi}{2} - x\right) = \cos x \qquad \cos\left(\frac{\pi}{2} - x\right) = \sin x$$

$$\tan\left(\frac{\pi}{2} - x\right) = \cot x \qquad \cot\left(\frac{\pi}{2} - x\right) = \tan x$$

$$\sec\left(\frac{\pi}{2} - x\right) = \csc x \qquad \csc\left(\frac{\pi}{2} - x\right) = \sec x$$

☆ Note: $\sin^2 x = (\sin x)^2$

$\sin x^2 = \sin(x^2)$

We can use identities to simplify a trigonometric expression or to verify a more complex identity

General procedure: start with more complicated expression and make it simpler, or to verify, turn more complicated expression into simpler expression.

Example: Simplify $\sin x \cos^2 x - \sin x$

$\sin x(\cos^2 x - 1)$ factor $\sin x$

$\sin x(-\sin^2 x)$ $\sin^2 x + \cos^2 x = 1$, so $\cos^2 x - 1 = -\sin^2 x$

$\boxed{-\sin^3 x}$

Strategies for simplifying trig expressions using identities:

1) Factor out trig functions as if they were variables

Ex: Simplify $\sin x\cos^2 x-\sin x$

$\sin x(\cos^2 x-1)$

2) When there is a squared term, think 'Pythagorean identity':

Ex: Factor the expression: $\sin x\left(1-\cos^2 x\right)$

$\sin x(\sin^2 x)$ Pythagorean identity; $\sin^2 x+\cos^2 x=1$

so, $\sin^2 x=1-\cos^2 x$

$\boxed{\sin^3 x}$

3) Convert everything to sin or cos

Ex: Simplify $\cot x\sin x$

$\frac{\cos x}{\sin x}\cdot\frac{\sin x}{1}$

$\frac{\sin x}{\sin x}\cdot\frac{\cos x}{1}$

$\boxed{\cos x}$

4) If you have fractions, combine with common denominator

Ex: Verify

$$\frac{\sin x}{1+\cos x}+\frac{\cos x}{\sin x}=\csc x$$

$\frac{\sin x}{\sin x}\cdot\frac{\sin x}{(1+\cos x)}+\frac{\cos x}{\sin x}\frac{(1+\cos x)}{(1+\cos x)}=\csc x$

$\frac{\sin^2 x}{\sin x(1+\cos x)}+\frac{\cos x+\cos^2 x}{\sin x(1+\cos x)}=\csc x$

$\frac{\sin^2 x+\cos^2 x+\cos x}{\sin x(1+\cos x)}=\csc x$

$\frac{1+\cos x}{\sin x(1+\cos x)}=\csc x$

$\frac{1}{\sin x}\cdot\frac{(1+\cos x)}{(1+\cos x)}=\csc x$

$\frac{1}{\sin x}=\csc x$

$\csc x=\csc x$ ✓

5) If there is a binomial in the denominator, multiply by the 'conjugate' if it creates a Pythagorean identity:

Ex: Rewrite $\dfrac{1}{1-\sin x}$ so it is not in fractional form.

$$\frac{1}{(1-\sin x)}\frac{(1+\sin x)}{(1+\sin x)}$$

$$\frac{1+\sin x}{1+\sin x-\sin x-\sin^2 x}$$

$$\frac{1+\sin x}{1-\sin^2 x}$$

$$\frac{1+\sin x}{\cos^2 x}$$

$$\frac{1+\sin x}{\cos^2 x}$$

$$\frac{1}{\cos^2 x}+\frac{\sin x}{\cos^2 x}$$ (separate fractions) (only if same denominator)

$$\sec^2 x+\frac{\sin x}{\cos x}\cdot\frac{1}{\cos x}$$

$$\boxed{\sec^2 x+\tan x\sec x}$$

(no fractions)

6) Look for factoring patterns treating trig functions like variables:

Ex: Factor the expression: $\sec^2 x-1$

$$(\sec x)^2-(1)^2$$

$$a^2-b^2$$

$$\boxed{(\sec x+1)(\sec x-1)}$$

$4\tan^2\theta+\tan\theta-3$

$4(\tan\theta)^2+(\tan\theta)-3$ $\quad u=\tan\theta$

$4u^2+u-3$ $\quad$ M: -12, A: 1; $4, -3$

$$\frac{(4u+4)}{4}\frac{(4u-3)}{1}$$

$$(u+1)(4u-3)$$

$$\boxed{(\tan\theta+1)(4\tan\theta-3)}$$

7) Don't forget the less-used identities:

Ex: Simplify $\dfrac{\sin\left(\dfrac{\pi}{2}-x\right)}{\cos\left(\dfrac{\pi}{2}-x\right)}$

$$\frac{\cos x}{\sin x}$$

$$\boxed{\cot x}$$

Practice: Simplify...

#1. $\left(1-\sin^2 x\right)\sec x$

$\cos^2 x \sec x$

$\frac{\cos^2 x}{1} \frac{1}{\cos x}$

$\frac{\cos x}{1} \frac{\cos x}{\cos x}$

$\boxed{\cos x}$

#2. $\cot\theta \sec\theta$

$\frac{\cancel{\cos\theta}}{\sin\theta} \frac{1}{\cancel{\cos\theta}}$

$\frac{1}{\sin\theta} \frac{\cancel{\cos\theta}}{\cancel{\cos\theta}}$

$\frac{1}{\sin\theta}$

$\boxed{\csc\theta}$

#3. $\dfrac{\cos^2\left(\dfrac{\pi}{2}-x\right)}{\cos x}$

$\frac{\sin^2(x)}{\cos x}$

$\frac{\sin x}{\cos x} \cdot \frac{\sin x}{1}$

$\boxed{\tan x \sin x}$

'Verifying' an identity means showing that it is true by using identities on one side only to convert that side into a copy of the other side.
(Usually, work with more complicated side to simplify into other side).

#4. Verify... $\cos x \sec x - \cos^2 x = \sin^2 x$

$\frac{\cos x}{1} \cdot \frac{1}{\cos x} - \cos^2 x = \sin^2 x$

$1 - \cos^2 x = \sin^2 x$

$\sin^2 x = \sin^2 x$ ✓

$\dfrac{\underbrace{\sec^2\theta - \tan^2\theta}_{=1} + \tan\theta}{\sec\theta} = \cos\theta + \sin\theta$

$\frac{\sin^2\theta + \cos^2\theta = 1}{\cos^2\theta \quad \cos^2\theta \quad \cos^2\theta}$

$\tan^2\theta + 1 = \sec^2\theta$

$1 = \sec^2\theta - \tan^2\theta$

$\frac{1 + \tan\theta}{\sec\theta} = \cos\theta + \sin\theta$

$\frac{1}{\sec\theta} + \frac{\tan\theta}{\sec\theta} = \cos\theta + \sin\theta$

$\cos\theta + \frac{\sin\theta}{\cos\theta} \cdot \frac{\cos\theta}{1} = \cos\theta + \sin\theta$

$\cos\theta + \sin\theta = \cos\theta + \sin\theta$ ✓

5.1 day 2/5.2: Verifying Trigonometric Identities/More Examples

Other problems using basic identities (1-13 in hw):

Ex: Use the given values to evaluate the remaining trig functions

$$\csc\theta = \frac{5}{3}, \quad \tan\theta = \frac{3}{4}$$

$\frac{1}{\sin\theta} = \frac{5}{3}$

$\boxed{\sin\theta = \frac{3}{5}}$

$\tan\theta = \frac{3}{4}$

$\frac{\sin\theta}{\cos\theta} = \frac{3}{4}$

$\frac{\left(\frac{3}{5}\right)}{\cos\theta} = \frac{3}{4}$

(cross multiply): $3\cos\theta = 4\left(\frac{3}{5}\right)$

$3\cos\theta = \frac{12}{5}$

$\cos\theta = \frac{12}{3\cdot 5}$

$\boxed{\cos\theta = \frac{4}{5}}$

$$\boxed{\begin{array}{ll} \sin\theta = \frac{3}{5} & \csc\theta = \frac{5}{3} \\ \cos\theta = \frac{4}{5} & \sec\theta = \frac{5}{4} \\ \tan\theta = \frac{3}{4} & \cot\theta = \frac{4}{3} \end{array}}$$

Perform an operation, then simplify: (need common denominator)

Ex: $$\frac{1}{\sec x+1} - \frac{1}{\sec x-1}$$

$$\frac{(\sec x-1)}{(\sec x-1)}\frac{1}{(\sec x+1)} - \frac{1}{(\sec x-1)}\frac{(\sec x+1)}{(\sec x+1)}$$

$$\frac{\sec x-1 - (\sec x+1)}{\sec^2 x - \sec x + \sec x - 1}$$

$$\frac{\sec x - 1 - \sec x - 1}{\sec^2 x - 1}$$

$$\frac{-2}{\tan^2 x} = \boxed{-2\cot^2 x}$$

Use trig substitution to write an algebraic expression:

Ex: $\sqrt{16-4x^2}, \quad x = 2\sin\theta$

$\sqrt{16-4(2\sin\theta)^2}$

$\sqrt{16-4\cdot 4\sin^2\theta}$

$\sqrt{16-16\sin^2\theta}$

$\sqrt{16(1-\sin^2\theta)}$

$\sqrt{16\cos^2\theta}$

$\sqrt{16}\sqrt{\cos^2\theta}$

$\boxed{4\cos\theta}$ (technically, $4|\cos\theta|$)

More general tips for verifying identities:

1) Work with one side of the equation at a time. Usually best to try to turn more complicated side into less complicated side.

2) Look for opportunities to factor, add fractions.

3) Look for opportunities to use the fundamental identities.

4) Use simple side as the 'goal' to help guide what to do next. Example: if goal has secants, try converting what you have to secants, if goal has two terms and you are starting with one, look for ways to split fractions, etc.

5) Try converting everything to sin or cos and see if anything cancels or combines.

6) Try something! The path to a dead end still reveals insights.

Simplifying Misconceptions / Common Errors

1) Numerator split:

$$\frac{2+3}{6}=\frac{2}{6}+\frac{3}{6}$$ ← good, common denominators

$$\frac{2}{3+6}\neq\frac{2}{3}+\frac{2}{6}$$ ← bad, different denominators

2) squared -> fourth:

$1+\cot^2 x=\csc^2 x$ but $1+\cot^4 x\neq\csc^4 x$

$1+\cot^4 x$
$(1)^2+(\cot^2 x)^2$
$(1+\cot^2 x)(1-\cot^2 x)$
ok

instead, use patterns or split

a^2-b^2 or maybe $1+\cot^2 x\cot^2 x$

3) jump in steps too large (skipping steps):

$$\frac{\tan x(\csc x-\cot x)}{\csc^2 x-\cot^2 x}$$

(steps)

$$\frac{\sec x-1}{1}$$

$$\cos^2\beta+\cos^2\beta\tan^2\beta$$

(steps)

$$\cos^2\beta\sec^2\beta$$

$$\frac{\tan x(\csc x-\cot x)}{\csc^2 x-\cot^2 x}$$
$$\frac{\tan x\csc x-\tan x\cot x}{1}$$
$$\frac{\frac{\sin x}{\cos x}\frac{1}{\sin x}-\frac{\sin x}{\cos x}\frac{\cos x}{\sin x}}{1}$$
$$\frac{\frac{1}{\cos x}-1}{1}$$
$$\boxed{\frac{\sec x-1}{1}}$$

$\cos^2\beta+\cos^2\beta\tan^2\beta$
$\cos^2\beta(1+\tan^2\beta)$
$\boxed{\cos^2\beta(\sec^2\beta)}$

4) cancelling part of numerator with denominator (vice versa):

$$\frac{1+\cancel{\cos^2 x}\sin^2 x}{\cancel{\cos^2 x}}$$

not OKAY

$$\frac{\cos^2 x+\cos^2 x\sin^2 x}{\cos^2 x}$$

$$\frac{\cancel{\cos^2 x}(1+\sin^2 x)}{\cancel{\cos^2 x}(1)}$$

OKAY

*Cancel only if you can factor out of entire numerator and denominator

5) denominator extended:

(correct)

$$\cos x + \sin x\left(\tan x\right)$$

$$\cos x + \sin x\left(\frac{\sin x}{\cos x}\right)$$

(incorrect)

$$\cos x + \sin x \tan x$$

$$\frac{\cos x + \sin x \sin x}{\cos x}$$

should only be under this sin x

6) replacement misplaced:

(correct)

$$\frac{\left(\tan x\right)}{\left(\csc x\right) + \left(\cot x\right)}$$

$$\frac{\left(\frac{\sin x}{\cos x}\right)}{\left(\frac{1}{\sin x}\right) + \left(\frac{\cos x}{\sin x}\right)}$$

(incorrect)

$$\frac{\tan x}{\csc x + \cot x}$$

$$\frac{1}{\csc x}\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}$$

5.3: Solving Trigonometric Equations

Solving Trig Equations: Primary goal is to get a single trig function on one side of the equation so you can find x.

Example...solve: $2\sin x - 1 = 0$

$2\sin x - 1 = 0$

$2\sin x = 1$

$\sin x = \frac{1}{2}$

"y" $= \frac{1}{2}$

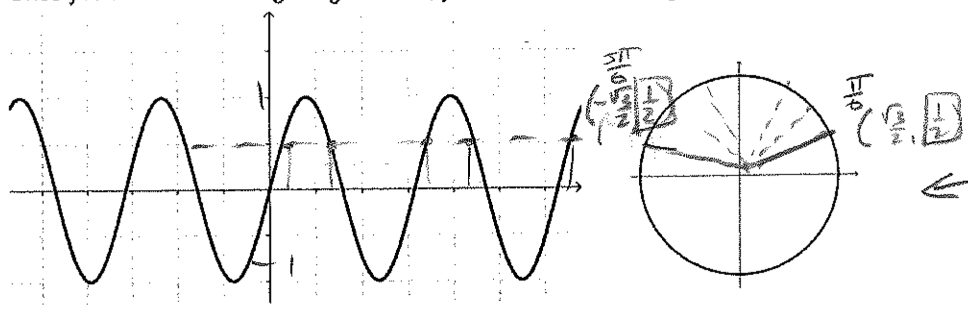
$\boxed{x = \frac{\pi}{6}, \frac{5\pi}{6}}$

(here 'x' = the angle)

to solve:

find the angles which make the equation true

Once you've isolated a single trig function, you can think of solving in a couple of ways:



Strategies...

1) Collect like terms

Ex: Find all solutions of $\sin x + \sqrt{2} = -\sin x$ in the interval $[0, 2\pi)$

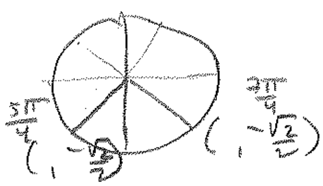
$+\sin x \qquad +\sin x$

$2\sin x + \sqrt{2} = 0$

$2\sin x = -\sqrt{2}$

$\sin x = \frac{-\sqrt{2}}{2}$

"y"



$\boxed{x = \frac{5\pi}{4}, \frac{7\pi}{4}}$

2) Extract square roots

Ex: Find all solutions of $3\tan^2 x - 1 = 0$ in the interval $[0, 2\pi)$

$3\tan^2 x = 1$

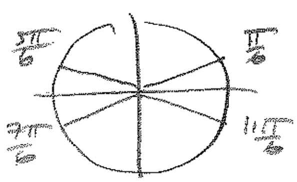
$\tan^2 x = \frac{1}{3}$

$\tan x = \pm\sqrt{\frac{1}{3}}$

$\tan x = \pm\frac{\sqrt{1}}{\sqrt{3}} = \pm\frac{1/2}{\sqrt{3}/2}$

(divide by 2 trick)

$\frac{\sin x}{\cos x} = \pm\frac{1/2}{\sqrt{3}/2}$ ← "y" ← "x"



$\boxed{x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}}$

3) Factoring (simple factoring, patterns, quadratic factoring)

Ex: Find all solutions of $\cot x \cos^2 x = 2\cot x$ in the interval $[0, 2\pi)$

** don't divide both sides by trig functions **

$-2\cot x \qquad -2\cot x$

$\cot x \cos^2 x - 2\cot x = 0$

$(\cot x)(\cos^2 x - 2) = 0$

1) $\cot x = 0$

2) $\cos^2 x - 2 = 0$

$\cos^2 x = 2$

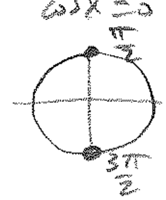
$\cos x = \pm\sqrt{2}$

1) $\cot x = 0$

$\frac{\cos x}{\sin x} = 0$

"x"

$\cos x = 0$



2) $\cos x = \pm\sqrt{2}$

larger than 1

not possible

$\boxed{x = \frac{\pi}{2}, \frac{3\pi}{2}}$

4) Square both sides to get a quadratic to factor

Ex: Find all solutions of $\cos x + 1 = \sin x$ in the interval $[0, 2\pi)$

$(\cos x+1)^2 = \sin^2 x$

$(\cos x+1)(\cos x+1) = \sin^2 x$

$\cos^2 x + 2\cos x + 1 = \sin^2 x$

$\cos^2 x + 2\cos x + 1 = 1 - \cos^2 x$

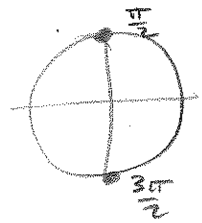
$2\cos^2 x + 2\cos x = 0$

$2\cos x(\cos x + 1) = 0$

1) $2\cos x = 0$ 2) $\cos x + 1 = 0$

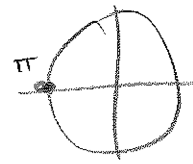
1) $2\cos x = 0$

$\cos x = 0$



2) $\cos x + 1 = 0$

$\cos x = -1$



$\boxed{x = \frac{\pi}{2}, \frac{3\pi}{2}, \pi}$

5) Function of multiple angles (sin3x, cos5x, etc.)

Ex: Find all solutions of $2\cos 3x - 1 = 0$ in the interval $[0, 2\pi)$

Solve for the argument given, then divide:

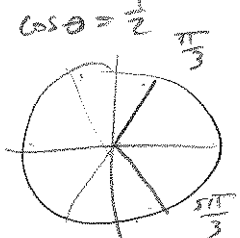
$2\cos 3x - 1 = 0$

substitute: $\theta = 3x$

$2\cos\theta - 1 = 0$

$2\cos\theta = 1$

$\cos\theta = \frac{1}{2}$



$\theta = \frac{\pi}{3}$ or $\theta = \frac{5\pi}{3}$

$3x = \frac{\pi}{3}$ $3x = \frac{5\pi}{3}$

$\boxed{x = \frac{\pi}{9} \text{ or } x = \frac{5\pi}{9}}$

6) Using inverse functions (use calculator if not a unit circle value)

Ex: Find all solutions of $(\sec^2 x) - 2\tan x = 4$ in the interval $[0, 2\pi)$

$\boxed{x = 1.249, 4.7124, \frac{3\pi}{4}, \frac{7\pi}{4}}$

$(1 + \tan^2 x) - 2\tan x = 4$

$\tan^2 x - 2\tan x - 3 = 0$

$(u = \tan x)$ $u^2 - 2u - 3 = 0$

$(u-3)(u+1) = 0$

$(\tan x - 3)(\tan x + 1) = 0$

1) $\tan x - 3 = 0$ 2) $\tan x + 1 = 0$

1) $\tan x - 3 = 0$

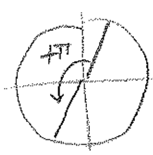
$\tan x = 3$

$\tan^{-1}(\tan x) = \tan^{-1}(3)$

$x = \tan^{-1}(3)$ calculator

$x = 1.249$

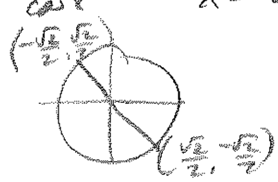
$x = 4.7124$



2) $\tan x + 1 = 0$

$\tan x = -1$

$\frac{\sin x}{\cos x} = -1$ $x = \frac{3\pi}{4}, \frac{7\pi}{4}$



7) Move everything to one side and use graphing calculator to find zeros

Ex: Find all solutions of $x = 2\sin x$ in the interval $[0, 2\pi)$

Some problems have no reasonable algebraic solution.

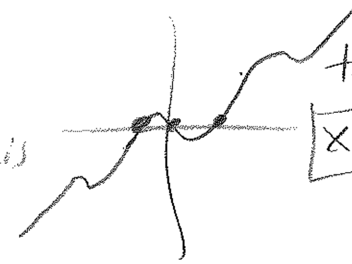
$x - 2\sin x = 0$

$Y1 = x - 2\sin x$

calculator

see where crosses x-axis

$(y = 0)$



trace or zero feature.

$\boxed{x = -1.895, 0, 1.895}$

5.4: Sum and Difference Formulas

$\left(\sin(u+v) \neq \sin u + \sin v\right)$

<u>Sum and Difference Formulas</u> (do not need to memorize)

$$\sin(u+v) = \sin u \cos v + \cos u \sin v$$

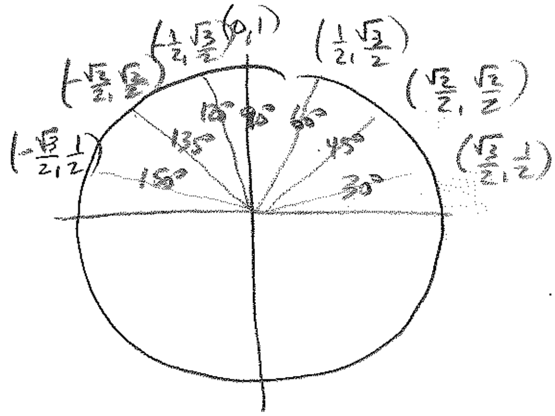
$$\sin(u-v) = \sin u \cos v - \cos u \sin v$$

$$\cos(u+v) = \cos u \cos v - \sin u \sin v$$

$$\cos(u-v) = \cos u \cos v + \sin u \sin v$$

$$\tan(u+v) = \frac{\tan u + \tan v}{1 - \tan u \tan v}$$

$$\tan(u-v) = \frac{\tan u - \tan v}{1 + \tan u \tan v}$$



Find $\sin 165°$ (hint: 165°=135°+30°)

$\sin(135° + 30°)$

$$\begin{aligned}
\sin(u+v) &= \sin u \cos v + \cos u \sin v \\
&= (\sin 135°)(\cos 30°) + (\cos 135°)(\sin 30°) \\
&= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{-\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) \\
&= \frac{\sqrt{6}}{4} + \frac{-\sqrt{2}}{4} \\
&= \boxed{\frac{\sqrt{6}-\sqrt{2}}{4}}
\end{aligned}$$

$\cos 165°$

$\cos(135° + 30°)$

$$\begin{aligned}
\cos(u+v) &= \cos u \cos v - \sin u \sin v \\
&= (\cos 135°)(\cos 30°) - (\sin 135°)(\sin 30°) \\
&= \left(-\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) \\
&= -\frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} \\
&= \boxed{\frac{-\sqrt{6}-\sqrt{2}}{4}}
\end{aligned}$$

$\tan 165°$

$\tan(135° + 30°)$

$$\begin{aligned}
\tan(u+v) &= \frac{(\tan u) + (\tan v)}{1 - (\tan u)(\tan v)} \\
&= \frac{(\tan 135°) + (\tan 30°)}{1 - (\tan 135°)(\tan 30°)} \\
&= \frac{(-1) + \left(\frac{1}{\sqrt{3}}\right)}{1 - (-1)\left(\frac{1}{\sqrt{3}}\right)} \\
&= \frac{\left(-1 + \frac{1}{\sqrt{3}}\right)\sqrt{3}}{\left(1 + \frac{1}{\sqrt{3}}\right)\sqrt{3}} \\
&= \boxed{\frac{-\sqrt{3}+1}{\sqrt{3}+1}}
\end{aligned}$$

$$\begin{aligned}
\tan 135° &= \frac{\sin 135°}{\cos 135°} \\
&= \frac{\sqrt{2}/2}{-\sqrt{2}/2} \\
&= -1
\end{aligned}$$

$$\begin{aligned}
\tan 30° &= \frac{\sin 30°}{\cos 30°} \\
&= \frac{1/2}{\sqrt{3}/2} = \frac{1}{\sqrt{3}}
\end{aligned}$$

Find $\sin\left(-\frac{\pi}{12}\right)$ (hint: $-\frac{\pi}{12}=\frac{\pi}{6}-\frac{\pi}{4}$)

$\sin\left(\frac{\pi}{6}-\frac{\pi}{4}\right)$

$\sin(u-v)=\sin u\cos v-\cos u\sin v$

$=\left(\sin\frac{\pi}{6}\right)\left(\cos\frac{\pi}{4}\right)-\left(\cos\frac{\pi}{6}\right)\left(\sin\frac{\pi}{4}\right)$

$=\left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)-\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$

$=\frac{\sqrt{2}}{4}-\frac{\sqrt{6}}{4}$

$=\boxed{\frac{\sqrt{2}-\sqrt{6}}{4}}$

$\tan\frac{\pi}{6}=\frac{\sin\frac{\pi}{6}}{\cos\frac{\pi}{6}}=\frac{1/2}{\sqrt{3}/2}=\frac{1}{\sqrt{3}}$

$\tan\frac{\pi}{4}=\frac{\sin\frac{\pi}{4}}{\cos\frac{\pi}{4}}=\frac{\sqrt{2}/2}{\sqrt{2}/2}=1$

$\cos\left(-\frac{\pi}{12}\right)$

$\cos\left(\frac{\pi}{6}-\frac{\pi}{4}\right)=\left(\cos\frac{\pi}{6}\right)\left(\cos\frac{\pi}{4}\right)+\left(\sin\frac{\pi}{6}\right)\left(\sin\frac{\pi}{4}\right)$

$=\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)+\left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$

$=\frac{\sqrt{6}}{4}+\frac{\sqrt{2}}{4}$

$=\boxed{\frac{\sqrt{6}+\sqrt{2}}{4}}$

$\tan\left(-\frac{\pi}{12}\right)$

$\tan\left(\frac{\pi}{6}-\frac{\pi}{4}\right)=\frac{\tan\frac{\pi}{6}-\tan\frac{\pi}{4}}{1+\tan\frac{\pi}{6}\tan\frac{\pi}{4}}$

$=\frac{\left(\frac{1}{\sqrt{3}}\right)-(1)}{1+\left(\frac{1}{\sqrt{3}}\right)(1)}=\frac{\left(\frac{1}{\sqrt{3}}-1\right)\sqrt{3}}{\left(1+\frac{1}{\sqrt{3}}\right)\sqrt{3}}$

$=\boxed{\frac{1-\sqrt{3}}{1+\sqrt{3}}}$

Use sum or difference formulas to write the expression as sin, cos or tan or an angle:

$\sin u\cos v+\cos u\sin v=\sin(u+v)$

$\sin 140^\circ\cos 50^\circ+\cos 140^\circ\sin 50^\circ=\sin(140^\circ+50^\circ)$

$=\boxed{\sin(190^\circ)}$

$\cos u\cos v+\sin u\sin v=\cos(u-v)$

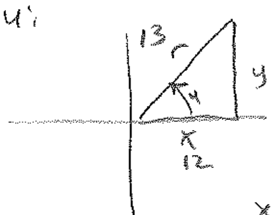
$\cos 3x\cos 2y+\sin 3x\sin 2y=\boxed{\cos(3x-2y)}$

Find the exact value of $\cos(u-v)$ given that:

$\sin u=\frac{5}{13}$, $\frac{y}{r}$ where $0<u<\frac{\pi}{2}$ and $\cos v=\frac{-3}{5}$, $\frac{x}{r}$ where $\frac{\pi}{2}<v<\pi$

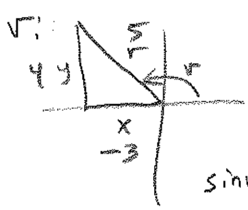
- Make 2 sketches (one for u, one for v)
- Use sketches to find the 'other' sine and cosine
- Use sum/difference formula and plug in values.

u:



$\cos u=\frac{x}{r}=\frac{12}{13}$

v:



$\sin v=\frac{4}{5}$

$\cos(u-v)=\cos u\cos v+\sin u\sin v$

$=\left(\frac{12}{13}\right)\left(\frac{-3}{5}\right)+\left(\frac{5}{13}\right)\left(\frac{4}{5}\right)$

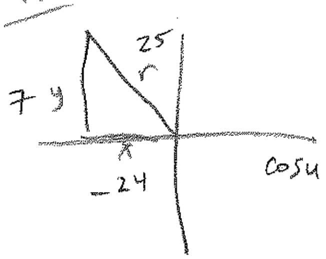
$=\frac{-36}{65}+\frac{20}{65}$

$=\boxed{\frac{-16}{65}}$

Practice: Find the exact value of $\boxed{\sin(u+v)}$ given that:

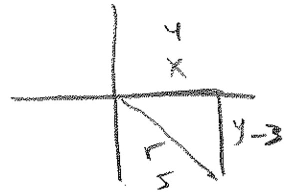
$\sin u = \frac{7}{25}$, $\frac{y}{r}$ *where* $\frac{\pi}{2} < u < \pi$ and $\cos v = \frac{4}{5}$, $\frac{x}{r}$ *where* $\frac{3\pi}{2} < v < 2\pi$

u:



$\cos u = \frac{-24}{25}$

v:



$\sin v = \frac{-3}{5}$

$\sin(u+v) = \sin u \cos v + \cos u \sin v$

$= \left(\frac{7}{25}\right)\left(\frac{4}{5}\right) + \left(\frac{-24}{25}\right)\left(\frac{-3}{5}\right)$

$= \frac{28}{125} + \frac{72}{125}$

$= \frac{100}{125} = \boxed{\frac{4}{5}}$

$\cos(u - v)$

Verify the identity: $\cos\left(\frac{5\pi}{4} - x\right) = -\frac{\sqrt{2}}{2}(\cos x + \sin x)$

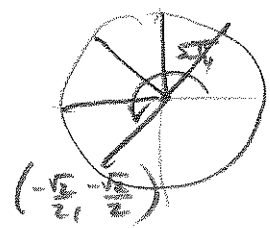
$\cos u \cos v + \sin u \sin v$

$\cos \frac{5\pi}{4} \cos x + \sin \frac{5\pi}{4} \sin x = -\frac{\sqrt{2}}{2}(\cos x + \sin x)$

$\left(-\frac{\sqrt{2}}{2}\right)\cos x + \left(-\frac{\sqrt{2}}{2}\right)\sin x = -\frac{\sqrt{2}}{2}(\cos x + \sin x)$

$-\frac{\sqrt{2}}{2}(\cos x + \sin x) = -\frac{\sqrt{2}}{2}(\cos x + \sin x)$

✓



$\sin(u - v)$

Verify the identity: $\sin(3\pi - x) = \sin x$

$\sin u \cos v - \cos u \sin v$

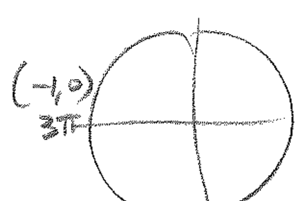
$\sin 3\pi \cos x - \cos 3\pi \sin x = \sin x$

$(0)\cos x - (-1)\sin x = \sin x$

$0 + 1 \sin x = \sin x$

$\sin x = \sin x$

✓



5.5: Double, Half-Angle, and Power-Reducing Formulas

Double Angle Formulas (do not need to memorize)

$$\sin 2u = 2\sin u\cos u$$

$$\cos 2u = \cos^2 u - \sin^2 u$$

$$\cos 2u = 2\cos^2 u - 1$$

$$\cos 2u = 1 - 2\sin^2 u$$

$$\tan 2u = \frac{2\tan u}{1-\tan^2 u}$$

Half Angle Formulas (do not need to memorize)

$$\sin\frac{u}{2} = \pm\sqrt{\frac{1-\cos u}{2}} = \pm\sqrt{\frac{1}{2}(1-\cos u)}$$

$$\cos\frac{u}{2} = \pm\sqrt{\frac{1+\cos u}{2}} = \pm\sqrt{\frac{1}{2}(1+\cos u)}$$

the sign depends upon the quadrant of u
(graph u, find u/2, determine sign)

$$\tan\frac{u}{2} = \frac{1-\cos u}{\sin u} = \frac{\sin u}{1+\cos u}$$

Power-Reducing Formulas (do not need to memorize)

$$\sin^2 u = \frac{1-\cos 2u}{2} \qquad \cos^2 u = \frac{1+\cos 2u}{2} \qquad \tan^2 u = \frac{1-\cos 2u}{1+\cos 2u}$$

Examples:

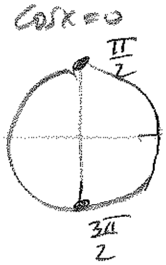
Solve: $\underbrace{\sin 2x} - \cos x = 0$

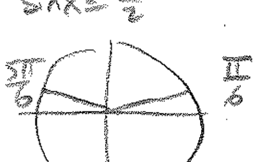
$2\sin x\cos x - \cos x = 0$

$\cos x(2\sin x - 1) = 0$

$\cos x = 0$ $\qquad$ $2\sin x - 1 = 0$

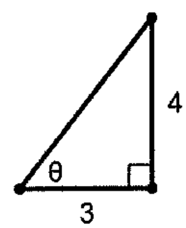
$\sin x = \frac{1}{2}$





$$\boxed{x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}}$$

Use the figure to find the exact value of the $\cot 2\theta = \frac{1}{\tan 2\theta}$



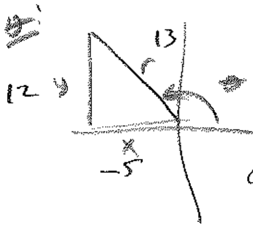
$\tan\theta = \frac{4}{3}$

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2\left(\frac{4}{3}\right)}{1-\left(\frac{4}{3}\right)^2} = \frac{\left(\frac{8}{3}\right)9}{\left(1-\frac{16}{9}\right)9} = \frac{24}{9-16}$$

$$\tan 2\theta = \frac{24}{-7} \quad \text{so} \quad \boxed{\cot 2\theta = \frac{-7}{24}}$$

Given $\sin \cancel{x}\,\theta = \frac{12}{13}$, $\frac{\pi}{2} < \cancel{x}\,\theta < \pi$ $\qquad$ Find $\sin 2\cancel{x}$ $\qquad$ change to θ to avoid confusion

$\sin 2\theta$



$\cos\theta = \frac{-5}{13}$

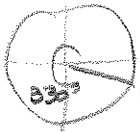
$$\sin 2\theta = 2\sin\theta\cos\theta = 2\left(\frac{12}{13}\right)\left(\frac{-5}{13}\right) = \frac{(2)(12)(-5)}{(13)(13)} = \boxed{\frac{-120}{169}}$$

Find the exact value of $\cos 165°$ $\quad 165° = \frac{330°}{2}$

$$\cos\left(\frac{u}{2}\right) = \pm\sqrt{\frac{1+\cos u}{2}}$$

$$\cos(165°) = \cos\left(\frac{330°}{2}\right) = \pm\sqrt{\frac{1+\cos(330°)}{2}}$$

$$= \pm\sqrt{\frac{1+\left(\frac{\sqrt{3}}{2}\right)}{2}}$$



$$\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right) \quad = \pm\sqrt{\frac{\left(1+\frac{\sqrt{3}}{2}\right)2}{(2)\,2}}$$

$$= \pm\sqrt{\frac{2+\sqrt{3}}{4}}$$

$$= \pm\frac{\sqrt{2+\sqrt{3}}}{\sqrt{4}}$$

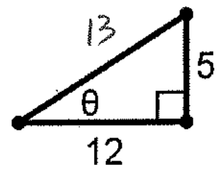
$$= \pm\frac{\sqrt{2+\sqrt{3}}}{2} \quad = \boxed{-\frac{\sqrt{2+\sqrt{3}}}{2}}$$

original angle 165°

cos is negative

which one?

Use the figure to find the exact value of $\sin\frac{\theta}{2}$



$$\cos\theta = \frac{adj}{hyp} = \frac{12}{13}$$

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$$

$$= \pm\sqrt{\frac{\left(1-\left(\frac{12}{13}\right)\right)}{(2)}}$$

$$= \pm\sqrt{\frac{\left(1-\frac{12}{13}\right)13}{(2)\,13}}$$

$$= \pm\sqrt{\frac{13-12}{26}}$$

$$= \pm\sqrt{\frac{1}{26}}$$

$$= \pm\frac{\sqrt{1}}{\sqrt{26}} = \pm\frac{1}{\sqrt{26}}\cdot\frac{\sqrt{26}}{\sqrt{26}}$$

in triangle lengths positive so

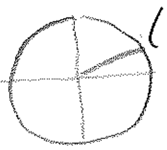
$$= \pm\frac{\sqrt{26}}{26}$$

$$\boxed{\sin\frac{\theta}{2} = \frac{\sqrt{26}}{26}}$$

Find the exact value of $\tan\frac{\pi}{12}$ $\quad \frac{\pi}{12} = \frac{\left(\frac{\pi}{6}\right)}{2}$

$$\tan\left(\frac{u}{2}\right) = \frac{1-\cos u}{\sin u}$$

$$\tan\left(\frac{\pi}{12}\right) = \tan\left(\frac{\left(\frac{\pi}{6}\right)}{2}\right) = \frac{1-\cos\frac{\pi}{6}}{\sin\frac{\pi}{6}}$$



$$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right) \quad \tan\left(\frac{\pi}{12}\right) = \frac{1-\left(\frac{\sqrt{3}}{2}\right)}{\left(\frac{1}{2}\right)}$$

$$= \frac{\left(1-\frac{\sqrt{3}}{2}\right)2}{\left(\frac{1}{2}\right)2}$$

$$= \frac{2-\sqrt{3}}{1}$$

$$\boxed{\tan\left(\frac{\pi}{12}\right) = 2-\sqrt{3}}$$

Rewrite in terms of the first power of the cosine:

$$\sin^4 x = (\sin^2 x)^2 = \left(\frac{1-\cos 2x}{2}\right)^2$$

$$= \frac{(1-\cos 2x)^2}{2^2} = \frac{1-2\cos 2x+\cos^2 2x}{4}$$

$$= \frac{1}{4}\left[1-2\cos 2x+\cos^2(2x)\right]$$

$$= \frac{1}{4}\left[1-2\cos 2x+\left[\frac{1+\cos 4x}{2}\right]\right]$$

$$= \frac{1}{4}\left[1-2\cos 2x+\frac{1}{2}+\frac{1}{2}\cos 4x\right]$$

$$= \frac{1}{4} - \frac{1}{2}\cos 2x + \frac{1}{8} + \frac{1}{2}\cos 4x$$

$$\boxed{\sin^4 x = \frac{3}{8} - \frac{1}{2}\cos 2x + \frac{1}{2}\cos 4x}$$

Graph using the power-reducing formulas: $\cos^2 x = \frac{1+\cos 2x}{2} = \frac{1}{2}+\frac{1}{2}\cos(2x)$

$$\frac{0}{2} \le \frac{2x}{2} \le \frac{2\pi}{2}$$

$$0 \le x \le \pi \quad \text{period}$$