Precalculus – Lesson Notes: Chapter 8 Matrices

8.1 day 1: Matrix terminology, Representing a system of equations with a matrix

A matrix - is a rectangular arrangement of real numbers:

$$\begin{array}{l} \text{row 1} \\[1em] \text{row 2} \\[1em] \text{row 3} \end{array} \begin{pmatrix} 2 & -3 & 4.5 \\ \frac{3}{2} & 8.3 & -15 \\ 0 & 2 & 0 \end{pmatrix} \qquad \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$\begin{array}{ccc} \text{col 1} & \text{col 2} & \text{col 3} \end{array}$$

Individual elements denoted by subscripts: $a_{row,column}$

Dimension or Order of a matrix: #rows x #columns

$$\begin{pmatrix} 2 & -3 & 4.5 \\ \frac{3}{2} & 8.3 & -15 \\ 0 & 2 & 0 \end{pmatrix} \qquad \begin{pmatrix} 2 & 3 \\ -4 & 5 \end{pmatrix} \qquad \begin{bmatrix} 2 & -5 & 3 \\ 0 & 1 & 6 \end{bmatrix}$$

3x3 2x2 2x3

$$\begin{bmatrix} 1 & 5 \\ 2 & 6 \\ 0 & 8 \\ 9 & 1 \end{bmatrix} \qquad \begin{bmatrix} 2 & -4 & 6 \end{bmatrix} \qquad \begin{bmatrix} 2 \\ -4 \\ 7 \end{bmatrix}$$

4x2 1x3 3x1

Representing systems of equations with a matrix:

$$\begin{cases} x + 3y = 9 \\ -y + 4z = -2 \\ x - 5z = 0 \end{cases} \rightarrow \left[\begin{array}{ccc:c} 1 & 3 & 0 & 9 \\ 0 & -1 & 4 & -2 \\ 1 & 0 & -5 & 0 \end{array}\right] \leftarrow \text{augmented matrix}$$

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & -1 & 4 \\ 1 & 0 & -5 \end{bmatrix} \qquad \begin{bmatrix} 9 \\ -2 \\ 0 \end{bmatrix}$$

Coefficient matrix Constant matrix

8.1 day 2: Gaussian and Gauss-Jordan Elimination with matrices

(day2)

Solving a system of equations using matrices (Gaussian Elimination, back-substitution):

$$\begin{cases} x-2y+3z=9 \\ -x+3y=-4 \\ 2x-5y+5z=17 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{array}\right]$$

$$\begin{array}{r} \\ R_1+R_2 \\ -2R_1+R_3 \end{array}\left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & -1 & -1 \end{array}\right]$$

Same elementary row operations from last chapter

$$\begin{array}{r} \\ \\ R_2+R_3 \end{array}\left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 4 \end{array}\right]$$

$$\begin{array}{r} \\ \\ \frac{1}{2}R_3 \end{array}\left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Row-Echelon Form (REF)

$\boxed{z=2}$

$y+3z=5$

$y+3(2)=5$

$\boxed{y=-1}$

$x-2y+3z=9$

$x-2(-1)+3(2)=9$

$x+2+6=9$

$\boxed{x=1}$

$\boxed{(1,-1,2)}$

Solving a system of equations using matrices (Gauss-Jordan Elimination):

$$\begin{cases} x-2y+3z=9 \\ -x+3y=-4 \\ 2x-5y+5z=17 \end{cases}$$

- Get coefficient 1 in upper left (swap equations, or divide row 1 by leading coefficient).
- Use this 1 to cancel this column's value to zero in all other rows.
- Move to next row and get a 1 on main diagonal.
- Use this 1 to cancel this column's value to zero in all other rows (below *and above*).
- Continue until left side is the identity matrix (zeros everywhere except 1s on diagonal).

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{array}\right]$$

$$\begin{array}{r} \\ R_1+R_2 \\ -2R_1+R_3 \end{array}\left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & -1 & -1 \end{array}\right]$$

$$\begin{array}{r} 2R_2+R_1 \\ \\ R_2+R_3 \end{array}\left[\begin{array}{ccc|c} 1 & 0 & 9 & 19 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 4 \end{array}\right]$$

$$\begin{array}{r} \\ \\ \frac{1}{2}R_3 \end{array}\left[\begin{array}{ccc|c} 1 & 0 & 9 & 19 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\begin{array}{r} -9R_3+R_1 \\ -3R_3+R_2 \\ \\ \end{array}\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$x=1$

$y=-1$

$z=2$

Reduced Row-Echelon Form (RREF)

$\boxed{(1,-1,2)}$

(no back-substitution necessary)

Examples/Practice: Solve system using Gaussian or Gauss-Jordan elimination

$$\begin{cases} 2x+6y=14 \\ 2x+3y=2 \end{cases}$$

$$\left[\begin{array}{cc|c} 2 & 6 & 14 \\ 2 & 3 & 2 \end{array}\right]$$

$$\begin{array}{c} \tfrac{1}{2}R_1 \\ \end{array}\left[\begin{array}{cc|c} 1 & 3 & 7 \\ 2 & 3 & 2 \end{array}\right]$$

$$\begin{array}{c} \\ -2R_1+R_2 \end{array}\left[\begin{array}{cc|c} 1 & 3 & 7 \\ 0 & -3 & -12 \end{array}\right]$$

$$\begin{array}{c} \\ -\tfrac{1}{3}R_2 \end{array}\left[\begin{array}{cc|c} 1 & 3 & 7 \\ 0 & 1 & 4 \end{array}\right]$$

$$\begin{array}{c} -3R_2+R_1 \\ \end{array}\left[\begin{array}{cc|c} 1 & 0 & -5 \\ 0 & 1 & 4 \end{array}\right]$$

$$\boxed{(-5,4)}$$

$$\begin{cases} 2x-y+2z=3 \\ x-y+z=1 \\ -x-2y-2z=-4 \end{cases}$$

$$\left[\begin{array}{ccc|c} 2 & -1 & 2 & 3 \\ 1 & -1 & 1 & 1 \\ -1 & -2 & -2 & -4 \end{array}\right]$$

$$\begin{array}{c} R_2 \\ R_1 \\ \end{array}\left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 2 & -1 & 2 & 3 \\ -1 & -2 & -2 & -4 \end{array}\right]$$

$$\begin{array}{c} \\ -2R_1+R_2 \\ R_1+R_3 \end{array}\left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & -3 & -1 & -3 \end{array}\right]$$

$$\begin{array}{c} R_2+R_1 \\ \\ 3R_2+R_3 \end{array}\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{array}\right]$$

$$\begin{array}{c} \\ \\ -R_3 \end{array}\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\begin{array}{c} -R_3+R_1 \\ \\ \end{array}\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$x=2$

$y=1$

$z=0$

$$\boxed{(2,1,0)}$$

$$\begin{cases} x+y-5z=3 \\ x-2z=1 \\ 2x-y-z=0 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -5 & 3 \\ 1 & 0 & -2 & 1 \\ 2 & -1 & -1 & 0 \end{array}\right]$$

$$\begin{array}{c} \\ -R_1+R_2 \\ -2R_1+R_3 \end{array}\left[\begin{array}{ccc|c} 1 & 1 & -5 & 3 \\ 0 & -1 & 3 & -2 \\ 0 & -3 & 9 & -6 \end{array}\right]$$

$$\begin{array}{c} \\ -R_2 \\ \end{array}\left[\begin{array}{ccc|c} 1 & 1 & -5 & 3 \\ 0 & 1 & -3 & 2 \\ 0 & -3 & 9 & -6 \end{array}\right]$$

$$\begin{array}{c} -R_2+R_1 \\ \\ 3R_2+R_3 \end{array}\left[\begin{array}{ccc|c} 1 & 0 & -2 & 1 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

(Consistent)

too few equations

$x-2z=1 \qquad y-3z=2$

$x=2z+1 \qquad y=3z+2$

general solution:

$$\boxed{(2z+1,\ 3z+2,\ z)}$$

a few specific solutions:

$$\boxed{\begin{array}{ll} (1,2,0) & \\ (3,5,1) & \\ (5,8,2) & \end{array}}$$

$(z=0)$

$(z=1)$

$(z=2$

8.2 day 1: Operations with Matrices

Matrix notation: A matrix is represented by a capital letter: $A = \begin{bmatrix} 2 & 4 \\ -1 & 3 \end{bmatrix}$

Equality of matrices: Two matrices are equal if:
- they have the same order (dimensions)
- all corresponding entries are equal.

Homework example: Given the following matrices are equal, find x, y, and z.

$$\begin{bmatrix} x+4 & 8 & -3 \\ 1 & 22 & 2y \\ 7 & -2 & z+2 \end{bmatrix} = \begin{bmatrix} 2x+9 & 8 & -3 \\ 1 & 22 & -2 \\ 7 & -2 & 11 \end{bmatrix}$$

$x+4=2x+9$ $\boxed{-5=x}$

$2y=-2$ $\boxed{y=-1}$

$z+2=11$ $\boxed{z=9}$

Matrix addition, subtraction: *to add or subtract, order of matrices must be the same*

Add (or subtract) corresponding entries:

$$\begin{bmatrix} 2 & -3 & 1 \\ 0 & 1 & 4 \end{bmatrix} + \begin{bmatrix} 4 & 2 & -7 \\ 2 & -2 & 4 \end{bmatrix} = \begin{bmatrix} 6 & -1 & -6 \\ 2 & -1 & 8 \end{bmatrix}$$

$$\underset{2\times 3}{\begin{bmatrix} 7 & 6 & -11 \\ 2 & -1 & 7 \end{bmatrix}} - \underset{2\times 2}{\begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix}}$$ (undefined) ← not possible

Multiplying a matrix by a scalar number: Multiply each element in the matrix by the number

$$3\begin{bmatrix} 2 & -3 & 1 \\ 0 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 6 & -9 & 3 \\ 0 & 3 & 12 \end{bmatrix}$$

Properties of Matrix Addition and Scalar Multiplication:

For matrices A, B, and C, and scalars c and d:

1. $A+B=B+A$ ***(commutative property of addition)***
2. $A+(B+C)=(A+B)+C$ ***(associative property of addition)***
3. $(cd)A=c(dA)$ ***(associative property of scalar multiplication)***
4. $1A=A$ ***(scalar identity)***
5. $c(A+B)=cA+cB$ ***(distributive property)***
6. $(c+d)A=cA+dA$ ***(distributive property)***

***<u>Summary</u>:* For simple operations, matrices behave the same as numbers. Order of operations is the same.**

Example: For the following matrices, find 3A - B:

$$A = \begin{bmatrix} 2 & 0 & 4 \\ 3 & -2 & 5 \\ -1 & 0 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 7 \\ -1 & 3 & 2 \\ -3 & 2 & 1 \end{bmatrix}$$

$$3\begin{bmatrix} 2 & 0 & 4 \\ 3 & -2 & 5 \\ -1 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 7 \\ -1 & 3 & 2 \\ -3 & 2 & 1 \end{bmatrix}$$ (multiply first)

(then subtract)

$$\begin{bmatrix} 6 & 0 & 12 \\ 9 & -6 & 15 \\ -3 & 0 & 6 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 7 \\ -1 & 3 & 2 \\ -3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & -2 & 5 \\ 10 & -9 & 13 \\ 0 & -2 & 5 \end{bmatrix}$$

Solving matrix equations:

1. Solve the equation using capital letter symbols for matrices.
2. Replace the symbols with matrices and find the final answer matrix.

Solve the matrix equation: $A - 2X = B$ for $A = \begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix}$ $B = \begin{bmatrix} -3 & -6 \\ 5 & 1 \end{bmatrix}$

$$A - 2X = B$$
$$-A \qquad -A$$
$$\frac{-2X}{-2} = \frac{B - A}{-2}$$

$$X = -\frac{1}{2}(B - A)$$

$$X = -\frac{1}{2}\left(\begin{bmatrix} -3 & -6 \\ 5 & 1 \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix}\right)$$

$$X = -\frac{1}{2}\begin{bmatrix} -5 & -5 \\ 1 & -2 \end{bmatrix}$$

$$X = \begin{bmatrix} \frac{5}{2} & \frac{5}{2} \\ -\frac{1}{2} & 1 \end{bmatrix}$$

8.2 day 2: Matrix Multiplication; Identity Matrix

Matrix multiplication (multiplying a matrix by another matrix):

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22} \end{bmatrix}$$

Easier to see with an example:

$$\begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix} \cdot \begin{bmatrix} -1 & 5 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 5 & \\ & \end{bmatrix} \quad (1)(-1)+(2)(3)=5$$

$$\begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix} \cdot \begin{bmatrix} -1 & 5 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 9 \\ & \end{bmatrix} \quad (1)(5)+(2)(2)=9$$

$$\begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix} \cdot \begin{bmatrix} -1 & 5 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 9 \\ 7 & \end{bmatrix} \quad (5)(-1)+(4)(3)=7$$

$$\begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix} \cdot \begin{bmatrix} -1 & 5 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 9 \\ 7 & 33 \end{bmatrix} \quad (5)(5)+(4)(2)=33$$

To compute an element:

1. Take the matching <u>row from the left matrix</u> and <u>column from the right matrix</u>,
2. Moving across row from left to right, and down column from top to bottom, multiply each pair of numbers.
3. Add the results...the sum is what you put in the corresponding space in the answer matrix.

Same procedure, regardless of the size of the matrices. But, for it to work:

<u>the number of columns of the left matrix must match the number of rows of the right matrix.</u>

$2\times3 \cdot 3\times3 = 2\times3$ answer

ok

answer

$$\begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & -2 \end{bmatrix} \cdot \begin{bmatrix} -2 & 4 & 2 \\ 1 & 0 & 0 \\ -1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} -5 & & \\ & & \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & -2 \end{bmatrix} \cdot \begin{bmatrix} -2 & 4 & 2 \\ 1 & 0 & 0 \\ -1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} -5 & 7 & -1 \\ -3 & & \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & -2 \end{bmatrix} \cdot \begin{bmatrix} -2 & 4 & 2 \\ 1 & 0 & 0 \\ -1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} -5 & 7 & \\ & & \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & -2 \end{bmatrix} \cdot \begin{bmatrix} -2 & 4 & 2 \\ 1 & 0 & 0 \\ -1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} -5 & 7 & -1 \\ -3 & 6 & \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & -2 \end{bmatrix} \cdot \begin{bmatrix} -2 & 4 & 2 \\ 1 & 0 & 0 \\ -1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} -5 & 7 & -1 \\ & & \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & -2 \end{bmatrix} \cdot \begin{bmatrix} -2 & 4 & 2 \\ 1 & 0 & 0 \\ -1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} -5 & 7 & -1 \\ -3 & 6 & 6 \end{bmatrix}$$

Example:

$$\begin{bmatrix}1&2\\1&1\end{bmatrix}\cdot\begin{bmatrix}-1&2\\1&-1\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix}$$

2x2 , 2x2

ok

answer

$$\begin{bmatrix}1&-2&-3\end{bmatrix}\cdot\begin{bmatrix}2\\-1\\1\end{bmatrix}=\begin{bmatrix}1\end{bmatrix}$$

1x3 , 3x1

OK

answer

$$\begin{bmatrix}1&-2&-3\end{bmatrix}\cdot\begin{bmatrix}5&-4&1\end{bmatrix}=$$

1x3 , 1x3

not possible

Identity Matrix: An identity matrix has 1's along the main diagonal, and 0's elsewhere:

$$\begin{bmatrix}1&0\\0&1\end{bmatrix}\qquad\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

main diagonal

Called the identity matrix because if you multiply a matrix by an identity matrix, you get the original matrix:

$$\begin{bmatrix}3&4&-2\\1&-6&2\\7&0&5\end{bmatrix}\cdot\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}3&4&-2\\1&-6&2\\7&0&5\end{bmatrix}$$

Identity matrix is like "1" but for matrices.

Properties of Matrix Multiplication:

For matrices $A, B,$ and C, and scalar c:

1. $A(BC)=(AB)C$ *(associative property of matrix multiplication)*
2. $A(B+C)=AB+AC$ *(distributive property)*
3. $(A+B)C=AC+AB$ *(distributive property)*
4. $c(AB)=(cA)B=A(cB)$ *(associative property of scalar multiplication)*

Note: no commutative property. In general, $AB \neq BA$

Matrices on the graphing calculator:

You can enter matrices into your calculator, then have the calculator perform operations, including matrix multiplication (and other more complex things we'll cover later.)

Examples to show capabilities and procedure:

Find 3A-B for $A=\begin{bmatrix}2 & 0 & 4\\ 3 & -2 & 5\\ -1 & 0 & 2\end{bmatrix}$ $B=\begin{bmatrix}1 & 2 & 7\\ -1 & 3 & 2\\ -3 & 2 & 1\end{bmatrix}$

Enter matrix A:

1. 2nd x^{-1} (MATRIX)
2. -> -> EDIT, with cursor on 1: [A] line, press 'enter'
3. '3', enter, '3', enter (specifying a 3x3 matrix)
4. type in each element and enter it by pressing the enter key, the cursor automatically jumps to the next element: e.g., '2', enter, '0', enter, '4', enter, '3', enter, etc. until all elements are entered. (use (-) key, not subtract, to enter negative numbers).
5. 2nd MODE (QUIT) to exit entry mode.

Enter matrix B:

6. 2nd x^{-1} (MATRIX)
7. -> -> EDIT, with cursor on 2: [B] line, press 'enter'
8. '3', enter, '3', enter (specifying a 3x3 matrix)
9. type in each element and enter it by pressing the enter key, the cursor automatically jumps to the next element: e.g., '1', enter, '2', enter, '7', enter, '-1', enter, etc. until all elements are entered. (use (-) key, not subtract, to enter negative numbers).
10. 2nd MODE (QUIT) to exit entry mode.

Compute expression 3A-B:

11. press '3'
12. 2nd x^{-1} (MATRIX)
13. with cursor on 1:[A], press 'enter'
14. press the subtraction key '-'
15. 2nd x^{-1} (MATRIX)
16. with cursor on 2:[B], press 'enter'
17. press 'enter' again to compute. Should display the answer matrix: $\begin{bmatrix}5 & -2 & 5\\ 10 & -9 & 13\\ 0 & -2 & 5\end{bmatrix}$

Using the same matrices, A and B, try to compute AB and BA. Answers should be:

$$AB=\begin{bmatrix}-10 & 12 & 18\\ -10 & 10 & 22\\ -7 & 2 & -5\end{bmatrix} \qquad BA=\begin{bmatrix}1 & -4 & 28\\ 5 & -6 & 15\\ -1 & -4 & 0\end{bmatrix}$$

8.3 day 1: Inverse of a Matrix

$3x = 6$

$\frac{1}{3} 3x = \frac{1}{3} 6$

"identity" $1 \cdot x = 2$

$x = 2$

What is an inverse?

Solve: $\frac{3x}{3} = \frac{6}{3}$

$x = 2$

really: $\frac{1}{3} 3x = \frac{1}{3} 6$

multiplying 3 by its "inverse"

Multiplying a number and its inverse produces identity (1).
Multiplying a matrix and its inverse produced identity (the identity matrix, I)

Example: Show that B is the inverse of A: (use calculator)

$$A = \begin{bmatrix} -5 & 6 \\ -1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -6 \\ 1 & -5 \end{bmatrix} \quad AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Note: Inverses are exceptions - inverses do obey the commutative property.

How to find the inverse of a matrix: 3 methods.

(Textbook example 2, p.583, has a general explanation of finding an inverse algebraically from its definition, but we will focus on the more commonly used procedures to find an inverse.)

1) Row-reducing an augmented matrix

1. Write the initial matrix, and augment that matrix with the identity matrix.
2. If possible, row-reduce the augmented matrix to produce the identity matrix on the left side.
3. The right side is the inverse of the initial matrix.

Example: Find the inverse of $A = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}$

$$\left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{array}\right]$$

$$R_1 + R_2 \quad \left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{array}\right]$$

$$-4R_2 + R_1 \rightarrow \left[\begin{array}{cc|cc} 1 & 0 & -3 & -4 \\ 0 & 1 & 1 & 1 \end{array}\right]$$

inverse of A

$A^{-1} = \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix}$

Symbol for inverse of a matrix

Check:

$A \cdot A^{-1} = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ✓

$A^{-1} A = \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ✓

2) Using the determinant

In section 8.4, we'll learn about determinants of matrices. For a 2x2 matrix, the determinant is given by:

$$A=\begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \textit{determinant}=\det(A)=|A|=ad-bc$$

Then the inverse of a matrix can be found using the determinant:

$$A^{-1}=\frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}=\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example: Find the inverse of $A=\begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}$

$$A^{-1}=\frac{1}{(1)(-3)-(4)(-1)}\begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix}$$

$$A^{-1}=\frac{1}{1}\begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix}=\begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix}$$

3) Using a calculator

1. Enter the initial matrix using the 2nd x^{-1} (MATRIX), EDIT, specifying the size and entering all the matrix entries.
2. 2nd quit, then select the matrix entered by using the 2nd x^{-1} (MATRIX), scrolling with cursor to entered matrix, then hitting enter.
3. Press the x^{-1} key, then hit enter.

Example: Find the inverse of $A=\begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}$

$$A^{-1}=\begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix}$$

Do all matrices have inverses?

No inverse if:

- Matrix is not square.
- Matrix has any row that is a multiple of any other row. ← because row operations will produce a row 0=0

8.3 day 2: Solving Matrix Equations

We now know many ways to solve systems of equations:

<u>Algebra 1-2</u>
- Graphing
- Substitution
- Elimination

<u>Precalculus</u>
- Gaussian Elimination (with equations)
- Gaussian Elimination (with matrices)
- Gauss-Jordan

Adding 1 more: - Using Matrix Inverse

Writing a system of equations as a matrix equation:

$$\begin{cases} x-2y=\ 1 \\ 2x-3y=-2 \end{cases} \rightarrow \underset{\text{coefficient matrix}}{\begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}} \cdot \underset{\text{variable matrix}}{\begin{bmatrix} x \\ y \end{bmatrix}} = \underset{\text{constants matrix}}{\begin{bmatrix} 1 \\ -2 \end{bmatrix}}$$

multiply out left side:

$$\begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$2\times\underbrace{2 \cdot 2}_{\text{ok}}\times 1$

answer

$$\begin{bmatrix} x-2y \\ 2x-3y \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

(reproduces original system equations)

Solving systems of equations using matrices: 2 methods

1) Write system as a matrix equation, solving using matrix inverse

$$\begin{cases} x-2y=1 \\ 2x-3y=-2 \end{cases} \qquad \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix} \bullet \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$A \cdot X \cdot B$$

$$A \bullet X = B$$

$$\underbrace{A^{-1} \cdot A}X = A^{-1} \cdot B$$

$$I \cdot X = A^{-1}B$$

$$X = A^{-1}B$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -7 \\ -4 \end{bmatrix} \quad \boxed{(-7,-4)}$$

2) Write coefficients and answers as augmented matrix, obtain reduced row-echelon form

$$\begin{cases} x-2y=\ 1 \\ 2x-3y=-2 \end{cases} \qquad \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 2 & -3 & -2 \end{array}\right]$$

$$-2R_1+R_2 \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & -4 \end{array}\right]$$

$$2R_2+R_1 \left[\begin{array}{cc|c} 1 & 0 & -7 \\ 0 & 1 & -4 \end{array}\right] \leftarrow \text{RREF}$$

$$\boxed{(-7,-4)}$$

This can also be done in calculator:
- Enter matrix in A
- MATRIX → MATH, B rref[A]

Examples from homework: Find solution using an inverse matrix. $\begin{cases} x-2y=5 \\ 2x-3y=10 \end{cases}$

$$\begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 5 \\ 10 \end{bmatrix}$$

$$A \cdot X = B$$

$$A^{-1}A \cdot X = A^{-1}B$$

$$I \cdot X = A^{-1}B$$

$$X = A^{-1}B$$

$$\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 10 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 5 \\ 0 \end{bmatrix} \quad \boxed{(5,0)}$$

How to 'show work' on a test

Use a calculator for large systems:

$$\begin{cases} 2x+5y \qquad\quad +w=11 \\ x+4y+2z-2w=-7 \\ 2x-2y+5z \;\; +w=3 \\ x \qquad\qquad\quad -3w=-1 \end{cases}$$

$$\begin{bmatrix} 2 & 5 & 0 & 1 \\ 1 & 4 & 2 & -2 \\ 2 & -2 & 5 & 1 \\ 1 & 0 & 0 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}=\begin{bmatrix} 11 \\ -7 \\ 3 \\ -1 \end{bmatrix}$$

$$A \cdot X = B$$

$$A^{-1}A \cdot X = A^{-1}B$$

$$IX = A^{-1}B$$

$$X = A^{-1}B$$

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}=\begin{bmatrix} 24/71 & -25/71 & 10/71 & 22/71 \\ 3/71 & 35/213 & 74/213 & -25/213 \\ -10/71 & 49/213 & 23/213 & -35/213 \\ 8/71 & -25/213 & 17/213 & -43/213 \end{bmatrix}\begin{bmatrix} 11 \\ -7 \\ 3 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}=\begin{bmatrix} 441/71 \\ -163/213 \\ -569/213 \\ 512/213 \end{bmatrix} \quad \boxed{\left(\frac{441}{71}, \frac{-163}{213}, \frac{-569}{213}, \frac{512}{213}\right)}$$

Find solution. $\begin{cases} 5x-3y+2z=2 \\ 2x+2y-3z=3 \\ -x+7y-8z=4 \end{cases}$

$$\begin{bmatrix} 5 & -3 & 2 \\ 2 & 2 & -3 \\ -1 & 7 & -8 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

$$A \cdot X = B$$

$$\vdots$$

$$X = A^{-1}B$$

A^{-1} doesn't exist

not all matrices, (even square matrices) have inverses

if can't find inverse must use other techniques like Gauss-Jordan (RREF)

trying RREF (Gauss-Jordan):

$$\left[\begin{array}{ccc|c} 5 & -3 & 2 & 2 \\ 2 & 2 & -3 & 3 \\ -1 & 7 & -8 & 4 \end{array}\right]$$

RREF in calculator

$$\left[\begin{array}{ccc|c} 1 & 0 & -5/16 & 13/16 \\ 0 & 1 & -19/16 & 11/16 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

infinitely many solutions!

$$x-\frac{5}{16}z=\frac{13}{16} \qquad y-\frac{19}{16}z=\frac{11}{16}$$

$$x=\frac{5}{16}z+\frac{13}{16} \qquad y=\frac{19}{16}z+\frac{11}{16}$$

general solution:

$$\boxed{\left(\frac{5}{16}a+\frac{13}{16}, \frac{19}{16}a+\frac{11}{16}, a\right)}$$

8.4: Determinants

Determinants:

- Only square matrices have determinants.
- Determinant is a scalar number, not a matrix.
- A determinant 'determines' if a matrix has an inverse.
- If the determinant of a matrix is zero, that matrix has no inverse (is non-invertible).

Finding determinant of a 2x2 matrix:

$$A=\begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \textit{determinant}=\det(A)=|A|=ad-bc$$

Examples/practice - Find the determinant:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$(1)(4)-(2)(3)$

$\boxed{-2}$

$$\begin{bmatrix} 4 & 6 \\ 1 & -3 \end{bmatrix}$$

$(4)(-3)-(1)(6)$

$-12-6$

$=\boxed{-18}$

$$\begin{bmatrix} -2 & -3 \\ 4 & x \end{bmatrix}$$

$(-2)(x)-(-3)(4)$

$=\boxed{-2x+12}$

$$\begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix}$$

$(1)(8)-(2)(4)$

$=\boxed{0}$

(this matrix has no inverse)

Finding determinants of larger matrices:

select a row

"minors"

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$+ \quad - \quad +$ "cofactors"

Example: Find determinant of $\begin{bmatrix} 2 & 4 & 6 \\ 0 & 3 & 1 \\ 0 & 0 & -5 \end{bmatrix}$ $=2\begin{vmatrix} 3 & 1 \\ 0 & -5 \end{vmatrix} - 4\begin{vmatrix} 0 & 1 \\ 0 & -5 \end{vmatrix} + 6\begin{vmatrix} 0 & 3 \\ 0 & 0 \end{vmatrix}$

$=2(-15-0)-4(0-0)+6(0-0)$

$=-30-0+0$

$=\boxed{-30}$

Can use any row or column.
Cofactors are positive or negative according to this pattern:
(Choose row or column with most zeros...easiest to compute.)

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

Example: Find determinant of $\begin{bmatrix} 2 & 4 & 6 \\ 0 & 3 & 1 \\ 0 & 0 & -5 \end{bmatrix}$ $=2\begin{vmatrix} 3 & 1 \\ 0 & -5 \end{vmatrix} - 0\begin{vmatrix} 4 & 6 \\ 0 & -5 \end{vmatrix} + 0\begin{vmatrix} 4 & 6 \\ 0 & -5 \end{vmatrix}$

$=2(-15-0)-0(\sim)+0(\sim)$

$=\boxed{-30}$

Triangular Matrices and determinants:

Triangular matrices have one or both 'halves' (above or below the main diagonal) zero:

For all triangular form matrices, the determinant is the product of the numbers on the main diagonal.

Upper triangular matrix:
$$\begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 3 & -5 & 4 \\ 0 & 0 & 9 & -3 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$det = 1 \cdot 3 \cdot 9 \cdot 2 = \boxed{54}$

Lower triangular matrix:
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 4 & 1 & 9 & 0 \\ 7 & 5 & 2 & 2 \end{bmatrix}$$

$det = \boxed{54}$

Diagonal matrix:
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$det = \boxed{54}$

Alternate (diagonal method) for finding determinants (works with 3x3 only):

1. Rewrite the first two columns at the end of the matrix.
2. Starting from the upper left corner, draw a diagonal to the bottom of the original matrix, and find the product of these 3 numbers.
3. Do the same for the other 2 numbers in the top row (diagonals down and to the right.)
4. Add up these products to a single number.
5. Now starting from the lower left corner, draw a diagonal up and to the right top of the original matrix, and find the product of these 3 numbers.
6. Do the same for the other 2 numbers in the bottom row (diagonals up and to the right.)
7. Add up these products to a single number.
8. The determinant is the number from the first diagonals, minus the number from the second diagonals.

Example: Find the determinant using the diagonal method:

$-112 - 30 - 54 = -196$

$$\begin{bmatrix} 5 & 6 & 2 \\ -1 & -8 & 3 \\ 7 & -2 & 9 \end{bmatrix} \begin{matrix} 5 & 6 \\ -1 & -8 \\ 7 & -2 \end{matrix}$$

$-360 + 126 + 4 = -230$

det = "bottom number" − "top number"

$det = -230 - (-196) = \boxed{-34}$

You can also use your calculator to find a determinant of a matrix:

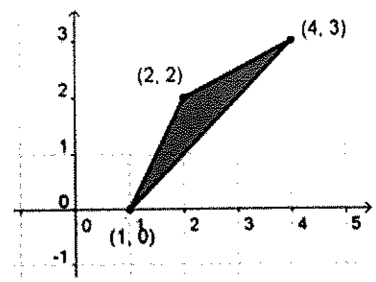
- Enter the matrix (e.g. in A)
- 2nd x^{-1} button -> MATH->det([A])

8.5 day 1: *Applications of Matrices and Determinants*

Area of a triangle, given coordinates of vertices:

$$Area_{triangle} = \pm\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Example:



$$Area = (\pm)\frac{1}{2}\begin{vmatrix} 1 & 0 & 1 \\ 2 & 2 & 1 \\ 4 & 3 & 1 \end{vmatrix} = (\pm)\frac{1}{2}(-3) = \boxed{\frac{3}{2}}$$

Test 3 points for collinearity:

What happens to the area, if the 3 points of a triangle are collinear?

$$3 \text{ points collinear if}: \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

Example:
Are the pts (0, 1), (4, 4), and (8,7) collinear?

$$\begin{vmatrix} 0 & 1 & 1 \\ 4 & 4 & 1 \\ 8 & 7 & 1 \end{vmatrix} = 0$$

(so area is zero)

so $\boxed{\text{Yes, points are collinear}}$

Find equation of a line given 2 points on the line:

$$\text{For equation of line, solve}: \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

Find equation of line passing through (10, 7), (-2, -7)

$-14 - 7x + 10y$

$$\begin{vmatrix} x & y & 1 \\ 10 & 7 & 1 \\ -2 & -7 & 1 \end{vmatrix} \begin{matrix} x & y \\ 10 & 7 \\ -2 & -7 \end{matrix}$$

$7x - 2y - 70$

$det = (7x - 2y - 70) - (-14 - 7x + 10y) = 0$

$7x - 2y - 70 + 14 + 7x - 10y = 0$

$\boxed{14x - 12y = 56}$

8.5 day 2: Applications of Matrices and Determinants (Cramer's Rule)

Cramer's Rule - another way to solve system of linear equations:

$$x=\frac{|A_x|}{|A|},\ y=\frac{|A_y|}{|A|},\ z=\frac{|A_z|}{|A|}$$

where A_n = matrix with nth column replaced with constants matrix column.

Example: Solve the system using Cramer's Rule: $\begin{cases}4x-2y=10\\3x-5y=11\end{cases}$ $\begin{bmatrix}4 & -2\\3 & -5\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}10\\11\end{bmatrix}$

$|A|=\begin{vmatrix}4 & -2\\3 & -5\end{vmatrix}=(-20-(-6))=-14$

$|A_x|=\begin{vmatrix}10 & -2\\11 & -5\end{vmatrix}=(-50-(-22))=-28$

$|A_y|=\begin{vmatrix}4 & 10\\3 & 11\end{vmatrix}=(44-30)=14$

$x=\frac{|A_x|}{|A|}=\frac{-28}{-14}=\boxed{2}$

$y=\frac{|A_y|}{|A|}=\frac{14}{-14}=\boxed{-1}$

$\boxed{(2,-1)}$

Example: Solve the system using Cramer's Rule: $\begin{cases}-x+2y-3z=1\\2x\qquad\ +z=0\\3x-4y+4z=2\end{cases}$ $\begin{bmatrix}-1 & 2 & -3\\2 & 0 & 1\\3 & -4 & 4\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1\\0\\2\end{bmatrix}$

$|A|=\begin{vmatrix}-1 & 2 & -3\\2 & 0 & 1\\3 & -4 & 4\end{vmatrix}=10$

$|A_x|=\begin{vmatrix}1 & 2 & -3\\0 & 0 & 1\\2 & -4 & 4\end{vmatrix}=8$

$|A_y|=\begin{vmatrix}-1 & 1 & -3\\2 & 0 & 1\\3 & 2 & 4\end{vmatrix}=-15$

$|A_z|=\begin{vmatrix}-1 & 2 & 1\\2 & 0 & 0\\3 & -4 & 2\end{vmatrix}=-16$

$x=\frac{|A_x|}{|A|}=\frac{8}{10}=\boxed{\frac{4}{5}}$

$y=\frac{|A_y|}{|A|}=\frac{-15}{10}=\boxed{\frac{-3}{2}}$

$z=\frac{|A_z|}{|A|}=\frac{-16}{10}=\boxed{\frac{-8}{5}}$

$\boxed{\left(\frac{4}{5},\frac{-3}{2},\frac{-8}{5}\right)}$

Matrix Application: Cryptography

Cryptography - 'hidden writing'

If you wanted to send a message to a friend, but encoded so that if your message was intercepted no one but your friend could read it, how would you encode your message?

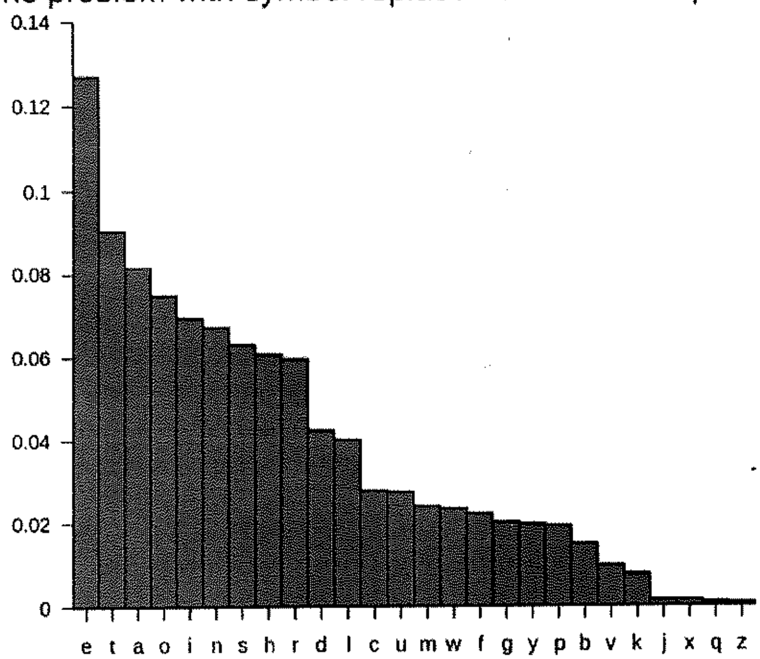
Symbol replacement -

The first codes people invented replaced the symbols (letters) in the original message with other symbols - either other letters, numbers or shapes, etc.

A B C D E F G H I J K L M N O P Q R S T U V W X Y Z
S R Z A N K O P B X Y C F Q D E V W G L I T M J H U

MEET ME MONDAY
FNNL FN FDQASH

The problem with symbol replacement - Susceptible to 'frequency analysis'



Encryption would be much stronger if a letter is encoded in different ways for different parts of the message.

Using a matrix can accomplish this by making the way letters are encrypted depend upon not just the letter, but the surrounding letters as well.

The matrix used for encryption becomes the 'key', and if the recipient also knows this matrix key, they can use the inverse of this matrix to decrypt the message.

Example: Let's encode the message MEET ME MONDAY using the encoding matrix $A = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix}$

1) Break the message into symbol groups each containing 3 letters:

[M E E] [T _ M] [E _ M] [O N D] [A Y _]

2) First, change the message into numbers by replacing each letter with a number as follows: (0 is assigned to a space).

A	B	C	D	E	F	G	H	I	J	K	L	M
1	2	3	4	5	6	7	8	9	10	11	12	13

N	O	P	Q	R	S	T	U	V	W	X	Y	Z
14	15	16	17	18	19	20	21	22	23	24	25	26

$[13\ \ 5\ \ 5]\ [20\ \ 0\ \ 13]\ [5\ \ 0\ \ 13]\ [15\ \ 14\ \ 4]\ [1\ \ 25\ \ 0]$

3) Multiply each symbol group by the encoding matrix to produce an encoded group:

$[13\ \ -26\ \ 21]\ [33\ \ -53\ \ -12]\ [18\ \ -23\ \ -42]\ [5\ \ -20\ \ 56]\ [-24\ \ 23\ \ 77]$

4) Recombine into a single string of numbers. This is the coded message:

13, -26, 21, 33, -53, -12, 18, -23, -42, 5, -20, 56, -24, 23, 77

To decode:

1) Find the inverse of the encoding matrix and store it in B to make a decoding matrix.

2) Break the encoded message into symbol groups of 3 numbers:

$[13\ \ -26\ \ 21]\ [33\ \ -53\ \ -12]\ [18\ \ -23\ \ -42]\ [5\ \ -20\ \ 56]\ [-24\ \ 23\ \ 77]$

3) Multiply each symbol group by the decoding matrix:

$[13\ \ 5\ \ 5]\ [20\ \ 0\ \ 13]\ [5\ \ 0\ \ 13]\ [15\ \ 14\ \ 4]\ [1\ \ 25\ \ 0]$

4) Use the letter-number map from above to convert the numbers back into letters:

[M E E] [T _ M] [E _ M] [O N D] [A Y _]

5) Recombine the original message.

MEET ME MONDAY