Precalculus – Lesson Notes: Chapter 9.1-9.3 Combinatorics

9.1 – Sequences and Series

Sequence – a list of numbers in a specific order. 1st term 2nd term 3rd term 1, 3, 5, 7, 9, 11 - - - -

$a_1, a_2, a_3, a_4, a_5, a_6 \dots a_n$

Sequences are like functions where 'n' is the input and the term is the output:

Infinite sequence: infinite number of terms (goes on forever) 1, 3, 5, 7, 9, ... ← continues
Finite sequence: finite number of terms (only n terms) 1, 3, 5, 7 ← only 4 terms

Some sequences have a 'rule' or 'expression' or 'formula' for finding a term given n:

$a_n = 2n-1$

$a_1 = 2(1)-1 = 2-1 = 1$
$a_2 = 2(2)-1 = 4-1 = 3$
$a_3 = 2(3)-1 = 6-1 = 5$

$\boxed{1, 3, 5, \dots}$

$a_n = \dfrac{(-1)^n}{2n-1}$

$a_1 = \dfrac{(-1)^1}{2(1)-1} = \dfrac{-1}{1} = -1$
$a_2 = \dfrac{(-1)^2}{2(2)-1} = \dfrac{1}{1} = 1$
$a_3 = \dfrac{(-1)^3}{2(3)-1} = \dfrac{-1}{5} = -\dfrac{1}{5}$
$a_4 = \dfrac{(-1)^4}{2(4)-1} = \dfrac{1}{7} = \dfrac{1}{7}$

$\boxed{-1, 1, -\frac{1}{5}, \frac{1}{7}, \dots}$

alternating sequence rule has $(-1)^n$ or $(-1)^{n+1}$

Some sequences have a rule for finding a term from previous terms (instead of from n)
These are called recursive sequences:
Example: The Fibonacci sequence... 1, 1, 2, 3, 5, 8, ...

What is the rule? each term is sum of 2 previous terms
(this term) = (previous term) + (term 2 back)

$a_1 = 1$
$a_2 = 1$
$a_k = a_{k-1} + a_{k-2}$

Factorials For positive integer n, $n! = n\cdot(n-1)\dots\cdot 3\cdot 2\cdot 1$ special case: $0! = 1$

Examples: $4! = 4\cdot 3\cdot 2\cdot 1 = 24$ $6! = 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1 = 720$

Simplifying factorials: $\dfrac{8!}{2!\cdot 6!} = \dfrac{8\cdot 7\cdot \cancel{6!}}{2\cdot 1\cdot \cancel{6!}} = \dfrac{8\cdot 7}{2} = \dfrac{\cancel{2}\cdot 4\cdot 7}{\cancel{2}} = \boxed{28}$

$\dfrac{(n+1)!}{n!}$

$\dfrac{(n+1)(n)(n-1)(n-2)(n-3)\dots(2)(1)}{(n)(n-1)(n-2)(n-3)\dots(2)(1)} = \boxed{(n+1)}$

On calculator: to find 6! ....

6, MATH, right arrow to PRB menu, down arrow to !, enter twice

try it

Finding terms of a sequence
Given a formula for nth term – just plug in n:
Example: Write the first 5 terms if $a_n = 5n-2$

$a_1 = 5(1)-2 = 3$
$a_2 = 5(2)-2 = 8$
$a_3 = 5(3)-2 = 13$
$a_4 = 5(4)-2 = 18$
$a_5 = 5(5)-2 = 23$

$\boxed{3, 8, 13, 18, 23 \dots}$

Given a rule for recursive sequence, write starting term(s), use rule to find next terms:
Example: Write the first 5 terms of recursive sequence: $a_1 = 5,\quad a_{k+1} = 3(a_k + 2)$

$\boxed{5, 21, 69, 213, 645 \dots}$

next term = 3(this term + 2)
$a_2 = 3(a_1 + 2)$
$a_3 = 3(a_2 + 2)$
$a_4 = 3(a_3 + 2)$

Finding a formula, given the sequence
Sometimes easy to see...can help writing a line of 'n' above matching terms:
Examples: Write an expression for the most apparent nth term of the sequence:

n: 1 2 3 4 5

$1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \ldots \frac{1}{n^2}$ $\boxed{a_n = \frac{1}{n^2}}$

$\frac{1}{1^2}\ 2^2\ 3^2\ 4^2\ 5^2 \ldots n^2$

n: 1 2 3 4

$\frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \frac{1}{120}, \ldots$ $a_n = \frac{1}{(n+1)!}$

2^n: 2 4
n!: 1 2 6 24
✓(n+1)!: 2 6 24 120

Sometimes difficult to see a pattern, so we look for matches in a table of patterns:
More examples:

n: 1 2 3 4

0, 3, 8, 15 ...

2n: 2 4 6
2^n: 2 4 8 16
n^2: 1 4 9 16
✓n^2-1: 0 3 8 15

$\boxed{a_n = n^2 - 1}$

n: 1 2 3 4 5

$1+\frac{1}{3}, 1+\frac{7}{9}, 1+\frac{25}{27}, 1+\frac{79}{81}, 1+\frac{241}{243}, \ldots$ $\boxed{a_n = 1 + \frac{3^n-2}{3^n}}$

Series = the sum of the terms in a sequence.

Sequence: 1, 3, 5, 7
Series: 1 + 3 + 5 + 7 = 16

Summation (Sigma) Notation

finite series $a_1 + a_2 + a_3 + a_4 + ... + a_i + ... + a_n = \sum_{i=1}^{n} a_i$

i = 'index of summation'

n = 'upper limit of summation'

infinite series $a_1 + a_2 + a_3 + a_4 + ... + a_i + ... = \sum_{i=1}^{\infty} a_i$

1 = 'lower limit of summation'
(does not have to be 1. sometimes 0 or other number)

Examples: Find $\sum_{i=1}^{5} 4i-3 = 1+5+9+13+17 = \boxed{45}$

$a_1 = 4(1)-3 = 1$
$a_2 = 4(2)-3 = 5$
$a_3 = 4(3)-3 = 9$
$a_4 = 4(4)-3 = 13$
$a_5 = 4(5)-3 = 17$

n: 1 2 3 4 ...

Use Sigma notation to write the sum: $\frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} .... + \frac{1}{128}$

① find rule for a_n:

2^n: 2 4 8
2^{n-1}: 1 2 4 8

$a_n = \frac{1}{2^{n-1}}$

② figure out what n is for last term:

$\frac{1}{2^{n-1}} = \frac{1}{128}$

$2^{n-1} = 128$

$2^{n-1} = 2^7$

$n-1 = 7$

$n = 8$

③ write in Σ form:

$\boxed{\sum_{n=1}^{8} \frac{1}{2^{n-1}}}$

$n = 1, \; 2, \; 3, \; 4, \; 5, \; 6, \ldots$

$n^2 = 1, \; 4, \; 9, \; 16, \; 25, \; 36, \ldots$

$(n+1)^2 = 4, \; 9, \; 16, \; 25, \; 36, \; 49, \ldots$

$(n-1)^2 = 0, \; 1, \; 4, \; 9, \; 16, \; 25, \ldots$

$n^3 = 1, \; 8, \; 27, \; 64, \; 125, \; 216, \ldots$

$(n+1)^3 = 8, \; 27, \; 64, \; 125, \; 216, \; 343, \ldots$

$2n = 2, \; 4, \; 6, \; 8, \; 10, \; 12, \ldots$

$2^n = 2, \; 4, \; 8, \; 16, \; 32, \; 64, \ldots$

$2^{n-1} = 1, \; 2, \; 4, \; 8, \; 16, \; 32, \ldots$

$2^{n+1} = 4, \; 8, \; 16, \; 32, \; 64, \; 128, \ldots$

$3n = 3, \; 6, \; 9, \; 12, \; 15, \; 18, \ldots$

$3^n = 3, \; 9, \; 27, \; 81, \; 243, \; 729, \ldots$

$3^{n-1} = 1, \; 3, \; 9, \; 27, \; 81, \; 243, \ldots$

$3^{n+1} = 9, \; 27, \; 81, \; 243, \; 729, \; 2187, \ldots$

$n! = 1, \; 2, \; 6, \; 24, \; 120, \; 720, \ldots$

$(n-1)! = 1, \; 1, \; 2, \; 6, \; 24, \; 120, \ldots$

$(n+1)! = 2, \; 6, \; 24, \; 120, \; 720, \; 5040, \ldots$

PreCalculus
~~HAlg3-4~~, **9.2 Notes – Arithmetic Sequences and Partial Sums**

Consider this sequence:

1st term → a_1

$$\underset{+3}{1, \; 4,} \; \underset{+3}{7,} \; \underset{+3}{10,} \; \underset{+3}{13,} \; 16, \ldots$$

+3 +3 +3 +3 +3 ← d = common difference

This is an **arithmetic sequence**. A sequence is arithmetic if the differences between consecutive terms is a constant, which is called the **common difference**.

Examples: Determine if the sequences are arithmetic and find the common differences.

#1. -12, -8, -4, 0, 4,....

+4 +4 +4 +4

$d=4$

#2. 9, 6, 3, 0, -3, ...

-3 -3 -3 -3

$d=-3$

#3. Find the first 5 terms and determine if the sequence is arithmetic: $a_n = (2^n)n$

$a_1 = (2^1)1 = 2$) +6

$a_2 = (2^2)2 = 8$) +16

$a_3 = (2^3)3 = 24$

$a_4 = (2^4)4 = 64$

$a_5 = (2^5)5 = 160$

<u>not</u> arithmetic

Formulas for nth term of arithmetic sequences: Three formulas...

1) $a_n = a_1 + (n-1)d$

← n-1 so $a_n = a_1$ when $n=1$

2) $a_n = dn + c, \quad where\ c = a_1 - d$ c is the 'zeroth' term (textbook's formula)

$a_n = a_1 + (n-1)d$

$= a_1 + dn - d$

$= dn + (a_1 - d)$ ← 'zeroth' term

3) $a_{n+1} = a_n + d$ (recursive formula)

Example: Find a formula for the nth term of the arithmetic sequence whose common difference is 3 and whose first term is 2.

$d = 3$

$a_1 = 2$

$a_n = a_1 + (n-1)d$

$\boxed{a_n = 2 + (n-1)3}$

or

$a_n = 2 + 3n - 3$

$\boxed{a_n = -1 + 3n}$

Example: The 4th term of an arithmetic sequence is 20, and the 13th term is 65. Find a formula for the nth term.

$a_1+(13-1)d =$ $\quad a_{13}=65$

$a_1+(4-1)d$ $\quad -\ a_4 = 20$

$9d =$ $\quad 45$ difference $=(13-4)d = 9d$

$\underbrace{a_4 \ a_5}_{d}\ \underbrace{a_6}_{d}\ \underbrace{a_7}_{d}\ \underbrace{a_8}_{d}\ \underbrace{a_9}_{d}\ \underbrace{a_{10}}_{d}\ \underbrace{a_{11}}_{d}\ \underbrace{a_{12}}_{d}\ \underbrace{a_{13}}_{d}$

$9d=45$ $\quad d=5$

$a_n = a_1+(n-1)d$

$20 = a_1+(4-1)5$

$20 = a_1 + \cancel{20} - 5$ $\quad a_1 = 5$

$\boxed{a_n = 5+(n-1)5}$

or

$a_n = 5+5n-5$

$\boxed{a_n = 5n}$

Example: Find the 7th term of the arithmetic sequence whose first two terms are 2 and 9.

$d = 9-2 = 7$ $\quad a_1 = 2$

$a_n = a_1+(n-1)d$

$a_n = 2+(n-1)7$

$a_7 = 2+(7-1)7$

$a_7 = 2+6\cdot 7$

$a_7 = 2+42 = \boxed{44}$

Example: The first two terms are given, find the missing term. $a_1 = 3,\quad a_2 = 9,\quad a_9 = ?$

$d = 9-3 = 6$

$a_n = 3+(n-1)6$

$a_9 = 3+(9-1)6$

$a_9 = 3+8\cdot 6 = \boxed{51}$

Practice: Find formulas for the arithmetic sequences:

#1. $a_1 = 15,\quad d = 4$

$a_n = a_1+(n-1)d$

$\boxed{a_n = 15+(n-1)4}$

or

$a_n = 15+4n-4$

$\boxed{a_n = 11+4n}$

#2. $\overset{a_1}{-6}, -2,\ 2,\ 6$

$+4\ +4\ +4 = d$

$\boxed{a_n = -6+(n-1)4}$

or

$a_n = -6+4n-4$

$\boxed{a_n = -10+4n}$

#3. $a_{\cancel{4}\,3} = \cancel{20}\ 94,\quad a_{\cancel{13}\,6} = \cancel{65}\ 85$

-9 for $(6-3$ terms$)$

$-9 = 3d$ $\quad d=-3$

$a_n = a_1+(n-1)(-3)$

$94 = a_1+(3-1)(-3)$

$94 = a_1 - 6$

$a_1 = 100$

$\boxed{a_n = 100-(n-1)3}$

or

$a_n = 100-3n+3 = \boxed{103-3n}$

Sum of a finite arithmetic series (partial sum of an infinite arithmetic series)

$S = 1 + 3 + 5 + 7 + 9 + 11 \quad = 36$

$S = 11+9+7+5+3+1$

$2S = 12+12+12+12+12+12$

$2S = 6(12)$

$S = \frac{1}{2}6(12) = \frac{1}{2}n(a_1+a_n)$

$$S_n = \frac{n}{2}(a_1+a_n)$$

$a_1 = -6$

Example: Find the partial sum: -6, -2, 2, 6, ... n=50

$+4 \quad +4 \quad +4 \qquad d = 4$

$a_n = a_1 + (n-1)d$

$a_n = -6 + (n-1)4$

$a_{50} = -6 + (50-1)4 = 190$

$S_{50} = \frac{n}{2}(a_1 + a_n)$

$S_{50} = \frac{50}{2}(-6 + 190) = \boxed{4600}$

Try these:

#4. What is the sum of integers from 1 to 100.

$a_1 = 1$

$a_{100} = 100$

$S_{100} = \frac{100}{2}(1 + 100) = \boxed{5050}$

$S_n = \frac{n}{2}(a_1 + a_n)$

#5. Find the sum: $\sum_{n=1}^{100} \frac{8-3n}{16}$

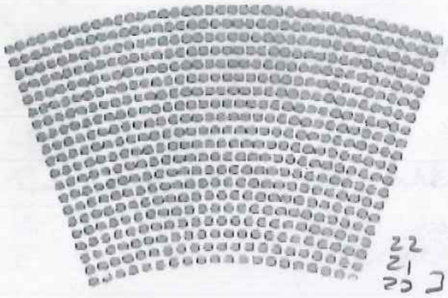
$a_1 = \frac{8-3}{16} = \frac{5}{16}$

$a_{100} = \frac{8-300}{16} = \frac{-292}{16}$

$S_{100} = \frac{100}{2}\left(\frac{5}{16} - \frac{292}{16}\right) = \boxed{\frac{100}{2} \cdot \frac{-287}{16}}$

$S_n = \frac{n}{2}(a_1 + a_n)$

Example: An auditorium has 40 rows of seats. There are 20 seats in the 1st row, 21 in the 2nd, 22 in the 3rd, etc. How many total seats are there?



$a_{40} = 59$ seats

22
21
20] $d = 1$

$a_1 = 20$

$a_n = a_1 + (n-1)d$

$a_n = 20 + (n-1)1$

$a_n = 20 + n - 1$

$a_n = 19 + n$

$a_{40} = 19 + 40 = 59$

$S = \frac{40}{2}(20 + 59) = \boxed{1580 \text{ seats}}$

Example: Consider a job offer with a starting salary of $36,800 and an annual raise of $1,750.

a) Determine the salary during the 6th year of employment.

b) Determine the total compensation from the company through 6 full years of employment.

$a_1 = 36800$

$d = 1750$

a) $a_n = a_1 + (n-1)d$

$a_n = 36800 + (n-1)1750$

$a_6 = 36800 + (6-1)1750 = \boxed{\$45{,}550}$

b) $S_6 = \frac{6}{2}(36800 + 45550) = \boxed{\$247{,}050}$

9.3 – *Geometric Sequences and Series*

Consider this sequence:

2, 4, 8, 16, 32, ...

$\times 2 \quad \times 2 \quad \times 2 \quad \times 2$ ← r=2 common ratio

$a_1, \quad a_1 r, \quad a_1 r^2, \quad a_1 r^3, \quad a_1 r^4, \ldots$

This is a **geometric sequence**. A sequence is geometric if the ratios of consecutive terms is a constant, which is called the **common ratio**. (r)

Examples: Determine if the sequences are geometric and find the common ratio.

#1. 12, 36, 108, 324, ...

$\times 3 \quad \times 3 \quad \times 3$

$\frac{36}{12} = 3$

$\frac{108}{36} = 3$

$\boxed{\text{yes, } r=3}$

#2. 1, 4, 9, 16, ...

$\times 4 \quad \times \frac{9}{4} \quad \times \frac{16}{9}$

$\boxed{\text{no}}$

Formula for nth term of geometric sequences: $\boxed{a_n = a_1 r^{n-1}}$

Example: Write the 1st 5 terms of the geometric sequence whose 1st term is 3 with common ratio of 2.

$a_1 = 3$

$r = 2$

$a_n = 3(2)^{n-1}$

$a_1 = 3(2)^{1-1} = 3(2)^0 = 3 \cdot 1 = 3$

$a_2 = 3(2)^{2-1} = 3(2)^1 = 3 \cdot 2 = 6$

$a_3 = 3(2)^{3-1} = 3(2)^2 = 3 \cdot 4 = 12$

$a_4 = 3(2)^{4-1} = 3(2)^3 = 3 \cdot 8 = 24$

$a_5 = 3(2)^{5-1} = 3(2)^4 = 3 \cdot 16 = 48$

$\boxed{3, 6, 12, 24, 48}$

Example: Find the 15th term of the geometric sequence whose 1st term is 20 and whose common ratio is 1.05.

$a_1 = 20$

$r = 1.05$

$a_n = 20(1.05)^{n-1}$

$a_{15} = 20(1.05)^{15-1}$

$= 20(1.05)^{14}$

$= \boxed{39.5986\ldots}$

Example: $a_4 = 125, \quad a_{10} = \dfrac{125}{64}$ Find the 14th term (assume terms of sequence are positive)

$a_{10} = a_1 (r)^{10-1}$

$a_1 (r)^9 = \frac{125}{64}$

$a_4 = a_1 (r)^{4-1}$

$a_1 (r)^3 = 125$

$\frac{a_{10}}{a_4} = \frac{a_1 r^9}{a_1 r^3} = \frac{(125/64)}{(125/1)}$

$= r^6 = \frac{125}{64} \cdot \frac{1}{125} = \frac{1}{64}$

$r^6 = \frac{1}{64}$

$r = \sqrt[6]{\frac{1}{64}} = \left(\frac{1}{64}\right)^{1/6} = \frac{1}{2}$

$r = \frac{1}{2}$ plug in →

$a_4 = 125$

$a_1 \left(\frac{1}{2}\right)^{4-1} = 125$

$a_1 = \frac{125}{(1/2)^3} = 125 \cdot 8 = 1000$ so

$a_n = a_1 (r)^{n-1}$

$a_n = 1000\left(\frac{1}{2}\right)^{n-1}$

and $a_{14} = 1000\left(\frac{1}{2}\right)^{14-1}$

$= \boxed{0.12207}$

$= \frac{125}{1024}$

Sum of a finite geometric sequence $$S_n = a_1\left(\frac{1-r^n}{1-r}\right)$$

Example: Find the sum: $\sum_{n=1}^{12} 4(0.3)^n = 1.2 + .36 + \ldots$

$$S_{12} = 1.2\left(\frac{1+(0.3)^{12}}{1-(0.3)}\right)$$

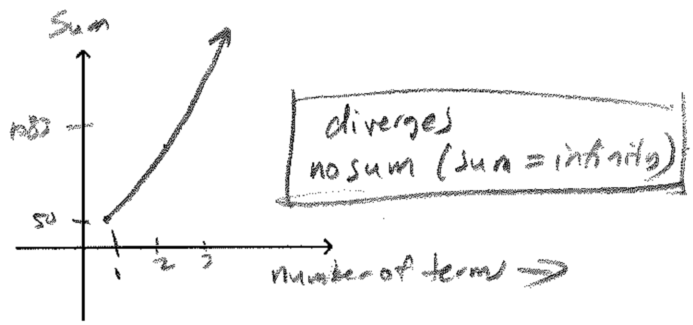
$$= 1.2(1.428577\ldots)$$

$$= \boxed{1.714\ldots}$$

Sum of an infinite geometric sequence

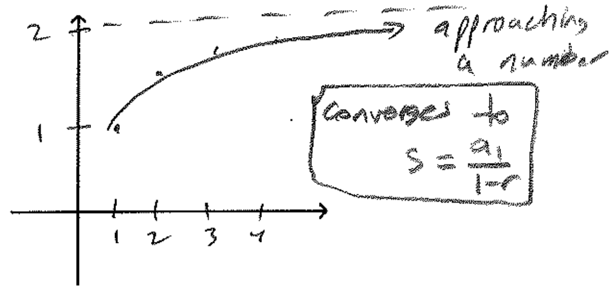
$$\sum_{n=1}^{\infty} 50(1.4)^{n-1} = 50 + 1033 + 29881 + \ldots$$

$|r| \geq 1$



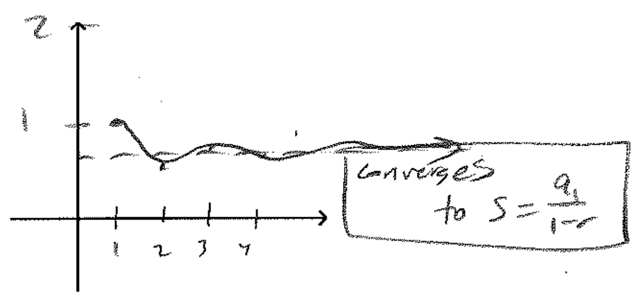
$$\sum_{n=1}^{\infty} 1\left(\frac{1}{2}\right)^{n-1} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$$

$|r| < 1$



$$\sum_{n=1}^{\infty} 1\left(-\frac{1}{2}\right)^{n-1} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \ldots$$

$|r| < 1$



For $a_n = a_1 r^{n-1}$, **converges if** $|r| < 1$ **If geometric series converges, it converges to:**

diverges if $|r| \geq 1$ $$S = \frac{a_1}{1-r}.$$

Examples: Find the sum: $\sum_{n=1}^{\infty} 40(0.6)^{n-1} = 40(0.6)^0 = 40$ a_1

$r = 0.6$

$|r| < 1$

converges to $S = \frac{a_1}{1-r}$

$$S = \frac{40}{1-0.6} = \frac{40}{0.4} = \boxed{100}$$

Find the sum: $\sum_{i=1}^{10} 5\left(-\frac{1}{3}\right)^{i-1} = \underset{a_1}{5} + \cdots$ finite

$S = a_1\left(\frac{1+r^n}{1-r}\right)$

$S = 5\left(\frac{1+(-\frac{1}{3})^{10}}{1-(-\frac{1}{3})}\right)$

$\boxed{S = 3.75}$

Find the sum: $\cancel{\sum_{k=0}^{10} 10\left(\frac{1}{2}\right)^k}$ $\sum_{n=0}^{\infty} 4\left(-\frac{1}{2}\right)^n = \underset{a_0}{4} + \cdots$ infinite

"a_1" "first term" $|r|<1$ converges

$S = \frac{a_1}{1-r}$

$= \frac{4}{1-(-\frac{1}{2})} = \frac{4}{1+\frac{1}{2}} = \frac{4}{\frac{3}{2}} = 4 \cdot \frac{2}{3} = \boxed{\frac{8}{3}}$

Use summation notation to express the sum: 7+14+28+....+896

$\times 2 \quad \times 2 \qquad n=8$

$a_1 = 7 \quad r = 2$

$a_n = a_1 (r)^{n-1}$

last term $a_n = 896 = 7(2)^{n-1}$

$128 = 2^{n-1}$

$2^7 = 2^{n-1}$

$7 = n-1$

$n = 8$

$\boxed{\sum_{n=1}^{8} 7(2)^{n-1}}$

Find the sum of the infinite geometric series: $\frac{2}{1} - \frac{4}{3} + \frac{8}{9} - \frac{16}{27} + \ldots$

$\times -\frac{2}{3} \quad \times -\frac{2}{3} \qquad r = -\frac{2}{3} \quad a_1 = 2$

$|r|<1$

converges so

$S = \frac{a_1}{1-r} = \frac{2}{1-(-\frac{2}{3})} = \frac{2}{1+\frac{2}{3}} = \frac{2}{\frac{5}{3}} = 2 \cdot \frac{3}{5} = \boxed{\frac{6}{5}}$

#91 (homework) A company buys a machine for $155,000 and it depreciates at a rate of 30% per year (at the end of each year, the value is 70% of what it was at the start of the year). Find the depreciated value of the machine after 5 full years.

$r = 0.7 \quad a_1 = 155000$

$a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5 \quad a_6$

1 yr 2 yrs 3 4 5 yrs

$a_n = 155000(0.7)^{n-1}$

$a_6 = 155000(0.7)^{6-1}$

$= \boxed{\$26050.85}$