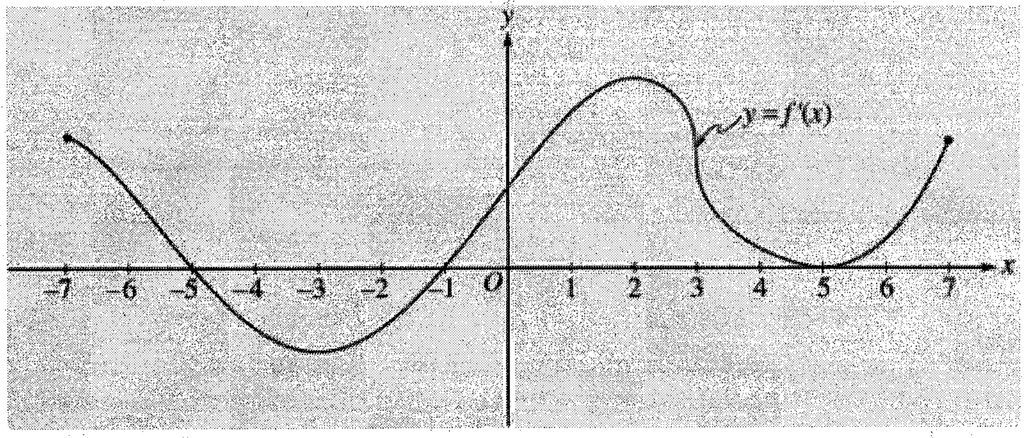
#1



The figure above shows the graph of f', the derivative of the function f, for $-7 \le x \le 7$. The graph of f' has horizontal tangent lines at $x = -3$, $x = 2$, and $x = 5$, and a vertical tangent line at $x = 3$.

Find all values of x, for $-7 < x < 7$, at which f attains a relative minimum. Justify your answer.

f relative minimum occurs at critical points where $f'(x)$ is 0 or DNE if there is a sign change (\ /) in $f'(x)$ from negative to positive.

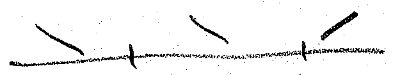
This occurs at $x = -1$.

Let h be a function defined for all $x \neq 0$ such that $h(4) = -3$ and the derivative of h is given by

$$h'(x) = \frac{x^2-2}{x} \text{ for all } x \neq 0.$$

#2 Find all values of x for which the graph of h has a horizontal tangent, and determine whether h has a local maximum, a local minimum, or neither at each of these values. Justify your answers.

$h(x)$ has horiz tangent when $h'(x)=0$, $x^2-2=0$, $x^2=2$, $\boxed{x=2,\ x=-2}$



-2 $\quad$ 2

$h'(-3)$ $\frac{(+)}{(-)}$ $\quad$ $h'(1)$ $\frac{(-)}{(+)}$ $\quad$ $h'(3)$ $\frac{(+)}{(+)}$

- $h(x)$ has a local minimum at $x=2$ because $h'(x)$ sign changes from negative to positive
- $h(x)$ has neither local max or min at $x=-2$ because sign of $h'(x)$ does not change.

Let h be a function defined for all $x \neq 0$ such that $h(4) = -3$ and the derivative of h is given by

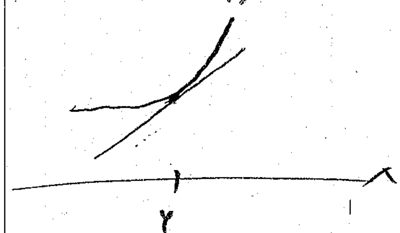
$$h'(x) = \frac{x^2-2}{x} \text{ for all } x \neq 0.$$

#3 Does the line tangent to the graph of h at $x = 4$ lie above or below the graph of h for $x > 4$? Why?

$$h'(4) = \frac{(4)^2-2}{4} = \frac{14}{4} > 0$$

$$h''(x) = \frac{x(2x)-(x^2-2)(1)}{(x)^2} = \frac{2x^2-x^2+2}{x^2} = \frac{x^2+2}{x^2}$$

$$h''(4) = \frac{(4)^2+2}{(4)^2} = \frac{18}{16} > 0$$



Since the slope of the tangent line at $x=4$ is positive and the concavity of $h(x)$ is concave up at $x=4$, the tangent line lies below the graph of h for $x>4$.

#4 Let f be the function defined by $f(x)=\sin^2 x-\sin x$ for $0 \le x \le 3\pi/2$

Find the absolute maximum value and the absolute minimum value of f. Justify your answer.

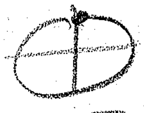
$f(x)$ abs. max and min will occur where $f'(x)=0$ or DNE or at interval ends. (no calculator!)

$$f'(x) = 2(\sin x)\cos x - \cos x = 0$$

$$\cos x\,[2\sin x - 1] = 0$$

$\cos x = 0$ $\sin x - 1 = 0$

$x = \frac{\pi}{2},\ x = \frac{3\pi}{2}$ $\sin x = 1$



$x = \frac{\pi}{2}$

critical pts at $x = \frac{\pi}{2},\ x = \frac{3\pi}{2}$

"candidates test!"

	x	$f(x)$
critical pt	$\frac{\pi}{2}$	$f(\frac{\pi}{2}) = (\sin(\frac{\pi}{2}))^2 - \sin(\frac{\pi}{2}) = (1)^2 - 1 = 0$
	$\frac{3\pi}{2}$	$f(\frac{3\pi}{2}) = (\sin(\frac{3\pi}{2}))^2 - \sin(\frac{3\pi}{2}) = (-1)^2 - (-1) = 2$
add interval ends	0	$f(0) = (\sin(0))^2 - \sin(0) = 0 - 0 = 0$

$\boxed{\begin{array}{l} f \text{ absolute max is } 2 \text{ which occurs at } x = \frac{3\pi}{2} \\ f \text{ absolute min is } 0 \text{ which occurs at } x = \frac{\pi}{2} \text{ and } x = 0 \end{array}}$

#5 Let f be the function defined by $f(x) = e^x \cos(x)$. (no calculator)

Find the average rate of change of f on the interval $0 \leq x \leq \pi$.

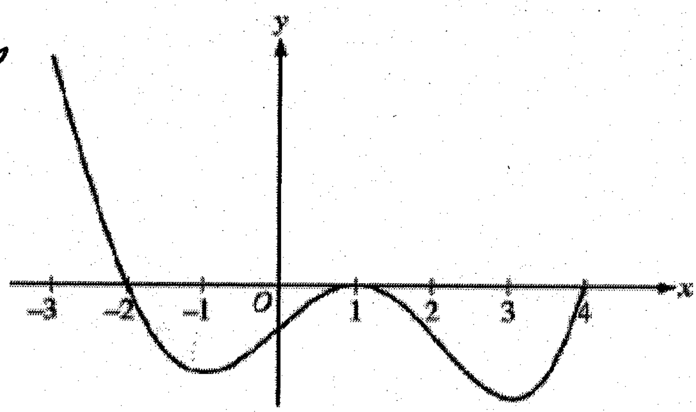
$$\text{Avg rate of change of } f = \frac{f(\pi) - f(0)}{\pi - 0}$$

$$= \frac{e^{\pi}\cos(\pi) - e^{0}\cos(0)}{\pi - 0}$$

$$= \frac{e^{\pi}(-1) - (1)(1)}{\pi}$$

$$= \boxed{\frac{-e^{\pi} - 1}{\pi}}$$

#6



Graph of f'

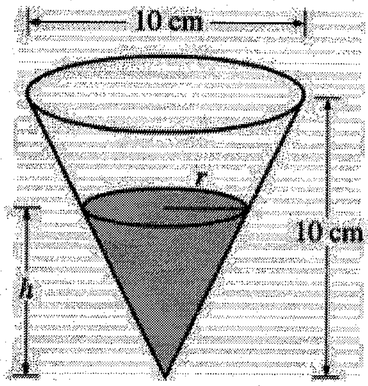
The figure above shows the graph of f', the derivative of a twice-differentiable function f, on the interval $[-3, 4]$. The graph of f' has horizontal tangents at $x = -1$, $x = 1$, and $x = 3$. The areas of the regions bounded by the x-axis and the graph of f' on the intervals $[-2, 1]$ and $[1, 4]$ are 9 and 12, respectively.

On what open intervals contained in $-3 < x < 4$ is the graph of f both concave down and decreasing? Give a reason for your answer.

- f is concave down where $f''(x) < 0$ which is where $f'(x)$ is decreasing, which occurs for $-3 < x < -1$ and $1 < x < 1$
- f is decreasing where $f'(x) < 0$ which occurs for $-2 < x < 1$, $1 < x < 4$.

$\boxed{f \text{ is both concave down and decreasing for } -2 < x < -1 \text{ and } 1 < x < 3}$

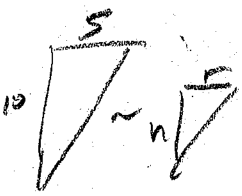
#7



A container has the shape of an open right circular cone, as shown in the figure above. The height of the container is 10 cm and the diameter of the opening is 10 cm. Water in the container is evaporating so that its depth h is changing at the constant rate of $\frac{-3}{10}$ cm/hr.

(Note: The volume of a cone of height h and radius r is given by $V = \frac{1}{3}\pi r^2 h$.)

(a) Find the volume V of water in the container when h = 5 cm. Indicate units of measure.



$\frac{10}{5} = \frac{h}{r}$ so $V = \frac{1}{3}\pi r^2 h$

$2 = \frac{h}{r}$ $\quad V(r) = \frac{1}{3}\pi\left(\frac{1}{2}h\right)^2 h$

$r = \frac{1}{2}h$ $\quad V(h) = \frac{\pi}{12}h^3$

$\boxed{V(5) = \frac{\pi}{12}(5)^3 \text{ cm}^3}$

(b) Find the rate of change of the volume of water in the container, with respect to time, when h = 5 cm. Indicate units of measure.

$\frac{d}{dt}[V] = \frac{d}{dt}\left[\frac{\pi}{12}h^3\right]$

$\frac{dV}{dt} = \frac{\pi}{12} 3h^2 \frac{dh}{dt}$

$\boxed{\left.\frac{dV}{dt}\right|_{h=5} = \frac{\pi}{12}3(5)^2\left(-\frac{3}{10}\right) \frac{\text{cm}^3}{\text{hr}}}$

(c) Show that the rate of change of the volume of water in the container due to evaporation is directly proportional to the exposed surface area of the water. What is the constant of proportionality?

$\frac{dV}{dt} = \frac{\pi}{12}3h^2\frac{dh}{dt}$ (from part b) $\qquad A = \pi r^2$ and $r = \frac{1}{2}h$ (from part a)

$\frac{dV}{dt} = \frac{\pi}{4}h^2\left(-\frac{3}{10}\right)$ $\qquad A = \pi\left(\frac{1}{2}h\right)^2$

$\frac{dV}{dt} = \frac{-3\pi}{40}h^2$ $\qquad A = \frac{\pi}{4}h^2$

$\frac{dV}{dt} \stackrel{?}{=} kA$

$-\frac{3\pi}{40}h^2 = k\frac{\pi}{4}h^2$

$-\frac{3\pi}{40} = k\frac{\pi}{4}$

$-\frac{3}{40} = k\frac{1}{4}$, $\boxed{K = \frac{-12}{40} = \frac{-3}{10}}$

#8

Traffic flow is defined as the rate at which cars pass through an intersection, measured in cars per minute. The traffic flow at a particular intersection is modeled by the function F defined by

$F(t) = 82 + 4\sin\left(\frac{t}{2}\right)$ for $0 \leq t \leq 30$, where $F(t)$ is measured in cars per minute and t is measured in minutes.

Is the traffic flow increasing or decreasing at $t = 7$? Give a reason for your answer.

$F'(7) = -1.873$ $\frac{\text{cars per min}}{\text{min}}$ (use math 8 on calculator)

$\boxed{F'(7) < 0, \text{ therefore traffic flow is decreasing at } t=7}$

#9

A particle moves along the y-axis so that its velocity at any time $t \geq 0$ is given by $v(t)=t \cos t$. At time $t=0$, the position of the particle is $y=3$.

For what values of t, $0 \leq t \leq 5$, is the particle moving upward?

the particle is moving upward when $v(t) > 0$

which occurs for $0 < t < 1.571$

and $4.712 < t \leq 5$.

#10 A particle moves along the x-axis so that its velocity at time $t, 0 \le t \le 5$, is given by $v(t) = 3(t-1)(t-3)$. At time $t=2$, the position of the particle is $x(2)=0$.

Find the minimum acceleration of the particle

$a(t) = 3(t-1)(1) + (t-3)(3)$

$a(t) = 3t - 3 + 3t - 9$

$a(t) = 6t - 12$

Over the interval $0 \le t \le 5$, $a(t)$ min may occur at critical points or interval ends,

$a'(t) = 6 \ne 0$ or DNE, so no critical pts

Since $a'(t) > 6$, $a(t)$ is increasing, therefore

minimum acceleration occurs at left end of interval $(t=0)$

$\boxed{\text{min accel} = a(0) = 6(0) - 12 = -12}$

#11

t (seconds)	0	10	40	60
$B(t)$ (meters)	100	136	9	49
$v(t)$ (meters per second)	2.0	2.3	2.5	4.6

Ben rides a unicycle back and forth along a straight east-west track. The twice-differentiable function B models Ben's position on the track, measured in meters from the western end of the track, at time t, measured in seconds from the start of the ride. The table above gives values for $B(t)$ and Ben's velocity, $v(t)$, measured in meters per second, at selected times t.

For $40 \le t \le 60$, must there be a time t when Ben's velocity is 2 meters per second? Justify your answer.

Yes. since $B(t)$ is twice-differentiable it is continuous and differentiable over $40 \le t \le 60$, $B(40) = 9$ meters and $B(60) = 49$ meters

so the average velocity $= \frac{B(60)-B(40)}{60-40} = \frac{49-9}{20} = \frac{40}{20} = 2 \frac{\text{meters}}{\text{second}}$.

The mean-value theorem guarantees at time c, $40 \le c \le 60$,

Such that $B'(c) = 2 \frac{\text{meters}}{\text{second}}$.

A water tank at Camp Newton holds 1200 gallons of water at time t = 0. During the time interval $0 \le t \le 18$ hours, water is pumped into the tank at the rate

$$W(t) = 95\sqrt{t}\sin^2\left(\frac{t}{6}\right) \text{ gallons per hour.}$$

During the same time interval, water is removed from the tank at the rate

$$R(t) = 275\sin^2\left(\frac{t}{3}\right) \text{ gallons per hour.}$$

#12 Is the amount of water in the tank increasing at time t=15 ? Why or why not?

amount of water, $A(t) = W(t) - R(t) + 1200$

so $A'(t) = W'(t) - R'(t)$

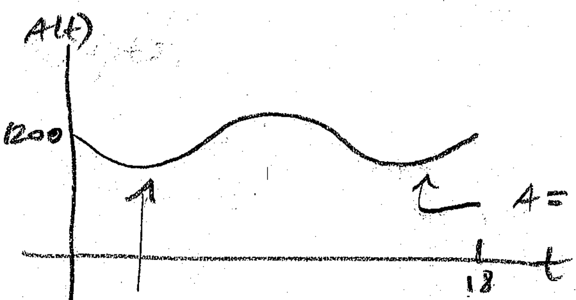
and $A'(15) = W'(15) - R'(15)$

$A'(15) = -54,41663609 - (-49,86859813) = -4,542 \frac{\text{gallons}}{\text{hr}}$

$\boxed{\text{The amount of water is decreasing at } t=15 \text{ because } A'(15) < 0}$

#13 At what time t, $0 \le t \le 18$, is the amount of water in the tank at an absolute minimum? Show the work that leads to your conclusion.

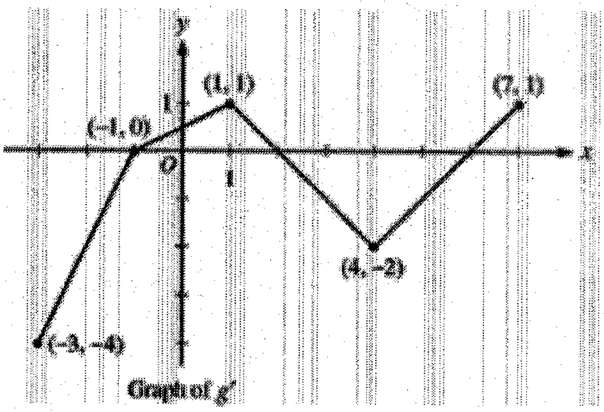
Graphing $A(t) = W(t) - R(t) + 1200$ over $0 \le t \le 18$:



A = 1078,714 gallons, at t = 15,08

A = 1012,837 gallons
at t = 4,032 hrs

$\boxed{\text{Absolute min water amount occurs at } t = 4,032 \text{ hrs}}$



Let g be a continuous function with $g(2) = 5$. The graph of the piecewise-linear function g', the derivative of g, is shown above for $-3 \le x \le 7$.

#4 Find the average rate of change of $g'(x)$ on the interval $-3 \le x \le 7$. Does the Mean Value Theorem applied on the interval $-3 \le x \le 7$ guarantee a value of c, for $-3 < c < 7$, such that $g''(c)$ is equal to this average rate of change? Why or why not?

avg rate of change of $g'(x)$ = $\dfrac{g'(7) - g'(-3)}{7-(-3)} = \boxed{\dfrac{1-(-4)}{7-(-3)}}$

No, the Mean Value Theorem does not guarantee c, $-3 < c < 7$, such that $g'(c)$ = avg rate of change because $g'(x)$ is not differentiable at $x = -1$, $x = 1$, and $x = 4$.

t (minutes)	0	2	5	7	11	12
$r'(t)$ (feet per minute)	5.7	4.0	2.0	1.2	0.6	0.5

The volume of a spherical hot air balloon expands as the air inside the balloon is heated. The radius of the balloon, in feet, is modeled by a twice-differentiable function r of time t, where t is measured in minutes. For $0 < t < 12$, the graph of r is concave down. The table above gives selected values of the rate of change, $r'(t)$, of radius of the balloon over the time interval $0 \leq t \leq 12$. The radius of the balloon is 30 feet when $t = 5$.

(Note: The volume of a sphere of radius r is given by $V = \frac{4}{3}\pi r^3$.)

#15 Estimate the radius of the balloon when $t = 5.4$ using the tangent line approximation at $t = 5$. Is your estimate greater than or less than the true value? Give a reason for your answer.

$r(5) = 30 \text{ ft}$

$r'(5) = 2 \text{ ft/min}$

$(r - 30) = 2(t - 5)$

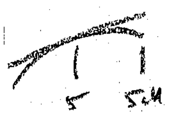
$r - 30 = 2t - 10$

$r = 2t + 20$

$r(5.4) \approx 2(5.4) + 20 = 10.8 + 20$

$\boxed{r(5.4) \approx 30.8 \text{ ft}}$

Since r is concave down for $0 < t < 12$ it is concave down near $t = 5$;



∴ the tangent line approximation is greater than the true value of the radius at $t = 5.4$.

#16 Find the rate of change of the volume of the balloon with respect to time when $t = 5$. Indicate units of measure.

$V = \frac{4\pi}{3} r^3$

$\frac{d}{dt}[V] = \frac{d}{dt}\left[\frac{4\pi}{3} r^3\right]$

$\frac{dV}{dt} = \frac{4\pi}{3}(3r^2)\frac{dr}{dt}$

$\boxed{\frac{dV}{dt} = 4\pi(30)^2(2) \ \frac{\text{ft}^3}{\text{min}}}$