AP Calculus BC – Unit 2 Extra Practice

2.1 – Extra Practice

Use the limit definition of the derivative to find the derivative:

#8b. $f(x) = -x^2 + 3x + 7$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{[-(x+h)^2 + 3(x+h) + 7] - [-x^2 + 3x + 7]}{h}$$

$$= \lim_{h\to 0} \frac{-x^2 - 2xh - h^2 + 3x + 3h + 7 + x^2 - 3x - 7}{h}$$

$$= \lim_{h\to 0} \frac{-2xh - h^2 + 3h}{h}$$

$$= \lim_{h\to 0} \frac{h(-2x - h + 3)}{h(1)}$$

$$= \lim_{h\to 0} (-2x - h + 3)$$

$$= -2x - (0) + 3 \quad \boxed{f'(x) = -2x + 3}$$

#9b. $r(t) = -t^2 - 4t$

$$r'(t) = \lim_{h\to 0} \frac{r(t+h) - r(t)}{h}$$

$$= \lim_{h\to 0} \frac{[-(t+h)^2 - 4(t+h)] - [-t^2 - 4t]}{h}$$

$$= \lim_{h\to 0} \frac{-t^2 - 2th - h^2 - 4t - 4h + t^2 + 4t}{h}$$

$$= \lim_{h\to 0} \frac{-2th - h^2 - 4h}{h}$$

$$= \lim_{h\to 0} \frac{h(-2t - h - 4)}{h(1)}$$

$$= \lim_{h\to 0} (-2t - h - 4)$$

$$= -2t - (0) - 4$$

$$\boxed{r'(t) = -2t - 4}$$

Use the limit definition of the derivative to find the derivative of f(x), then find f'(x) at the given x-value and write the equation of a tangent line to the function at this point of tangency:

#10b. $f(x) = x^2 + 5x - 6$ at $x = 3$

$$f'(x) = \lim_{h\to 0} \frac{[(x+h)^2 + 5(x+h) - 6] - [x^2 + 5x - 6]}{h}$$

$$= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 + 5x + 5h - 6 - x^2 - 5x + 6}{h}$$

$$= \lim_{h\to 0} \frac{2xh + h^2 + 5h}{h}$$

$$= \lim_{h\to 0} \frac{h(2x + h + 5)}{h}$$

$$= \lim_{h\to 0} (2x + h + 5) = 2x + (0) + 5$$

$f'(x) = 2x + 5$

$f'(3) = 2(3) + 5 = 11$

$f(3) = (3)^2 + 5(3) - 6 = 18$

tangent line:

$$\boxed{(y - 18) = 11(x - 3)}$$

#11b. $f(x) = \sqrt{x-1}$ at $x = 5$

use the conjugate

$$f'(x) = \lim_{h\to 0} \frac{\sqrt{(x+h)-1} - \sqrt{x-1}}{h}$$

$$= \lim_{h\to 0} \frac{(\sqrt{x+h-1} - \sqrt{x-1})}{h} \cdot \frac{(\sqrt{x+h-1} + \sqrt{x-1})}{(\sqrt{x+h-1} + \sqrt{x-1})}$$

$$= \lim_{h\to 0} \frac{(x+h-1) - (x-1)}{h(\sqrt{x+h-1} + \sqrt{x-1})}$$

$$= \lim_{h\to 0} \frac{h(1)}{h(\sqrt{x+h-1} + \sqrt{x-1})}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{x+h-1} + \sqrt{x-1}} = \frac{1}{\sqrt{x+0-1} + \sqrt{x-1}}$$

$$f'(x) = \frac{1}{2\sqrt{x-1}}$$

$$f'(5) = \frac{1}{2\sqrt{5-1}} = \frac{1}{2(2)} = \frac{1}{4}$$

$f(5) = \sqrt{5-1} = 2$

tangent line: $\boxed{(y - 2) = \frac{1}{4}(x - 5)}$

Use the limit definition of the derivative to find the derivative of f(x), then find f'(x) at the given x-value and write the equation of a tangent line to the function at this point of tangency:

#12b. $f(x)=\dfrac{6}{x+2}$ at $x=0$

$$f'(x)=\lim_{h\to 0}\frac{\frac{6}{x+h+2}-\frac{6}{x+2}}{h}$$

$$=\lim_{h\to 0}\frac{\left(\frac{6}{x+h+2}-\frac{6}{x+2}\right)}{h}\frac{(x+2)(x+h+2)}{(x+2)(x+h+2)}$$

$$=\lim_{h\to 0}\frac{6(x+2)-6(x+h+2)}{h(x+2)(x+h+2)}$$

$$=\lim_{h\to 0}\frac{\cancel{6x}+\cancel{12}-\cancel{6x}-6h-\cancel{12}}{h(x+2)(x+h+2)}$$

$$=\lim_{h\to 0}\frac{-6}{(x+2)(x+h+2)}=\frac{-6}{(x+2)(x+0+2)}$$

$$f'(x)=\frac{-6}{(x+2)^2}$$

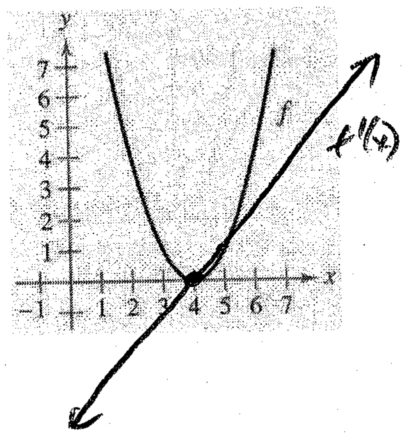
$$f'(0)=\frac{-6}{(0+2)^2}=\frac{-6}{4}=\frac{-3}{2}$$

$$f(0)=\frac{6}{0+2}=3$$

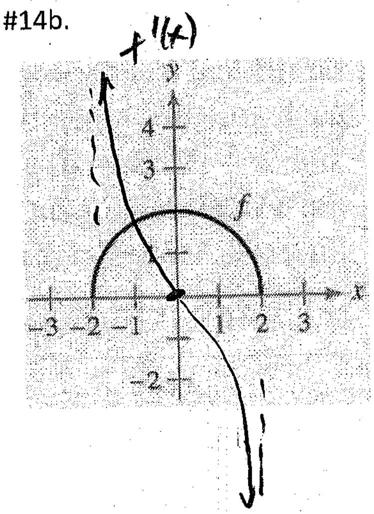
tangent line: $\boxed{(y-3)=\frac{-3}{2}(x-0)}$

Given the graph of f(x), sketch the graph of f'(x) on the same axes:

#13b.



#14b.



#15b. The given limit gives the value of the derivative of a function at a particular x-value. Identify the function and x-value:

$$\lim_{h\to 0}\frac{\tan 3(x+h)-\tan(3x)}{h}=f'(x) \text{ for } f(x)=\tan(3x)$$

at any x-value (not specified)

2.2 – Extra Practice

Find the derivative:

#8b. $g(t)=t^2-\frac{4}{t^3}$ $=t^2-4t^{-3}$

$\boxed{g'(t)=2t+12t^{-4}=2t+\frac{12}{t^4}}$

#9b. $f(x)=\frac{2x^4+4}{x^3}$ $=2x+4x^{-3}$

$\boxed{f'(x)=2-12x^{-4}=2-\frac{12}{x^4}}$

#10b. $g(x)=x^{\frac{1}{4}}-x^{\frac{1}{2}}+3x$

$\boxed{g'(x)=\frac{1}{4}x^{-3/4}-\frac{1}{2}x^{-1/2}+3}$

#11b. $f(x)=3\sqrt{x}+\sqrt[4]{x}+5\sqrt[3]{x^2}$ $=3x^{1/2}+x^{1/4}+5x^{2/3}$

$$\boxed{\begin{aligned}f'(x)&=\frac{3}{2}x^{-1/2}+\frac{1}{4}x^{-3/4}+\frac{10}{3}x^{-1/3}\\&=\frac{3}{2\sqrt{x}}+\frac{1}{4\sqrt[4]{x^3}}+\frac{10}{3\sqrt[3]{x}}\end{aligned}}$$

#12b. $g(x)=3\cos x+5e^x$

$\boxed{g'(x)=3(-\sin x)+5e^x}$

#13b. $f(x)=7^x+4\sin x$

$\boxed{f'(x)=7^x(\ln 7)+4\cos x}$

#14b. $r(t)=\frac{5t^3-t^6}{t^4}+\frac{6t}{t^3}-\frac{7t^3}{t^3}$ $=5t^{-1}-t^2+6t^{-2}-7$

$$\boxed{\begin{aligned}r'(t)&=-5t^{-2}-2t-12t^{-3}\\&=\frac{-5}{t^2}-2t-\frac{12}{t^3}\end{aligned}}$$

#15b. $r(t)=\frac{2t^5}{t^3}+\frac{t^2-6t}{t^4}$ $=2t^2+t^{-2}-6t^{-3}$

$$\boxed{\begin{aligned}r'(t)&=4t-2t^{-3}+18t^{-4}\\&=4t-\frac{2}{t^3}+\frac{18}{t^4}\end{aligned}}$$

2.3 – Extra Practice

Find the derivative:

#10b. $g(x)=(x^3+2x)^3(4x-x^3)^5$

$$\boxed{g'(x)=(x^3+2x)^3\left[5(4x-x^3)^4(4-3x^2)\right]+(4x-x^3)^5\left[3(x^3+2x)^2(3x^2+2)\right]}$$

#11b. $h(x)=\dfrac{x^2}{2\sqrt{x}+1}=\dfrac{x^2}{2x^{1/2}+1}$

$$\boxed{h'(x)=\frac{(2\sqrt{x}+1)(2x)-x^2(x^{-1/2})}{(2\sqrt{x}+1)^2}}$$

#12b. Find $f'(4)$ for $f(x)=\dfrac{3x^3-5x}{x^2-2}$

$$f'(x)=\frac{(x^2-2)(9x^2-5)-(3x^3-5x)(2x)}{(x^2-2)^2}$$

$$\boxed{f'(4)=\frac{(4^2-2)(9(4)^2-5)-(3(4)^3-5(4))2(4)}{(4^2-2)^2}=\frac{570}{196}=\frac{285}{98}}$$

#13b. Find $f'(\pi)$ for $f(x)=e^x\cos x$

$$f'(x)=e^x(-\sin x)+\cos x\, e^x$$

$$\boxed{\begin{aligned}f'(\pi)&=e^\pi(-\sin\pi)+\cos\pi\, e^\pi\\&=e^\pi(-1)(0)+(-1)e^\pi\\&=-e^\pi\end{aligned}}$$

#13b. Find (and fully simplify) $f'(x)$ for $f(x)=\dfrac{x^2+5x+6}{x^2-4}$

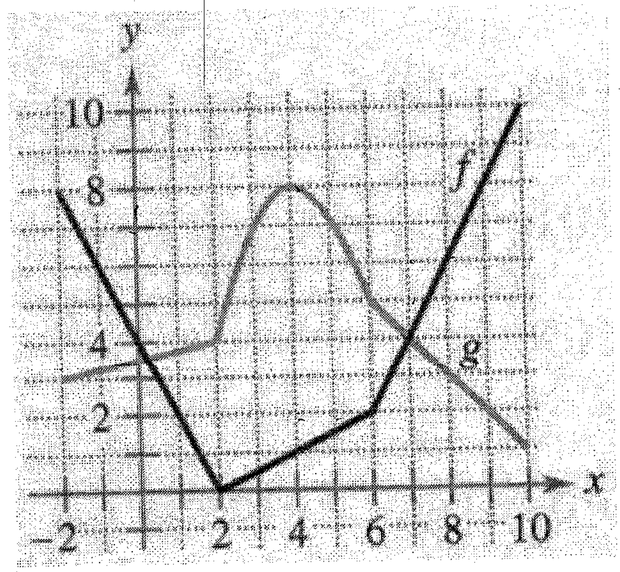
$$f'(x)=\frac{(x^2-4)(2x+5)-(x^2+5x+6)(2x)}{(x^2-4)^2}$$

$$=\frac{2x^3+5x^2-8x-20-2x^3-10x^2-12x}{[(x-2)(x+2)]^2}$$

$$=\frac{-5x^2-20x-20}{(x-2)^2(x+2)^2}$$

$$f'(x)=\frac{-5(x^2+4x+4)}{(x-2)^2(x+2)^2}=\frac{-5(x+2)(x+2)}{(x-2)^2(x+2)^2}=\boxed{\frac{-5}{(x-2)^2}}$$

Let $p(x)=f(x)g(x)$ and $q(x)=\dfrac{f(x)}{g(x)}$ If the graphs of $f(x)$ and $g(x)$ are:



#14. Find $p'(4)$

$$p'(x)=f(x)g'(x)+g(x)f'(x)$$

$$p'(4)=f(4)g'(4)+g(4)f'(4)$$

$$\boxed{p'(4)=(1)(0)+(8)\left(\tfrac{1}{2}\right)=4}$$

#15. Find $q'(7)$

$$q'(x)=\frac{g(x)f'(x)-f(x)g'(x)}{[g(x)]^2}$$

$$q'(7)=\frac{g(7)f'(7)-f(7)g'(7)}{[g(7)]^2}$$

$$\boxed{q'(7)=\frac{(4)(2)-(4)(-1)}{[4]^2}=\frac{8+5}{16}=\frac{13}{16}}$$

2.4 – Extra Practice

Find the derivative:

#9b. $y = 5\left(2-x^3\right)^4$

$$\boxed{\begin{aligned} y' &= 5\left(4(2-x^3)^3(-3x^2)\right) \\ &= -60(2-x^3)^3x^2 \end{aligned}}$$

#10b. $f(t) = \sqrt{5-t} = (5-t)^{1/2}$

$$\boxed{\begin{aligned} f'(t) &= \tfrac{1}{2}(5-t)^{-1/2}(-1) \\ &= \frac{-1}{2\sqrt{5-t}} \end{aligned}}$$

#11b. $f(t) = \left(\dfrac{1}{t-3}\right)^2 = (t-3)^{-2}$

$$\boxed{\begin{aligned} f'(t) &= -2(t-3)^{-3}(1) \\ &= \frac{-2}{(t-3)^3} \end{aligned}}$$

#12b. $h(t) = \left(\dfrac{t^2}{t^3+2}\right)^2$

$$\boxed{h'(t) = 2\left(\frac{t^2}{t^3+2}\right)^1\left[\frac{(t^3+2)(2t)-t^2(3t^2)}{(t^3+2)^2}\right]}$$

#13b. $y = \dfrac{e^x - e^{-x}}{2} = \frac{1}{2}e^x - \frac{1}{2}e^{-x}$

$$\boxed{\begin{aligned} y' &= \tfrac{1}{2}e^x - \tfrac{1}{2}e^{-x}(-1) \\ &= \tfrac{1}{2}e^x + \tfrac{1}{2}e^{-x} \end{aligned}}$$

#14b. $f(x) = \ln\left(\dfrac{2x}{x+3}\right)$

$$\boxed{\begin{aligned} f'(x) &= \frac{1}{\left(\frac{2x}{x+3}\right)}\left[\frac{(x+3)(2)-2x(1)}{(x+3)^2}\right] \\ &= \frac{x+3}{2x}\left[\frac{2x+6-2x}{(x+3)^2}\right] \\ &= \frac{6}{2x(x+3)} \\ &= \frac{3}{x(x+3)} \end{aligned}}$$

#15b. $y=\ln|\csc(x)|$

$$\boxed{\begin{aligned} y' &= \frac{1}{\csc x}\left[-\csc x \cot x\right] \\ &= -\cot x \end{aligned}}$$

#16b. Write an equation for the tangent line to

$y=\sqrt[5]{3x^3+4x}$ at $x=2$

$y=(3x^3+4x)^{1/5}$

$y'=\frac{1}{5}(3x^3+4x)^{-4/5}(9x^2+4)$

$y'(2)=\frac{1}{5}\left(3(2)^3+4(2)\right)^{-4/5}\left(9(2)^2+4\right)$

$=\frac{1}{5}\,\frac{1}{(\sqrt[5]{32})^4}\,40=8\,\frac{1}{16}=\frac{1}{2}=m$

$y(2)=\sqrt[5]{3(2)^3+4(2)}=\sqrt[5]{32}=2$

tangent line; $\boxed{(y-2)=\frac{1}{2}(x-2)}$

#17b. Write an equation for the tangent line to

$=2(\tan x)^3$

$f(x)=2\tan^3(x)$ at $x=\frac{\pi}{4}$

$f'(x)=2(3\tan^2(x))\sec^2 x$

$f'(\frac{\pi}{4})=6\left[\frac{\sin \pi/4}{\cos \pi/4}\right]^2\frac{1}{[\cos \pi/4]^2}$

$m=f'(\frac{\pi}{4})=6\left[\frac{\sqrt{2}/2}{\sqrt{2}/2}\right]^2\frac{1}{[\sqrt{2}/2]^2}=6(1)^2\frac{4}{2}=12$

$f(\frac{\pi}{4})=2\left[\tan\frac{\pi}{4}\right]^3=2(1)^3=2$

tangent line; $\boxed{(y-2)=12(x-\frac{\pi}{4})}$

#18b. Find $f''(x)$ for $f(x)=6(x^3+4)^3$

$f'(x)=\underline{18(x^3+4)^2}\,\underline{(3x^2)}$ now, product rule)

$\boxed{f''(x)=18(x^3+4)^2(6x)+3x^2(36(x^3+4)^1(3x^2))}$

#19b. Find $f''(x)$ for $f(x)=\sin(x^2)$

$f'(x)=\underline{\cos(x^2)}\,\underline{2x}$ now, product rule...

$\boxed{f''(x)=\cos(x^2)(2)+2x(-\sin(x^2)2x)}$

#20 (hint) First, take the derivative of *g(x)* and remember that the Chain Rule is required.

2.5 – Extra Practice

#6b. Find $\frac{dy}{dx}$ if $2x^3+3y^3=64$

$\frac{d}{dx}[2x^3]+\frac{d}{dx}[3y^3]=\frac{d}{dx}[64]$

$6x^2+9y^2\frac{dy}{dx}=0$

$9y^2\frac{dy}{dx}=-6x^2$

$\boxed{\frac{dy}{dx}=\frac{-6x^2}{9y^2}=\frac{-2x^2}{3y^2}}$

#7b. Find $\frac{dy}{dx}$ if $x^3y^3-y=x$

$x^3\frac{d}{dx}[y^3]+y^3\frac{d}{dx}[x^3]-\frac{d}{dx}[y]=\frac{d}{dx}[x]$

$x^3(3y^2\frac{dy}{dx})+y^3(3x^2)-1\frac{dy}{dx}=1$

$(3x^3y^2-1)\frac{dy}{dx}=1-3x^2y^3$

$\boxed{\frac{dy}{dx}=\frac{1-3x^2y^3}{3x^3y^2-1}}$

#8b. Find $\frac{dy}{dx}$ if $3e^{xy}-x=y^2$

$\frac{d}{dx}[3e^{xy}]-\frac{d}{dx}[x]=\frac{d}{dx}[y^2]$

$3e^{(xy)}\frac{d}{dx}[xy]-1=2y\frac{dy}{dx}$

$3e^{(xy)}(x\frac{d}{dx}[y]+y\frac{d}{dx}[x])-1=2y\frac{dy}{dx}$

$3e^{xy}(x(1\frac{dy}{dx})+y(1))-1=2y\frac{dy}{dx}$

$3xe^{xy}\frac{dy}{dx}+3ye^{xy}-1=2y\frac{dy}{dx}$

$(3xe^{xy}-2y)\frac{dy}{dx}=1-3ye^{xy}$

$\boxed{\frac{dy}{dx}=\frac{1-3ye^{xy}}{3xe^{xy}-2y}}$

#9b. Find $\frac{dy}{dx}$ if $xy=6$

$x\frac{d}{dx}[y]+y\frac{d}{dx}[x]=\frac{d}{dx}[6]$

$x(1\frac{dy}{dx})+y(1)=0$

$x\frac{dy}{dx}=-y$

$\frac{dy}{dx}=\frac{-y}{x}$

—or— could just solve for y here

$y=\frac{6}{x}=6x^{-1}$

$\frac{dy}{dx}=-6x^{-2}$

$\frac{dy}{dx}=\frac{-6}{x^2}$

equivalent, and both are correct

$\left(\frac{dy}{dx}=\frac{-y}{x}=\frac{-(\frac{6}{x})}{x}=\frac{-6}{x^2}\right)$

#10b. Find $\frac{d^2y}{dx^2}$ (*the 2nd derivative*) if $x^2-y^2=36$

$\frac{d}{dx}[x^2]-\frac{d}{dx}[y^2]=\frac{d}{dx}[36]$

$2x-2y\frac{dy}{dx}=0$

$-2y\frac{dy}{dx}=-2x$

$\frac{dy}{dx}=\frac{-2x}{-2y}=\frac{x}{y}$

now $\frac{d^2y}{dx^2}=\frac{d}{dx}\left[\frac{x}{y}\right]$ quotient rule

$=\frac{y\frac{d}{dx}[x]-x\frac{d}{dx}[y]}{y^2}$

$=\frac{y(1)-x(1\boxed{\frac{dy}{dx}})}{y^2}$ ← replace this w/ result from first part

$=\boxed{\frac{y-x(\frac{x}{y})}{y^2}}$

#11b. Find $\frac{dy}{dx}$ if $y=(1+x)^{\frac{1}{x}}$ (requires logarithmic diff)

$\ln y = \ln (1+x)^{1/x}$

$\ln y = \left(\frac{1}{x}\right) \ln(1+x) = (x^{-1}) \ln(1+x)$

$\frac{d}{dx}[\ln y] = \left(\frac{1}{x}\right) \frac{d}{dx}[\ln(1+x)] + \ln(1+x) \frac{d}{dx}[x^{-1}]$

$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x} \frac{1}{1+x}(1) + \ln(1+x)(-x^{-2})$

$\frac{dy}{dx} = \left[\frac{1}{x(1+x)} - \frac{\ln(1+x)}{x^2}\right] y$

$\boxed{\frac{dy}{dx} = \left[\frac{1}{x(1+x)} - \frac{\ln(1+x)}{x^2}\right] (1+x)^{1/x}}$

#12b. Find $\frac{dy}{dx}$ if $y=(\ln(x))^{\ln(x)}$ (requires logarithmic diff)

$\ln y = \ln \left(\ln(x)^{\ln(x)}\right)$

$\ln y = (\ln x)(\ln(\ln x))$

$\frac{d}{dx}[\ln y] = (\ln x)\frac{d}{dx}[\ln(\ln x)] + \ln(\ln x) \frac{d}{dx}(\ln x)$

$\frac{1}{y}\frac{dy}{dx} = (\ln x) \frac{1}{\ln x}\left[\frac{1}{x}\right] + \ln(\ln x) \frac{1}{x}$

$\frac{dy}{dx} = \left(\frac{1}{x} + \frac{\ln(\ln x)}{x}\right] y$

$\boxed{\frac{dy}{dx} = \left(\frac{1}{x} + \frac{\ln(\ln x)}{x}\right] (\ln(x))^{\ln(x)}}$

2.6 – Extra Practice

Find the derivative:

#5b. $f(x) = \arcsin\left(x^2\right)$

$$\boxed{\begin{aligned} f'(x) &= \frac{1}{\sqrt{1-(x^2)^2}}(2x) \\ &= \frac{2x}{\sqrt{1-x^4}} \end{aligned}}$$

#6b. $g(x) = \dfrac{\arccos(x)}{x+1}$

$$\boxed{g'(x) = \frac{(x+1)\frac{-1}{\sqrt{1-x^2}}(1) - \arccos(x)(1)}{(x+1)^2}}$$

Find the derivative:

#7b. $y = e^{2t}\ln\left(t^2+4\right) - \dfrac{1}{2}\arctan\left(\dfrac{t}{3}\right)$

$$\boxed{\begin{aligned} y' &= e^{2t}\frac{1}{t^2+4}(2t) + \ln(t^2+4)e^{2t}(2) - \frac{1}{2}\frac{1}{1+\left(\frac{t}{3}\right)^2}\left(\frac{1}{3}\right) \\ &= \frac{2te^{2t}}{t^2+4} + 2e^{2t}\ln(t^2+4) - \frac{1}{6}\frac{1}{1+\left(\frac{t}{3}\right)^2} \end{aligned}}$$

Find an equation of the tangent line to the graph of the function at the given x-value:

#8b. $y = \dfrac{1}{2}\arccos(x) \qquad x = -\dfrac{\sqrt{2}}{2}$

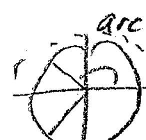
$$m = y' = \frac{1}{2}\frac{-1}{\sqrt{1-x^2}}\Bigg|_{x=-\frac{\sqrt{2}}{2}} = \frac{1}{2}\frac{-1}{\sqrt{1-\left(-\frac{\sqrt{2}}{2}\right)^2}} = \frac{1}{2}\frac{-1}{\sqrt{1-\frac{1}{2}}} = \frac{1}{2}\frac{-1}{\sqrt{\frac{1}{2}}} = -\frac{1}{2}\sqrt{\frac{2}{1}} = -\frac{\sqrt{2}}{2}$$

$$\begin{aligned} y\left(-\tfrac{\sqrt{2}}{2}\right) &= \tfrac{1}{2}\arccos\left(-\tfrac{\sqrt{2}}{2}\right) \\ &= \tfrac{1}{2}\left(\tfrac{3\pi}{4}\right) = \tfrac{3\pi}{8} \end{aligned}$$

$\arccos\left(-\frac{\sqrt{2}}{2}\right) = \theta$

$\cos\theta = -\frac{\sqrt{2}}{2}$

$\arccos\left(-\frac{\sqrt{2}}{2}\right) = \frac{3\pi}{4}$



arccos

$\theta = \frac{3\pi}{4}$

tangent line: $\boxed{\left(y - \frac{3\pi}{8}\right) = -\frac{\sqrt{2}}{2}\left(x + \frac{\sqrt{2}}{2}\right)}$