AP Calculus BC – Unit 3, Part 1 Extra Practice

3.1 – Extra Practice

Evaluate the limit, using L'Hopital's Rule if necessary.

#18b. $\lim_{x\to4}\frac{x^2-2x-8}{x-4}$

$= \lim_{x\to4}\frac{(x-4)(x+2)}{x-4}$

$= \lim_{x\to4}(x+2)$

$= 4+2$

$= 6$

#19b. $\lim_{x\to0}\frac{\sin(4x)}{\sqrt{x}}$

$\lim_{x\to0}\sin(4x)=0$ $\left(\frac{0}{0}\right)$ use L'Hopital's rule

$\lim_{x\to0}\sqrt{x}=0$

$= \lim_{x\to0}\frac{4\cos(4x)}{\frac{1}{2}x^{-1/2}}$

$= \lim_{x\to0} 8\cos(4x)\sqrt{x}$

$= 8(1)(\sqrt{0})$

$= 0$

#20b. $\lim_{x\to0}\frac{\sin(7x)}{x}$

$\lim_{x\to0}\sin(7x)=0$

$\lim_{x\to0}x=0$ $\left(\frac{0}{0}\right)$ use L'Hopital's rule

$= \lim_{x\to0}\frac{7\cos(7x)}{1}$

$= \frac{7(1)}{1}$

$= 7$

#21b. $\lim_{x\to\infty}\frac{x^3}{e^{(x^2)}}$

$\lim_{x\to\infty}x^3\to\infty$ $\left(\frac{\infty}{\infty}\right)$

$\lim_{x\to\infty}e^{x^2}\to\infty$ use L'Hopital's rule

$= \lim_{x\to\infty}\frac{3x^2}{e^{(x^2)}2x}$

$= \lim_{x\to\infty}\frac{3x}{2e^{x^2}}$ $\left(\frac{\infty}{\infty}\right)$

$= \lim_{x\to\infty}\frac{3}{2e^{x^2}(2x)} = 0$

#22b. $\lim_{x\to3^-}\frac{x^2+2x}{x-3}$

$\lim_{x\to3^-}(x^2+2x)=15$

$\lim_{x\to3^-}(x-3)=0$

$\left(\frac{15}{0}\right)$ vertical asymptote

$x\to3^-$ (2.9) $\frac{x(x+2)}{(x-3)}$

$\frac{(+)(+)}{(-)}<0$

$\therefore \lim_{x\to3^-}\frac{x^2+2x}{x-3}\to-\infty$

or

DNE

#23. $\lim_{x\to0}\frac{x}{\arctan(2x)}$

$\lim_{x\to0}x=0$

$\lim_{x\to0}\arctan(2x)=0$

$\left(\frac{0}{0}\right)$ use L'Hopital's rule

$= \lim_{x\to0}\frac{1}{\left(\frac{1}{1+(2x)^2}\right)(2)}$

$= \frac{1}{\left(\frac{1}{1+0}\right)(2)} = \frac{1}{2}$

#24b. $\lim_{x\to\infty} x\tan\left(\frac{1}{x}\right)$

$\lim_{x\to\infty} x = \infty$, $\lim_{x\to\infty} \tan(\frac{1}{x}) = 0$ $(\infty)(0)$

$= \lim_{x\to\infty} \frac{\tan(\frac{1}{x})}{(\frac{1}{x})}$

$\lim_{x\to\infty} \tan(\frac{1}{x}) = 0$, $\lim_{x\to\infty} (\frac{1}{x}) = 0$

$(\frac{0}{0})$ use L'Hopital's rule

$= \lim_{x\to\infty} \frac{\sec^2(\frac{1}{x})(-x^{-2})}{(-x^{-2})}$

$= \lim_{x\to\infty} \sec^2(\frac{1}{x})$

$= \frac{1}{(\cos(0))^2} = \frac{1}{1^2} = 1$

#25b. $\lim_{x\to 0^+} (x-2)^3 \ln(x)$

$\lim_{x\to 0^+} (x-2)^3 = -8$

$\lim_{x\to 0^+} \ln(x) \to -\infty$

$(-8)(-\infty) \to \infty$

$\therefore \lim_{x\to 0^+} (x-2)^3 \ln(x) \to \infty$ or DNE

#26b. $\lim_{x\to 4^+} (3(x-4))^{x-4} = y$

$\ln y = \ln(\lim_{x\to 4^+} (3(x-4))^{x-4})$

$\ln y = \lim_{x\to 4^+} (x-4)\ln(3(x-4))$

$(0)(-\infty)$

$= \lim_{x\to 4^+} \frac{\ln(3(x-4))}{(\frac{1}{x-4})}$ $\lim_{x\to 4^+} \ln(3(x-4)) \to -\infty$ $\lim_{x\to 4^+} \frac{1}{x-4} \to \infty$

$(\frac{-\infty}{\infty})$ use L'Hopital's rule:

$\ln y = \lim_{x\to 4^+} \frac{\frac{1}{3(x-4)}(3)}{-(x-4)^{-2}} = \lim_{x\to 4^+} \frac{-(x-4)^2}{(x-4)}$

$\ln y = \lim_{x\to 4^+} -(x-4) = 0$

$\ln y = 0$

$y = e^0 = 1$

$\therefore \lim_{x\to 4^+} (3(x+4))^{x-4} = 1$

#26c. $\lim_{x\to\infty} x^{\left(\frac{1}{x}\right)} = y$

$\ln y = \ln(\lim_{x\to\infty} x^{(1/x)})$

$\ln y = \lim_{x\to\infty} (\frac{1}{x})\ln(x)$

$(0)(\infty)$

$= \lim_{x\to\infty} \frac{\ln x}{x}$ $\lim_{x\to\infty} \ln x \to \infty$ $\lim_{x\to\infty} x \to \infty$

$(\frac{\infty}{\infty})$ use L'Hopital's rule:

$\ln y = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \frac{0}{1} = 0$

$y = e^0 = 1$

$\therefore \lim_{x\to\infty} x^{(\frac{1}{x})} = 1$

3.2 – Extra Practice

Without using a calculator, find the intervals where the function is increasing and decreasing, and find all relative maxima and minima.

#8b. $f(x) = x^4 - 2x^2$

$f'(x) = 4x^3 - 4x = 4x(x^2-1) = 4x(x-1)(x+1)$

$\underline{f'(x) = 0}$ $\qquad$ $\underline{f'(x)\ DNE}$

$4x(x-1)(x+1) = 0$

$x = 0,\ x = 1,\ x = -1$

$-\infty$ —— -1 —— 0 —— 1 —— ∞

$f'(-2)$ $\quad$ $f'(-\frac{1}{2})$ $\quad$ $f'(\frac{1}{2})$ $\quad$ $f'(2)$

$(+)(-)(-)(-)$ $\quad$ $(+)(-)(-)(+)$ $\quad$ $(+)(+)(-)(+)$ $\quad$ $(+)(+)(+)(+)$

$\searrow$ $\quad$ $\nearrow$ $\quad$ $\searrow$ $\quad$ $\nearrow$

f increasing: $(-1,0) \cup (1,\infty)$

f decreasing: $(-\infty,-1) \cup (0,1)$

f relative max at $x = 0$ $(0,0)$

f relative min at $x = -1$ and $x = 1$

$(-1,-1)$ $\quad$ $(1,-1)$

#9b. $f(x) = \dfrac{x^2}{2x-1}$

$f'(x) = \dfrac{(2x-1)(2x) - x^2(2)}{(2x-1)^2} = \dfrac{4x^2 - 2x - 2x^2}{(2x-1)^2} = \dfrac{2x^2-2x}{(2x-1)^2}$

$= \dfrac{2x(x-1)}{(2x-1)^2}$

$\underline{f'(x) = 0}$ $\qquad$ $\underline{f'(x)\ DNE}$

$2x(x-1) = 0$ $\qquad$ $(2x-1)^2 = 0$

$x = 0,\ x = 1$ $\qquad$ $x = \frac{1}{2}$

$-\infty$ —— 0 —— $\frac{1}{2}$ —— 1 —— ∞

$f'(-1)$ $\quad$ $f'(\frac{1}{4})$ $\quad$ $f'(\frac{3}{4})$ $\quad$ $f'(2)$

$\frac{(+)(-)(-)}{(+)}$ $\quad$ $\frac{(+)(+)(-)}{(+)}$ $\quad$ $\frac{(+)(+)(-)}{(+)}$ $\quad$ $\frac{(+)(+)(+)}{(+)}$

$\nearrow$ $\quad$ $\searrow$ $\quad$ $\searrow$ $\quad$ $\nearrow$

f increasing: $(-\infty,0) \cup (1,\infty)$

f decreasing: $(0,\frac{1}{2}) \cup (\frac{1}{2},1)$

f rel. max at $x = 0$ $(0,0)$

f rel. min at $x = 1$ $(1,1)$

#10b. $f(x) = x - 2\cos x$ for $0 < x < 2\pi$

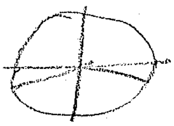
$f'(x) = 1 + 2\sin x$

$\underline{f'(x) = 0}$

$1 + 2\sin x = 0$

$2\sin x = -1$

$\sin x = -\frac{1}{2}$



$x = \frac{7\pi}{6},\ x = \frac{11\pi}{6}$

0 —— $\frac{7\pi}{6}$ —— $\frac{11\pi}{6}$ —— 2π

$f'(\pi)$ $\quad$ $f'(\frac{3\pi}{2})$ $\quad$ $f'(\frac{23\pi}{12})$

$= 1 + 2\sin\pi$ $\quad$ $= 1 + 2\sin(\frac{3\pi}{2})$ $\quad$ $= 1 + 2\sin(\frac{23\pi}{12})$

$= 1 + 2(0)$ $\quad$ $= 1 + 2(-1)$ $\quad$ $= 1 + 2$(small neg. #)

$(+)$ $\quad$ $(-)$ $\quad$ $(+)$

$\nearrow$ $\quad$ $\searrow$ $\quad$ $\nearrow$

f increasing: $(0, \frac{7\pi}{6}) \cup (\frac{11\pi}{6}, 2\pi)$

f decreasing: $(\frac{7\pi}{6}, \frac{11\pi}{6})$

f relative maximum at $x = \frac{7\pi}{6}$ $\quad (\frac{7\pi}{6}, \frac{7\pi}{6} + \sqrt{3})$ $\quad f(\frac{7\pi}{6}) = \frac{7\pi}{6} - 2(-\frac{\sqrt{3}}{2})$

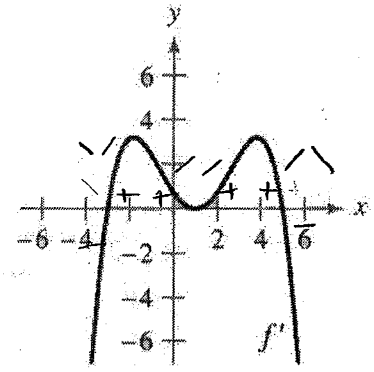
f relative minimum at $x = \frac{11\pi}{6}$ $\quad (\frac{11\pi}{6}, \frac{11\pi}{6} - \sqrt{3})$ $\quad f(\frac{11\pi}{6}) = \frac{11\pi}{6} - 2(\frac{\sqrt{3}}{2})$

#11b. Use the graph of $f'(x)$ to find:

a) critical numbers of *f*

b) intervals on which *f* is increasing or decreasing

c) for each critical number state whether *f* has a relative maximum, relative minimum, or neither



a) $x=-3$, $x=1$, $x=5$ (where $f'(x)=0$ or DNE)

b) increasing: $(-3,1)\cup(1,5)$ (where $f'(x)>0$)

decreasing: $(-\infty,-3)\cup(5,\infty)$ (where $f'(x)<0$)

c) $x=-3$ \ / relative minimum

$x=1$ / / neither

$x=5$ / \ relative maximum

Without using a calculator, find the intervals where the function is concave up and concave down, and find all inflection points.

#12b. $f(x)=x^5-5x+2$

$f'(x)=5x^4-5$

$f''(x)=20x^3$

$f''(x)=0$ $\quad$ $f''(x)$ DNE

$20x^3=0$

$x=0$

$-\infty$ ——— 0 ——— ∞

$f''(-1)$ $\quad$ $f''(1)$

$(+)(+)$ $\quad$ $(+)(-)$

f concave up: $(-\infty,0)$

f concave down: $(0,\infty)$

inflection pt at $x=0$ $(0,2)$

#13b. $f(x)=x+\dfrac{2}{\sin x}$ *for* $\dfrac{-\pi}{2}<x<\dfrac{\pi}{2}$

$f(x)=x+2\csc x$

$f'(x)=1+2(-\csc x\cot x)=1+(-2\csc x)\cot x$

$f''(x)=-2\csc x(-\csc^2 x)+\cot x(-2(-\csc x\cot x$

$=2\csc^3 x+2\csc x\cot^2 x$

$=2\csc x(\csc^2 x+\cot^2 x)$

$=2\dfrac{1}{\sin x}\left(\dfrac{1}{(\sin x)^2}+\dfrac{(\cos x)^2}{(\sin x)^2}\right)=\dfrac{2(1+(\cos x)^2)}{(\sin x)^3}$

$f''(x)=0$ $\quad$ $f''(x)$ DNE

$1+(\cos x)^2=0$ $\quad$ $\sin x=0$

$(\cos x)^2=-1$ $\quad$ $x=0$

$-\frac{\pi}{2}$ ——— 0 ——— $\frac{\pi}{2}$

$f''(-\frac{\pi}{4})$ $\quad$ $f''(\frac{\pi}{4})$

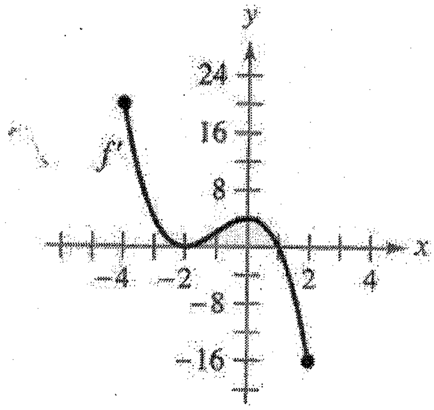
$\dfrac{(+)(+)}{(-)}$ $\quad$ $\dfrac{(+)(+)}{(+)}$

f concave up $(-\frac{\pi}{2},0)$

f concave down $(0,\frac{\pi}{2})$

No inflection pt at $x=0$ because there is a vertical asymptote

#14b. Using the graph of the <u>derivative</u> of the function *f* on the interval [-4, 2]]:



a) On what interval(s) is *f* increasing / decreasing? Explain.

f is increasing where $f'(x)>0$ which occurs over $[-4,-2)\cup(-2,1)$

f is decreasing where $f'(x)<0$ which occurs over $(1,2]$

b) On what interval(s) is *f* concave up / concave down? Explain.

f is concave up where $f''(x)>0$ which is where $f'(x)$ is increasing which occurs over $(-2,0)$

f is concave down where $f''(x)<0$ which is where $f'(x)$ is decreasing which occurs over $(-4,2)\cup(0,2)$

c) At what x-value(s) does *f* have relative extrema?

f has relative extrema where the sign of $f'(x)$ changes which occurs at $x=1$

d) At what x-value(s) does *f* have inflection points?

f has inflection pts where the sign of $f''(x)$ changes which is where the $f'(x)$ curve changes direction (increasing to decreasing or vice versa). This occurs at $x=-2$ and $x=1$

3.3 – Extra Practice

~~#7 Hint: Remember for the MVT or Rolle's Theorem to apply the function must be continuous on the closed interval and differentiable on the open interval.~~

Determine whether the Mean Value Theorem or Rolle's Theorem can be applied for the specified function and interval, and if it can be applied, find all *x*-values in the interval where the instantaneous rate of change equals the average rate of change:

#8b. $f(x) = x^2 - x - 12 \quad over \quad [-2,4]$

f(x) is continuous over $[-2,4]$ and differentiable over $(-2,4)$

$\therefore$ the Mean Value Theorem does guarantee c, $-2 \le c \le 4$ such that $f'(c) =$ avg rate of change

avg rate of change $= \dfrac{f(4)-f(-2)}{4-(-2)} = \dfrac{0-(-6)}{4+2} = \dfrac{6}{6} = 1$

$f'(x) = 2x - 1$

$f'(c) = 2c - 1 = 1$

$2c = 2$

when $c = 1$

#9b. $f(x) = (x+3)\ln(x+3) \quad over \quad [-2,-1]$

f(x) is continuous over $[-2,-1]$ and differentiable over $(-2,-1)$ $\therefore$ the Mean Value Theorem does guarantee c, $-2 \le c \le -1$, such that $f'(c) =$ avg rate of change.

avg rate of change $= \dfrac{f(-1)-f(-2)}{-1-(-2)} = \dfrac{1.3863 - 0}{1} = 1.3863$

$f'(x) = (x+3)\dfrac{1}{x+3}(1) + \ln(x+3)(1) = 1 + \ln(x+3)$

$f'(c) = 1 + \ln(c+3) = 1.3863$

$\ln(c+3) = 0.3863$

$c+3 = e^{0.3863}$

$c = e^{0.3863} - 3 = -1.528$

#10b. $f(x) = \tan x \quad over \quad [0,\pi]$

f(x) is discontinuous at $x = \frac{\pi}{2}$

$\therefore$ the Mean Value Theorem/Rolle's Theorem do not apply.

#11b. $f(x) = (x-1)(x-2)(x-3) \quad over \quad [1,3]$

$f(3) = 0$
$f(1) = 0$ slope $= 0 \rightarrow$ Rolle's case

f(x) is continuous over $[1,3]$ and differentiable over $(1,3)$

$\therefore$ Rolle's Theorem guarantees c, $1 \le c \le 3$, such that $f'(c) = 0$.

$f(x) = (x-1)(x^2-5x+6)$

$f(x) = x^3 - 5x^2 + 6x - x^2 + 5x - 6$

$f(x) = x^3 - 6x^2 + 11x - 6$

$f'(x) = 3x^2 - 12x + 11$

$f'(c) = 3c^2 - 12c + 11 = 0$

$c = 1.423$ or $c = 2.577$

3.4 – Extra Practice

Without using a calculator, find the absolute maximum and absolute minimum value of the function over the given interval:

#3b. $f(x)=x^3-\frac{3}{2}x^2$ *over* $[-1,2]$

$f'(x)=3x^2-3x$

$=3x(x-1)$

$\underline{f'(x)=0}$ $\quad$ $\underline{f'(x)\ DNE}$

$x=0,\ x=1$ $\quad$ —

x	$f(x)=x^3-\frac{3}{2}x^2$
0	0
1	$1-\frac{3}{2}=\frac{2}{2}-\frac{3}{2}=-\frac{1}{2}$
-1	$-1-\frac{3}{2}=-\frac{2}{2}-\frac{3}{2}=-\frac{5}{2}$ abs min
2	$8-\frac{3}{2}(4)=8-6=2$ abs max

abs. max of f on $[-1,2]$ is 2 (which occurs at $x=2$)

abs. min of f on $[-1,2]$ is $-\frac{5}{2}$ (which occurs at $x=-1$)

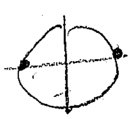
#4b. $f(x)=3\cos x$ *over* $[0,2\pi]$

$f'(x)=-3\sin x$

$\underline{f'(x)=0}$ $\quad$ $\underline{f'(x)\ DNE}$

$-3\sin x=0$ $\quad$ —

$\sin x=0$



$x=0,\ x=\pi,\ x=2\pi$

x	$f(x)=3\cos x$
0	$3(1)=3$ abs max
π	$3(-1)=-3$ abs min
2π	$3(1)=3$ abs max

abs max of f over $[0,2\pi]$ is 3 (which occurs at $x=0$ and $x=2\pi$)

abs min of f over $[0,2\pi]$ is -3 (which occurs at $x=\pi$)