AP Calculus BC – Unit 4, Part 2 Extra Practice

4.5 day 1 – Extra Practice

Evaluate using u-substitution or Reverse Chain Rule (whichever you prefer)

#17b. $\int 2x\left(x^2-9\right)^3\,dx$

Inside: x^2-9

$2x$

$$\boxed{\frac{(x^2-9)^4}{4}+C}$$

#18b. $\int \frac{x^3}{\sqrt{1+x^4}}\,dx$

$u=1+x^4$

$\frac{du}{dx}=4x^3$

$du=4x^3dx$

$x^3dx=\frac{1}{4}du$

$\frac{1}{4}\int u^{-1/2}du$

$\frac{1}{4}(2)u^{1/2}+C$

$\frac{1}{2}(1+x^4)^{1/2}+C$

$$\boxed{\frac{1}{2}\sqrt{1+x^4}+C}$$

#19b. $\int e^{3x}\sec\left(e^{3x}\right)\tan\left(e^{3x}\right)dx$

$u=e^{3x}$

$\frac{du}{dx}=3e^{3x}$

$du=3e^{3x}dx$

$e^{3x}dx=\frac{1}{3}du$

$\frac{1}{3}\int \sec u\tan u\,du$

$\frac{1}{3}\sec(u)+C$

$$\boxed{\frac{1}{3}\sec(e^{3x})+C}$$

#20b. $\int e^{\ln(5x+3)}\,dx$

$=\int(5x+3)\,dx$

$5\int x\,dx+\int 3\,dx$

$$\boxed{\frac{5}{2}x^2+3x+C}$$

4.5 day 2 – Extra Practice

Evaluate the definite integral.

#3b. $\int_0^1 x\sqrt{1-x^2}\,dx$

$u=1-x^2$

$\frac{du}{dx}=-2x$

$du=-2x\,dx$

$x\,dx=-\frac{1}{2}du$

$-\frac{1}{2}\int_1^0 u^{1/2}\,du$

$-\frac{1}{2}\frac{2}{3}\left(u^{3/2}\right]_1^0$

$\boxed{-\frac{1}{3}\left[(0)^{3/2}-(1)^{3/2}\right]}$

$(=\frac{1}{3})$

#4b. $\int_0^4 \frac{1}{\sqrt{2x+1}}\,dx$

$u=2x+1$

$\frac{du}{dx}=2$

$du=2dx$

$dx=\frac{1}{2}du$

$\frac{1}{2}\int_1^9 u^{-1/2}\,du$

$\frac{1}{2}\frac{2}{1}\left[u^{1/2}\right]_1^9$

$\boxed{\left((9)^{1/2}-(1)^{1/2}\right)}$

$\sqrt{9}-\sqrt{1}$

$3-1$

$(=2)$

#5b. $\int_0^{\sqrt{2}} xe^{\left(\frac{-x^2}{2}\right)}\,dx$

$u=-\frac{1}{2}x^2$

$\frac{du}{dx}=-x$

$du=-x\,dx$

$x\,dx=-du$

$-\int_0^{-1} e^u\,du$

$-\left(e^u\right]_0^{-1}$

$\boxed{-\left[e^{-1}-e^0\right]}$

#6b. $\int_1^5 \frac{x}{\sqrt{2x-1}}\,dx$

$u=2x-1$

$2x=u-1$

$x=\frac{1}{2}(u-1)$

$\frac{du}{dx}=2$

$du=2dx$

$dx=\frac{1}{2}du$

$\int_1^9 u^{-1/2}\left(\frac{1}{2}\right)(u-1)\frac{1}{2}du$

$\frac{1}{4}\int_1^9\left(u^{1/2}-u^{-1/2}\right)du$

$\frac{1}{4}\left[\frac{2}{3}u^{3/2}-2u^{1/2}\right]_1^9$

$\left[\frac{1}{6}u^{3/2}-\frac{1}{2}u^{1/2}\right]_1^9$

$\boxed{\left(\frac{1}{6}(9)^{3/2}-\frac{1}{2}(9)^{1/2}\right]-\left(\frac{1}{6}(1)^{3/2}-\frac{1}{2}(1)^{1/2}\right)}$

$(=\frac{16}{3})$

4.6 – Extra Practice

Evaluate the integral.

#7b. $\int \frac{5x^3-2x}{\sqrt[3]{x^2}}\,dx$

$5\int \frac{x^3}{x^{2/3}}dx - 2\int \frac{x}{x^{2/3}}dx$

$5\int x^{7/3}dx - 2\int x^{1/3}dx$

$\boxed{5\left(\frac{3}{10}\right)x^{10/3} - 2\left(\frac{3}{4}\right)x^{4/3} + c}$

#8b. $\int \frac{x-3}{\sqrt{4-3x^2}}\,dx$

$\int \frac{x}{\sqrt{4-3x^2}}dx - 3\int \frac{1}{\sqrt{4-3x^2}}dx$

$u=4-3x^2 \quad du=-6x\,dx$ | $u=\sqrt{3}x \quad du=\sqrt{3}dx \quad a=2$

$\frac{du}{dx}=-6x \quad x\,dx=-\frac{1}{6}du$ | $\frac{du}{dx}=\sqrt{3} \quad dx=\frac{1}{\sqrt{3}}du$

$-\frac{1}{6}\int \frac{1}{\sqrt{u}}du \qquad -3\int \frac{1}{\sqrt{(2)^2-(\sqrt{3}x)^2}}dx$

$-\frac{1}{6}\int u^{-1/2}du \qquad -3\int \frac{1}{\sqrt{a^2-u^2}}du$

$-\frac{1}{6}u^{1/2} \qquad -3\arcsin\left(\frac{u}{a}\right)+c$

$\boxed{-\frac{1}{6}(4-3x^2)^{1/2} - 3\arcsin\left(\frac{\sqrt{3}x}{2}\right)+c}$

#9b. $\int 3x^4\cot\left(4x^5+1\right)dx$

$\int 3x^4 \frac{\cos(4x^5+1)}{\sin(4x^5+1)}dx$

$u=\sin(4x^5+1)$

$\frac{du}{dx}=\cos(4x^5+1)20x^4$

$du=20x^4\cos(4x^5+1)dx$

$x^4\cos(4x^5+1)dx=\frac{1}{20}du$

$\frac{3}{20}\int \frac{1}{u}du$

$\boxed{\frac{3}{20}\ln|\sin(4x^5+1)|+c}$

#10b. $\int\left(\cot^2(5x)+1\right)dx$

$\frac{\cos^2\theta}{\sin^2\theta}+\frac{\sin^2\theta}{\sin^2\theta}=\frac{1}{\sin^2\theta}$

$\cot^2\theta+1=\csc^2\theta$

$\int \csc^2(5x)dx$

$\frac{1}{5}\int 5\csc^2(5x)dx$

$\boxed{-\frac{1}{5}\cot(5x)+c}$

#11b. $\int \frac{1}{x^2+4x+11}\,dx$

$x^2+4x+4+11-4$

$(x+2)^2+7$

$\int \frac{1}{7+(x+2)^2}\,dx \qquad u=x+2 \qquad a=\sqrt{7}$

$du=dx$

$\int \frac{1}{a^2+u^2}\,du$

$$\boxed{\frac{1}{\sqrt{7}}\arctan\left(\frac{x+2}{\sqrt{7}}\right)+C}$$

#12b. $\int \frac{1}{x^2+4x+3}\,dx$

$x^2+4x+4+3-4$

$(x+2)^2-1$

$\int \frac{1}{(x+2)^2-1}\,dx$

minus doesn't match arctan form so we don't have any way to evaluate this integral :(

(but we won't do this on test, of course) :)

#13b. $\int \frac{3x^2-2x+3}{x^2+1}\,dx$

$$\begin{array}{r} 3 \\ x^2+1\,\overline{)\,3x^2-2x+3} \\ -(3x^2 \quad\;\; +3) \\ \hline -2x \end{array}$$

$\frac{3x^2-2x+3}{x^2+1}=3+\frac{-2x}{x^2+1}$

$\int \left[3+\frac{-2x}{x^2+1}\right]dx$

$\int 3\,dx-\int \frac{2x}{x^2+1}\,dx \qquad u=x^2+1$

$\frac{du}{dx}=2x$

$du=2x\,dx$

$\int 3\,dx-\int \frac{1}{u}\,du$

$$\boxed{3x-\ln|x^2+1|+C}$$

#14b. $\int \frac{6}{3e^{-x}+5}\,dx$

$u=3e^{-x}+5$

$\frac{du}{dx}=-3e^{-x}$

$du=-3e^{-x}\,dx$

(need this in numerator for u-sub)

$6\int \frac{1}{3e^{-x}+5}\,dx$

$\frac{6}{5}\int \frac{5}{3e^{-x}+5}\,dx$

$\frac{6}{5}\int \frac{5+3e^{-x}-3e^{-x}}{3e^{-x}+5}\,dx$

$\frac{6}{5}\int \frac{3e^{-x}+5}{3e^{-x}+5}\,dx-\frac{6}{5}\int \frac{3e^{-x}}{3e^{-x}+5}\,dx$

$\frac{6}{5}\int 1\,dx-\frac{6}{5}\int \frac{3e^{-x}}{3e^{-x}+5}\,dx \qquad u=3e^{-x}+5 \quad du=-3e^{-x}\,dx$

$\frac{6}{5}\int 1\,dx+\frac{6}{5}\int \frac{1}{u}\,du$

$$\boxed{\frac{6}{5}x+\frac{6}{5}\ln|3e^{-x}+5|+C}$$

4.7 day 1 – Extra Practice

Evaluate the integral.

#7b. $\int 4xe^{2x}\,dx$

$u = 4x \quad dv = e^{2x}dx$

$\frac{du}{dx} = 4 \quad \int dv = \int e^{2x}dx$

$du = 4dx \quad v = \frac{1}{2}e^{2x}$

$uv - \int v\,du$

$(4x)(\frac{1}{2}e^{2x}) - \int(\frac{1}{2}e^{2x})(4dx)$

$2xe^{2x} - 2\int e^{2x}dx$

$2xe^{2x} - 2(\frac{1}{2}e^{2x}) + C$

$\boxed{2xe^{2x} - e^{2x} + C}$

#8b. $\int 3x^2 \ln(x)\,dx$

$u = \ln x \quad dv = 3x^2dx$

$\frac{du}{dx} = \frac{1}{x} \quad \int dv = \int 3x^2dx$

$du = \frac{1}{x}dx \quad v = x^3$

$uv - \int v\,du$

$(\ln x)(x^3) - \int(x^3)(\frac{1}{x}dx)$

$x^3\ln x - \int x^2dx$

$\boxed{x^3\ln x - \frac{1}{3}x^3 + C}$

#9b. $\int 7\ln(3x)\,dx$

$u = \ln(3x) \quad dv = 7dx$

$\frac{du}{dx} = \frac{1}{3x}(3) \quad \int dv = \int 7dx$

$du = \frac{1}{x}dx \quad v = 7x$

$uv - \int v\,du$

$(\ln(3x))(7x) - \int(7x)(\frac{1}{x}dx)$

$7x\ln(3x) - 7\int 1\,dx$

$\boxed{7x\ln(3x) - 7x + C}$

#10b. $\int x^2 \sin(6x)\,dx$

$u = x^2 \quad dv = \sin(6x)dx$

$\frac{du}{dx} = 2x \quad \int dv = \int \sin(6x)dx$

$uv - \int v\,du \quad du = 2xdx \quad v = -\frac{1}{6}\cos(6x)$

$(x^2)(-\frac{1}{6}\cos(6x)) - \int(-\frac{1}{6}\cos(6x))(2xdx)$

$-\frac{1}{3}x\cos(6x) + \frac{1}{3}\left[\int x\cos(6x)dx\right]$

again...

$u = x \quad dv = \cos(6x)dx$

$\frac{du}{dx} = 1 \quad \int dv = \int \cos(6x)dx$

$du = dx \quad v = \frac{1}{6}\sin(6x)$

$-\frac{1}{3}x\cos(6x) + \frac{1}{3}\left[uv - \int v\,du\right]$

$-\frac{1}{3}x\cos(6x) + \frac{1}{3}\left[\frac{1}{6}x\sin(6x) - \frac{1}{6}\int \sin(6x)dx\right]$

$-\frac{1}{3}x\cos(6x) + \frac{1}{3}\left[\frac{1}{6}x\sin(6x) + \frac{1}{36}\cos(6x)\right] + C$

$\boxed{-\frac{1}{3}x\cos(6x) + \frac{1}{18}x\sin(6x) + \frac{1}{108}\cos(6x) + C}$

#11b. $\int t^2 \ln(t+1)\,dt$

$u = \ln(t+1) \qquad dv = t^2 dt$

$\frac{du}{dt} = \frac{1}{t+1}(1) \qquad \int dv = \int t^2 dt$

$du = \frac{1}{t+1} dt \qquad v = \frac{1}{3}t^3$

$uv - \int v\,du$

$(\ln(t+1))(\frac{1}{3}t^3) - \int (\frac{1}{3}t^3)(\frac{1}{t+1})\,dt$

$\frac{1}{3}t^3\ln(t+1) - \frac{1}{3}\left[\int \frac{t^3}{t+1}\,dt\right]$

u-sub

$u = t+1 \rightarrow t = u-1$

$\frac{du}{dt} = 1$

$du = dt$

$\frac{1}{3}t^3\ln(t+1) - \frac{1}{3}\left[\int \frac{(u-1)^3}{u}\,du\right]$

$\frac{1}{3}t^3\ln(t+1) - \frac{1}{3}\left[\int \frac{(u-1)(u^2-2u+1)}{u}\,du\right]$

$\frac{1}{3}t^3\ln(t+1) - \frac{1}{3}\left[\int \frac{u^3-2u^2+u-u^2+2u-1}{u}\,du\right]$

$\frac{1}{3}t^3\ln(t+1) - \frac{1}{3}\left[\int \frac{u^3-3u^2+3u-1}{u}\,du\right]$

$\frac{1}{3}t^3\ln(t+1) - \frac{1}{3}\left[\int (u^2-3u+3-\frac{1}{u})\,du\right]$

$\frac{1}{3}t^3\ln(t+1) - \frac{1}{3}\left[\frac{1}{3}u^3 - \frac{3}{2}u^2 + 3u - \ln|u|\right] + C$

$$\boxed{\frac{1}{3}t^3\ln(t+1) - \frac{1}{3}\left[\frac{1}{3}(t+1)^3 - \frac{3}{2}(t+1)^2 + 3(t+1) - \ln|t+1|\right] + C}$$

4.7 day 2 – Extra Practice

Evaluate the integral.

#5b. $\int e^{-3x}\sin(5x)\,dx$

$u=\sin(5x)$ $\quad dv=e^{-3x}dx$

$\frac{du}{dx}=5\cos(5x)$ $\quad \int dv=\int e^{-3x}dx$

$du=5\cos(5x)dx$ $\quad v=-\frac{1}{3}e^{-3x}$

$uv-\int v\,du$

$-\frac{1}{3}e^{-3x}\sin(5x)-\int\left(-\frac{1}{3}e^{-3x}\right)(5\cos(5x)dx)$

$-\frac{1}{3}e^{-3x}\sin(5x)+\frac{5}{3}\left[\int e^{-3x}\cos(5x)dx\right]$

$u=\cos(5x)$ $\quad dv=e^{-3x}dx$

$\frac{du}{dx}=-5\sin(5x)$ $\quad \int dv=\int e^{-3x}dx$

$du=-5\sin(5x)dx$ $\quad v=-\frac{1}{3}e^{-3x}$

$-\frac{1}{3}e^{-3x}\sin(5x)+\frac{5}{3}\left[uv-\int v\,du\right]$

$-\frac{1}{3}e^{-3x}\sin(5x)+\frac{5}{3}\left[-\frac{1}{3}e^{-3x}\cos(5x)-\int\left(-\frac{1}{3}e^{-3x}\right)(-5\sin(5x))dx\right]$

$-\frac{1}{3}e^{-3x}\sin(5x)+\frac{5}{3}\left[-\frac{1}{3}e^{-3x}\cos(5x)-\frac{5}{3}\underbrace{\int e^{-3x}\sin(5x)dx}_{\text{original problem!}}\right]$

$\int e^{-3x}\sin(5x)dx=-\frac{1}{3}e^{-3x}\sin(5x)-\frac{5}{9}e^{-3x}\cos(5x)-\frac{25}{9}\int e^{-3x}\sin(5x)dx$

$+\frac{25}{9}\int e^{-3x}\sin(5x)dx$ $\qquad +\frac{25}{9}\int e^{-3x}\sin(5x)dx$

$\frac{34}{9}\int e^{-3x}\sin(5x)dx=-\frac{1}{3}e^{-3x}\sin(5x)-\frac{5}{9}e^{-3x}\cos(5x)$

$\left(1+\frac{25}{9}\right)$

$\int e^{-3x}\sin(5x)dx=\boxed{\frac{9}{34}\left(-\frac{1}{3}e^{-3x}\sin(5x)-\frac{5}{9}e^{-3x}\cos(5x)\right)+C}$

#6b. $\int x^3 e^{-2x}\,dx$

sign	u	dv
+	x^3	e^{-2x}
−	$3x^2$	$-\frac{1}{2}e^{-2x}$
+	$6x$	$\frac{1}{4}e^{-2x}$
−	6	$-\frac{1}{8}e^{-2x}$
+	0	$\frac{1}{16}e^{-2x}$

$$\boxed{-\frac{1}{2}x^3e^{-2x}-\frac{3}{4}x^2e^{-2x}-\frac{3}{4}xe^{-2x}-\frac{3}{8}e^{-2x}+C}$$

#7b. $\int x^4\cos(x)\,dx$

sign	u	dv
+	x^4	$\cos x$
−	$4x^3$	$\sin x$
+	$12x^2$	$-\cos x$
−	$24x$	$-\sin x$
+	24	$\cos x$
−	0	$\sin x$

$$\boxed{x^4+4x^3\cos x-12x^2\sin x-24x\cos x+24\sin x+C}$$

#8 (hints):

for a) if $u=x^2,\ dv=\dfrac{x}{\sqrt{4+x^2}}dx$

The answers for parts a and b will look different:

a) $x^2\sqrt{4+x^2}-\frac{2}{3}\left(4+x^2\right)^{3/2}+C$

b) $\frac{1}{3}\left(4+x^2\right)^{3/2}-4\left(4+x^2\right)^{1/2}+C$

...but if you graph these in a calculator, they are the same curve :)

4.8 – Extra Practice

Evaluate:

#10b. $\int \sin^5 x \cos^2 x \, dx$

$u = \cos x$

$\frac{du}{dx} = -\sin x$

$du = -\sin x dx$

$\sin x dx = -du$

$\int \sin^4 x \cos^2 x \sin x dx$

$\int \sin^2 x \sin^2 x \cos^2 x \sin x dx$

$\int (1-\cos^2 x)(1-\cos^2 x)(\cos^2 x) \sin x dx$

$-\int (1-u^2)(1-u^2) u^2 du$

$-\int (1-2u^2+u^4)u^2 du$

$-\int (u^2-2u^4+u^6) du$

$-\frac{1}{3}u^3 + \frac{2}{5}u^5 - \frac{1}{7}u^7 + c$

$\boxed{-\frac{1}{3}\cos^3 x + \frac{2}{5}\cos^5 x - \frac{1}{7}\cos^7 x + C}$

#11b. $\int \sin^5 (3x) \cos(3x) \, dx$

$u = \cos(3x)$

$\frac{du}{dx} = -3\sin(3x)$

$du = -3\sin(3x)dx$

$\sin(3x)dx = -\frac{1}{3}du$

$\int \sin^2(3x) \sin^2(3x) \cos(3x) \sin(3x) dx$

$\int (1-\cos^2(3x))(1-\cos^2(3x))\cos(3x)\sin(3x)dx$

$\int (1-u^2)(1-u^2)u(-\frac{1}{3}du)$

$-\frac{1}{3}\int (1-2u^2+u^4)u\, du$

$-\frac{1}{3}\int (u-2u^3+u^5)\, du$

$-\frac{1}{3}(\frac{1}{2}u^2) + \frac{2}{3}(\frac{1}{4}u^4) - \frac{1}{3}(\frac{1}{6}u^6) + C$

$\boxed{-\frac{1}{6}\cos^2(3x) + \frac{1}{6}\cos^4(3x) - \frac{1}{18}\cos^6(3x) + C}$

#12b. $\int \sin^2 (9x) \, dx$

$u = 9x$

$\frac{du}{dx} = 9$

$du = 9dx$

$dx = \frac{1}{9}du$

$\frac{1}{9}\int \sin^2(u) du$

$\frac{1}{9}\int \frac{1-\cos(2u)}{2} du$

$\frac{1}{18}\int du - \frac{1}{18}\int \cos(2u) du$

$\frac{1}{18}u - \frac{1}{36}\sin(2u) + C$

$\frac{1}{18}(9x) - \frac{1}{36}\sin(2 \cdot 9x) + C$

$\boxed{\frac{1}{2}x - \frac{1}{36}\sin(18x) + C}$

#13b. $\int 7\csc(3x) \, dx$

$u = 3x$

$du = 3dx$

$dx = \frac{1}{3}du$

$\frac{7}{3}\int \csc(u) du$

$\frac{7}{3}\ln|\csc(u) - \cot(u)| + C$

$\boxed{\frac{7}{3}\ln|\csc(3x) - \cot(3x)| + C}$

#14b. $\int \cot^3(2x)\csc^3(2x)\,dx$

$u = \csc(2x)$

$\frac{du}{dx} = -\csc(2x)\cot(2x)\cdot 2$

$du = -2\csc(2x)\cot(2x)dx$

$\csc(2x)\cot(2x)\,dx = -\frac{1}{2}du$

$\int \cot^2(2x)\csc^2(2x)\csc(2x)\cot(2x)\,dx$

$\int (\csc^2(2x)-1)\csc^2(2x)\csc(2x)\cot(2x)\,dx$

$\int (u^2-1)u^2\left(-\frac{1}{2}du\right)$

$-\frac{1}{2}\int (u^4-u^2)\,du$

$-\frac{1}{2}\left(\frac{1}{5}u^5\right)+\frac{1}{2}\left(\frac{1}{3}u^3\right)+C$

$\boxed{-\frac{1}{10}\csc^5(2x)+\frac{1}{6}\csc^3(2x)+C}$

#15b. $\int 2\frac{\tan^2(3x)}{\sec(3x)}\,dx$

$2\int \frac{\sec^2(3x)-1}{\sec(3x)}\,dx$

$2\int \left(\frac{\sec^2(3x)}{\sec(3x)}-\frac{1}{\sec(3x)}\right)dx$

$2\int \sec(3x)\,dx - 2\int \cos(3x)\,dx$

$u=3x$

$du=3dx$

$dx=\frac{1}{3}dx$

$\boxed{\frac{2}{3}\ln|\sec(3x)+\tan(3x)|-\frac{2}{3}\sin(3x)+C}$

#16b. $\int \sec^4(7x)\,dx$

$u=\tan(7x)$

$\frac{du}{dx}=7\sec^2(7x)$

$\sec^2(7x)\,dx=\frac{1}{7}du$

$\int \sec^2(7x)\sec^2(7x)\,dx$

$\int (1+\tan^2(7x))\sec^2(7x)\,dx$

$\int (1+u^2)\left(\frac{1}{7}du\right)$

$\frac{1}{7}\int (1+u^2)\,du$

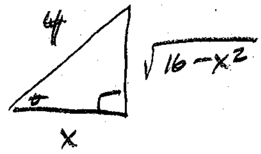
$\frac{1}{7}u+\frac{1}{7}\left(\frac{1}{3}u^3\right)+c$

$\boxed{\frac{1}{7}\tan(7x)+\frac{1}{21}\tan^3(7x)+C}$

4.9 – Extra Practice

Evaluate the integral.

#5b. $\int \frac{4}{x^2\sqrt{16-x^2}}\,dx$



$\cos\theta = \frac{x}{4}$ $\quad$ $\sin\theta = \frac{\sqrt{16-x^2}}{4}$

$x = 4\cos\theta$ $\quad$ $\sqrt{16-x^2} = 4\sin\theta$

$dx = -4\sin\theta\,d\theta$

$\tan\theta = \frac{\sqrt{16-x^2}}{x}$

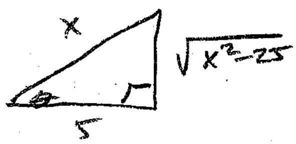
$\int \frac{4(-4)\sin\theta}{(4\cos\theta)^2\, 4\sin\theta}\,d\theta$

$-\frac{1}{4}\int \frac{1}{\cos^2\theta}\,d\theta = -\frac{1}{4}\int \sec^2\theta\,d\theta$

$= -\frac{1}{4}\tan\theta + C$

$= \boxed{-\frac{1}{4}\left(\frac{\sqrt{16-x^2}}{x}\right) + C}$

#6b. $\int \frac{x^3}{\sqrt{x^2-25}}\,dx$



$\cos\theta = \frac{5}{x}$ $\quad$ $\tan\theta = \frac{\sqrt{x^2-25}}{5}$

$x = \frac{5}{\cos\theta} = 5\sec\theta$ $\quad$ $\sqrt{x^2-25} = 5\tan\theta$

$dx = 5\sec\theta\tan\theta\,d\theta$

$\int \frac{(5\sec\theta)^3\, 5\sec\theta\tan\theta}{5\tan\theta}\,d\theta$

$125\int \sec^4\theta\,d\theta$

$125\int \sec^2\theta\sec^2\theta\,d\theta$ $\quad$ $u = \tan\theta$

$125\int (\tan^2\theta+1)\sec^2\theta\,d\theta$ $\quad$ $du = \sec^2\theta\,d\theta$

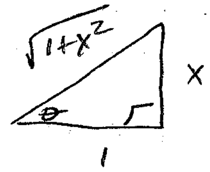
$125\int (u^2+1)\,du$

$\frac{125}{3}u^3 + 125u + C$

$\frac{125}{3}\tan^3\theta + 125\tan\theta + C$

$\boxed{\frac{125}{3}\left(\frac{\sqrt{x^2-25}}{5}\right)^3 + 125\left(\frac{\sqrt{x^2-25}}{5}\right) + C}$

#7b. $\int \frac{x^2}{\left(1+x^2\right)^2}\,dx$



$\tan\theta = \frac{x}{1}$

$x = \tan\theta$

$dx = \sec^2\theta\, d\theta$

$\cos\theta = \frac{1}{\sqrt{1+x^2}}$

$\sqrt{1+x^2} = \frac{1}{\cos\theta} = \sec\theta$

$1+x^2 = \sec^2\theta$

$\int \frac{\tan^2\theta}{(\sec^2\theta)^2}\sec^2\theta\, d\theta$

$\int \frac{\tan^2\theta}{\sec^2\theta}\,d\theta = \int \frac{\sin^2\theta}{\cos^2\theta}\cos^2\theta\, d\theta = \int \sin^2\theta\, d\theta$

$\theta = \arctan(x)$

$\int \left(\frac{1-\cos(2\theta)}{2}\right) d\theta$

$\sin(2\theta) = 2\sin\theta\cos\theta$

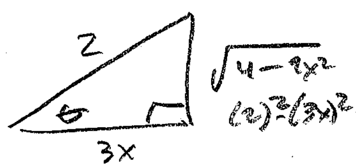
$\int \frac{1}{2}\,d\theta - \frac{1}{2}\int \cos(2\theta)\,d\theta$

$\frac{1}{2}\theta + \frac{1}{4}\sin(2\theta) + C$

$\frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta + C$

$\boxed{\frac{1}{2}\arctan(x) + \frac{1}{2}\left(\frac{x}{\sqrt{1+x^2}}\right)\left(\frac{1}{\sqrt{1+x^2}}\right) + C}$

#8b. $\int \sqrt{4-9x^2}\,dx$



$\cos\theta = \frac{3x}{2}$

$x = \frac{2}{3}\cos\theta$

$dx = -\frac{2}{3}\sin\theta\, d\theta$

$\theta = \arccos\left(\frac{3}{2}x\right)$

$\sin\theta = \frac{\sqrt{4-9x^2}}{2}$

$\sqrt{4-9x^2} = 2\sin\theta$

$\int 2\sin\theta\left(-\frac{2}{3}\sin\theta\right)d\theta$

$-\frac{4}{3}\int \sin^2\theta\, d\theta$

$-\frac{4}{3}\int \left(\frac{1-\cos(2\theta)}{2}\right) d\theta$

$-\frac{2}{3}\int d\theta + \frac{2}{3}\int \cos(2\theta)\,d\theta$

$\sin(2\theta) = 2\sin\theta\cos\theta$

$-\frac{2}{3}\theta + \frac{1}{3}\sin(2\theta) + C$

$-\frac{2}{3}\theta + \frac{2}{3}\sin\theta\cos\theta + C$

$\boxed{-\frac{2}{3}\arccos\left(\frac{3}{2}x\right) + \frac{2}{3}\left(\frac{\sqrt{4-9x^2}}{2}\right)\left(\frac{3}{2}x\right) + C}$

4.10 – Extra Practice

Evaluate the integral.

#8b. $\int \frac{2}{9x^2-1}dx$

$9x^2-1$

$(3x)^2-(1)^2$

a^2-b^2

$(3x-1)(3x+1)$

$\frac{2}{(3x-1)(3x+1)} = \frac{A}{3x-1} + \frac{B}{3x+1}$

(you can drop the denominator in this work...)

$A(3x+1)+B(3x-1)=2$

$3Ax+A+3Bx-B=2$

$(3A+3B)x+(A-B)=(0)x+(2)$

$\begin{cases} 3A+3B=0 \\ A-B=2 \end{cases}$

(no calculator on ~~test~~ — do by hand)

$3A+3B=0$

$3A-3B=6$

$6A=6$

$A=1$

$1-B=2$

$-B=1$

$B=-1$

$1\int \frac{1}{3x-1}dx - \int \frac{1}{3x+1}dx$

$\boxed{\ln|3x-1|-\ln|3x+1|+C}$

#9b. $\int \frac{3-x}{3x^2-2x-1}dx$

$3x^2-2x-1$

M A

-3 -2

$(-3)(1)$

$(3x+1)(3x-3)$

$\frac{1}{}\quad\frac{3}{}$

$(3x+1)(x-1)$

$\frac{3-x}{(3x+1)(x-1)} = \frac{A}{3x+1} + \frac{B}{x-1}$

$A(x-1)+B(3x+1)=3-x$

$Ax-A+3Bx+B=-x+3$

$(A+3B)x+(-A+B)=(-1)x+(3)$

$\begin{cases} A+3B=-1 \\ -A+B=3 \end{cases}$

$4B=2$

$B=\frac{1}{2}$

$-A+B=3$

$-A+\frac{1}{2}=3$

$-2A+1=6$

$-2A=5$

$A=-\frac{5}{2}$

$-\frac{5}{2}\int \frac{1}{3x+1}dx + \frac{1}{2}\int \frac{1}{x-1}dx$

$\boxed{-\frac{5}{2}\ln|3x+1| + \frac{1}{2}\ln|x-1|+C}$

#10b. $\int \frac{4x^2+3x+1}{x^3+2x^2-3x}\,dx$

x^3+2x^2-3x

$x(x^2+2x-3)$

$x(x+3)(x-1)$

$$\frac{4x^2+3x+1}{x^3+2x^2-3x} = \frac{A}{x} + \frac{B}{x+3} + \frac{C}{x-1}$$

$$= \frac{A(x+3)(x-1)}{x(x+3)(x-1)} + \frac{Bx(x-1)}{x(x+3)(x-1)} + \frac{Cx(x+3)}{x(x+3)(x-1)}$$ get a common denominator

$A(x+3)(x-1) + Bx(x-1) + Cx(x+3) = 4x^2+3x+1$

$A(x^2+2x-3)$

$Ax^2+2Ax-3A+Bx^2-Bx+Cx^2+3Cx = 4x^2+3x+1$

$(A+B+C)x^2+(2A-B+3C)x+(-3A) = (4)x^2+(3)x+(1)$

$$\begin{cases} A+B+C = 4 \\ 2A-B+3C = 3 \\ -3A = 1 \end{cases}$$ for systems this large, use RREF:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 2 & -1 & 3 & 3 \\ -3 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1/3 \\ 0 & 1 & 0 & 7/3 \\ 0 & 0 & 1 & 2 \end{array}\right] \begin{array}{l} =A \\ =B \\ =C \end{array}$$

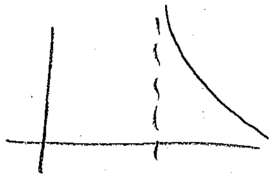
$$-\frac{1}{3}\int \frac{1}{x}dx + \frac{7}{3}\int \frac{1}{x+3}dx + 2\int \frac{1}{x-1}dx$$

$$\boxed{-\frac{1}{3}\ln|x| + \frac{7}{3}\ln|x+3| + 2\ln|x-1| + C}$$

4.11 – Extra Practice

Evaluate the integral or state that it diverges.

#7b. $\int_3^4 \frac{1}{(x-3)^{3/2}}\,dx$



$x=3$

$\lim_{b\to 3^+} \int_b^4 (x-3)^{-3/2}\,dx$

$\lim_{b\to 3^+} \left[-2(x-3)^{-1/2}\right]_b^4 = \lim_{b\to 3^+} \left[\frac{-2}{\sqrt{x-3}}\right]_b^4$

$= \lim_{b\to 3^+} \left[\frac{2}{\sqrt{4-3}} - \frac{2}{\sqrt{b-3}}\right]$

$= \frac{2}{1} - 2\left[\lim_{b\to 3^+} \frac{1}{\sqrt{b-3}}\right]$ ∞

$\boxed{\text{diverges}}$

#8b. $\int_{-\infty}^0 e^{3x}\,dx$

$\lim_{b\to -\infty^+} \int_b^0 e^{3x}\,dx$

$\lim_{b\to -\infty^+} \left[\frac{1}{3}e^{3x}\right]_b^0$

$\frac{1}{3}e^{3(0)} - \lim_{b\to -\infty^+}\left[\frac{1}{3}e^{3b}\right]$

$\frac{1}{3}e^{(0)} - 0$

$\frac{1}{3}(1)$

$\boxed{\frac{1}{3}}$

#9b. $\int_1^{\infty} \frac{6}{x^4}\,dx = \lim_{b\to\infty} 6\int_1^b x^{-4}\,dx$

$6\lim_{b\to\infty}\left[\frac{x^{-3}}{-3}\right]_1^b$

$6\lim_{b\to\infty}\left[-\frac{3}{x^3}\right]_1^b$

$6\lim_{b\to\infty}\left[\frac{-3}{b^3}-\frac{-3}{1^3}\right]$

$6[0+3]$

$\boxed{18}$

#10b. $\int_0^{\infty} e^{\left(\frac{1}{3}x\right)}\,dx = \lim_{b\to\infty}\int_0^b e^{(1/3x)}\,dx$

$=\lim_{b\to\infty}\left[\frac{e^{1/3x}}{(1/3)}\right]_0^b$

$=\lim_{b\to\infty}\left[3e^{1/3(b)}-3e^{1/3(0)}\right]$

∞

$\boxed{\text{diverges}}$