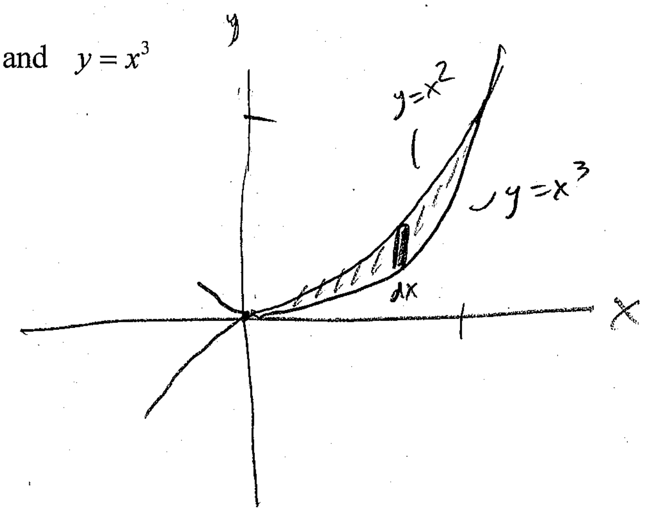
AP Calculus BC – Unit 5, Part 1 Extra Practice

5.1 – Extra Practice

On #4b and #5b, sketch the region by hand (no calculator) and find the area enclosed by the curves (integrate by hand).

#4b. $y = x^2$ and $y = x^3$



Intersections

$\begin{cases} y = x^2 \\ y = x^3 \end{cases}$

$x^3 = x^2$

$x^3 - x^2 = 0$

$x^2(x-1) = 0$

$x = 0 \quad x = 1$

$y = 0 \quad y = 1$

$(0,0) \quad (1,1)$

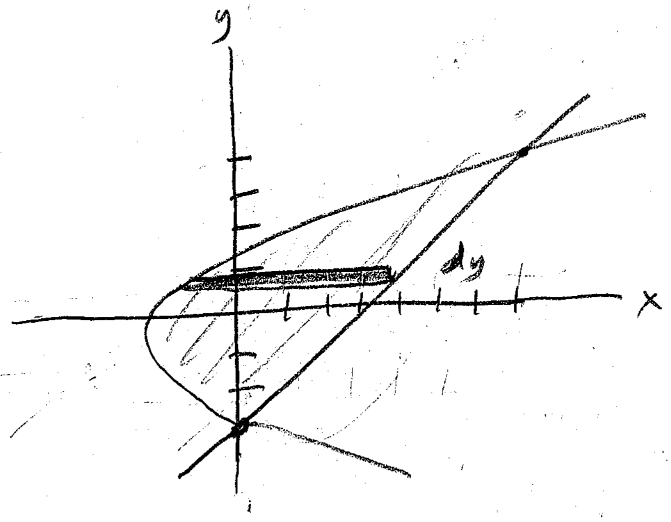
$$\boxed{A = \int_0^1 [x^2 - x^3]\,dx}$$

$$= \left[\tfrac{1}{3}x^3 - \tfrac{1}{4}x^4\right]_0^1$$

$$= \boxed{\left[\tfrac{1}{3}(1)^3 - \tfrac{1}{4}(1)^4\right] - \left[\tfrac{1}{3}(0)^3 - \tfrac{1}{4}(0)^4\right]}$$

On #4b and #5b, sketch the region by hand (no calculator) and find the area enclosed by the curves (integrate by hand).

#5b. $x = -9 + y^2$ and $x = y + 3$



Intersections

$\begin{cases} x = -9+y^2 \\ x = y+3 \end{cases}$

$-9+y^2 = y+3$

$y^2 - y - 12 = 0$

$(y-4)(y+3) = 0$

$y=4 \quad y=-3$

$x=7 \quad x=0$

$(7,4) \quad (0,-3)$

$$\boxed{A = \int_{-3}^{4} [(y+3) - (-9+y^2)]\, dy}$$

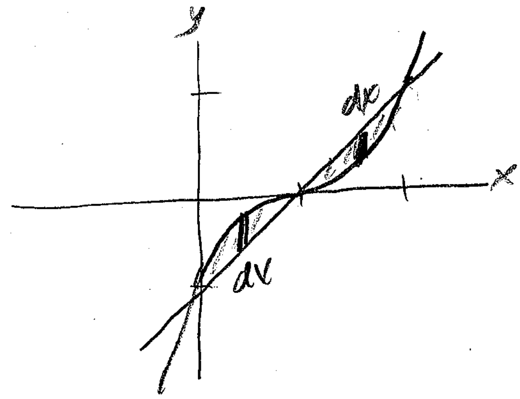
$$= \int_{-3}^{4} [y + 12 - y^2]\, dy$$

$$= \left(\frac{1}{2}y^2 + 12y - \frac{1}{3}y^3\right]_{-3}^{4}$$

$$= \boxed{\left[\frac{1}{2}(4)^2 + 12(4) - \frac{1}{3}(4)^3\right] - \left\{\frac{1}{2}(-3)^2 + 12(-3) - \frac{1}{3}(-3)^3\right\}}$$

On the rest of this assignment, sketch the curves and find the area enclosed (use your calculator for the sketch and the integral evaluation).

#6b. $y=(x-1)^3$ and $y=x-1$

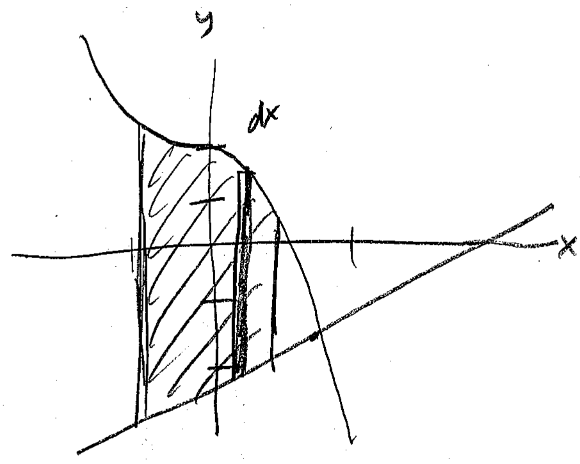


$$\boxed{A=\int_0^1 [(x-1)^3-(x-1)]\,dx+\int_1^2 [(x-1)-(x-1)^3]\,dx}$$

$= .25 + .25$

$\boxed{=.5}$

#7b. $y=-x^3+2$, $y=x-3$, $x=-1$, and $x=1$

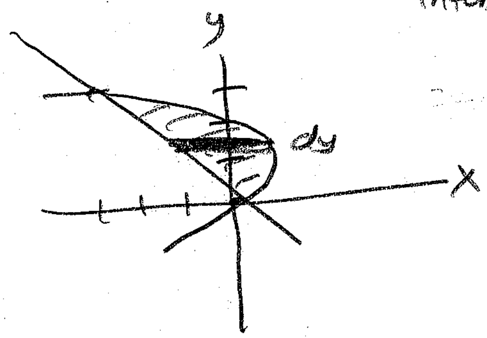


$$\boxed{A=\int_{-1}^{1} [(-x^3+2)-(x-3)]\,dx}$$

$\boxed{=10}$

#8b. $x = 2y - y^2$ and $x = -y$

I find it easier to graph these by hand:



intersections: $\begin{cases} x = 2y - y^2 \\ x = -y \end{cases}$

$-y = 2y - y^2$

$y^2 - 3y = 0$

$y(y-3) = 0$

$y = 0 \quad y = 3$

$x = 0 \quad x = -3$

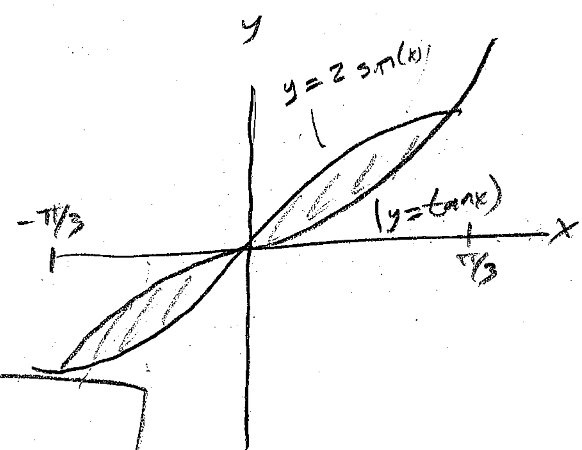
$(0,0) \quad (-3,3)$

$$\boxed{A = \int_0^3 [(2y - y^2) - (-y)]\,dy}$$

$$= \int_0^3 (3y - y^2)\,dy$$

$$\boxed{= 4.5}$$

#9b. $y = 2\sin(x)$ and $y = \tan(x)$ $\quad -\frac{\pi}{3} \le x \le \frac{\pi}{3}$



$$\boxed{A = \int_{-\pi/3}^{0} [\tan(x) - 2\sin(x)]\,dx + \int_0^{\pi/3} [2\sin(x) - \tan(x)]\,dx}$$

$$= .3068528194 + .3068528194$$

$$\boxed{= 0.614}$$

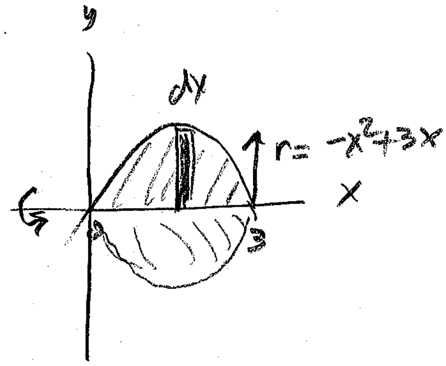
5.2 – Extra Practice

Sketch and find the volume (use your calculator for the sketch and the integral evaluation).

#7b. $y=-x^2+3x,\quad y=0\quad$ *about the x – axis*

$$V=\int_a^b \pi r^2\,dh$$

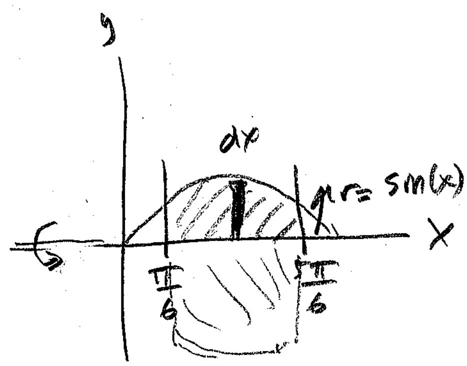
$$=\boxed{\int_0^3 \pi\left(-x^2+3x\right)^2 dx = 25.447}$$



#7c. $y=\sin(x),\ x=\frac{\pi}{6},\ x=\frac{5\pi}{6},\ y=0\quad$ *about the x – axis*

$$V=\int_a^b \pi r^2\,dh$$

$$\boxed{\int_{\pi/6}^{5\pi/6} \pi(\sin(x))^2 dx = 4.650}$$



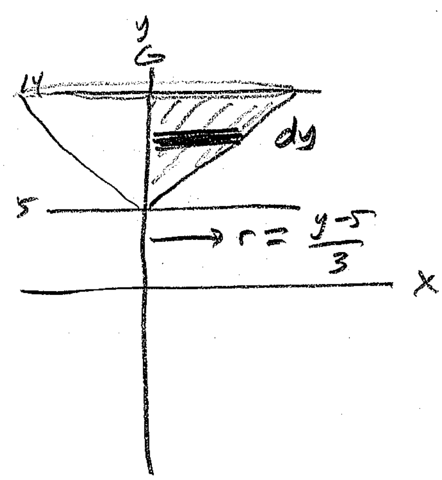
#8b. $y=3x+5,\ y=5,\ y=14,\ x=0\quad$ *about the y – axis*

$3x=y-5$

$x=\frac{y-5}{3}$

$$V=\int_a^b \pi r^2\,dh$$

$$\boxed{\int_5^{14} \pi\left(\frac{y-5}{3}\right)^2 dy = 84.823}$$



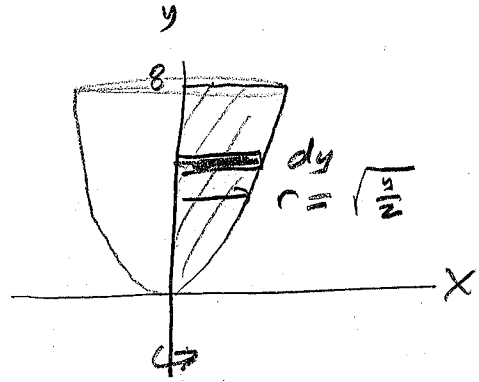
#9b. $y=2x^2,\ y=8,\ x=0$ *about the* $y-axis$

$x^2 = \frac{y}{2}$

$x = \pm\sqrt{\frac{y}{2}}$

$V = \int_a^b \pi r^2 \, dh$

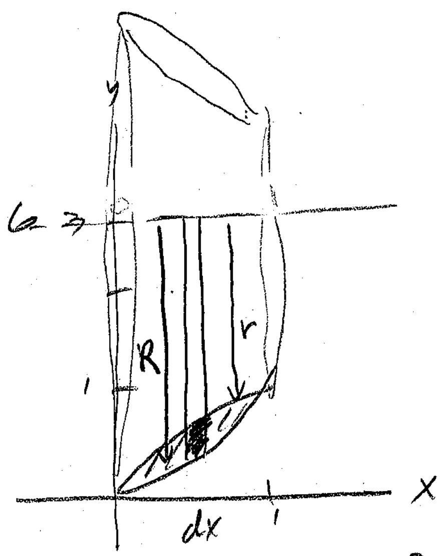
$$\boxed{\int_0^8 \pi\left(\sqrt{\frac{y}{2}}\right)^2 dy = 50.265}$$



#10b. $y=x^2,\ y=\sqrt{x}$ *around* $y=3$

$V = \int_a^b \pi R^2 \, dh - \int_a^b \pi r^2 \, dh$

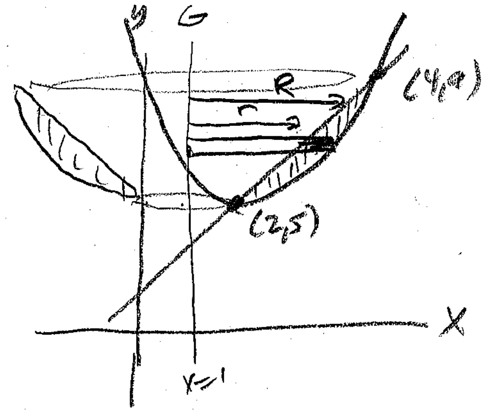
$$\boxed{\int_0^1 \pi(3-x^2)^2 dx - \int_0^1 \pi(3-\sqrt{x})^2 dx = 5.341}$$



$r + \sqrt{x} = 3 \rightarrow r = 3 - \sqrt{x}$

$R + x^2 = 3 \rightarrow R = 3 - x^2$

#11b. $y = x^2 - 4x + 9, \ y = 2x + 1 \quad around \ x = 1$



$y = 2x+1 \rightarrow 2x = y-1, \ x = \frac{y-1}{2}$

$y = x^2 - 4x + 9 \rightarrow x^2 - 4x^{+4} = y - 9^{+4}$

$(x-2)^2 = y-5$

$x - 2 = \pm\sqrt{y-5}$

$x = 2 \pm \sqrt{y-5}$

(solving for x requires completing the square)

$V = \int_a^b \pi R^2 dh - \int_a^b \pi r^2 dh$

$r + 1 = \frac{y-1}{2}$

so $r = \left(\frac{y-1}{2}\right) - 1$

$R + 1 = (2 + \sqrt{y-5})$

so $R = (2 + \sqrt{y-5}) - 1$

$$\boxed{\int_5^9 \pi\left[(2+\sqrt{y-5})-1\right]^2 dy - \int_5^9 \pi\left[\left(\frac{y-1}{2}\right)-1\right]^2 dy = 16{,}755}$$

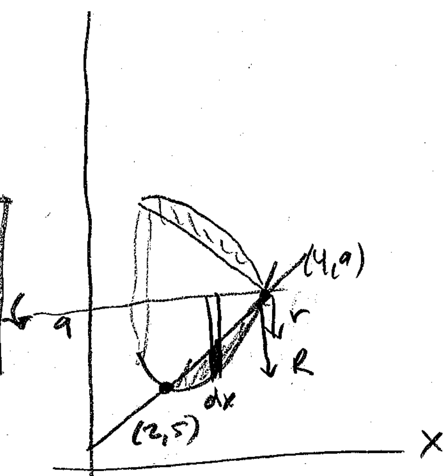
$71.20843395 - 54.45427266$

#12b. $y = x^2 - 4x + 9, \ y = 2x + 1 \quad around \ y = 9$

$V = \int_a^b \pi R^2 dh - \int_a^b \pi r^2 dh$

$$\boxed{\int_2^4 \pi\left[9 - (x^2 - 4x + 1)\right]^2 dx - \int_2^4 \pi\left[9 - (2x+1)\right]^2 dx = 20.106}$$

$53.61651462 - 33.51032164$



$r + (2x+1) = 9$

so $r = 9 - (2x+1)$

$R + (x^2 - 4x + 9) = 9$

so $R = 9 - (x^2 - 4x + 9)$

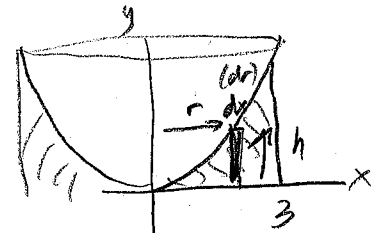
5.3 – Extra Practice

Sketch and find the volume using shell method (use your calculator for the sketch and the integral evaluation).

#7b. $y = x^2 + 3x, \quad x = 0, \quad x = 3, \quad y = 0 \quad$ *around the y – axis*

$$V = \int_a^b 2\pi r h \, dr$$

$$\boxed{\int_0^3 2\pi(x)(x^2+3x)\,dx = 296.881}$$



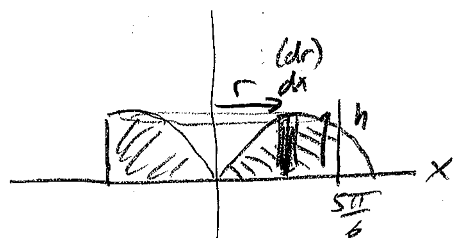
$r = x$

$h = x^2 + 3x$

#7c. $y = \sin(x), \quad x = 0, \quad x = \frac{5\pi}{6}, \quad y = 0 \quad$ *around the y – axis*

$$V = \int_a^b 2\pi r h \, dr$$

$$\boxed{\int_0^{5\pi/6} 2\pi(x)(\sin x)\,dx = 17.387}$$



$r = x$

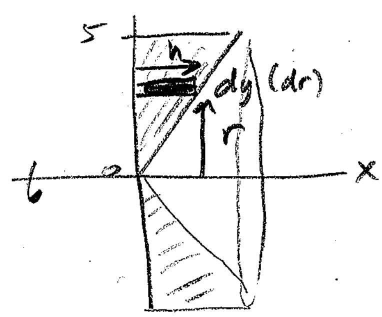
$h = \sin(x)$

#8b. $y = 3x, \quad y = 5, \quad x = 0 \quad$ *around the x – axis*

$x = \frac{y}{3}$

$$V = \int_a^b 2\pi r h \, dr$$

$$\boxed{\int_0^5 2\pi(y)\left(\frac{y}{3}\right)dy = 87.266}$$



$h = \frac{y}{3}$

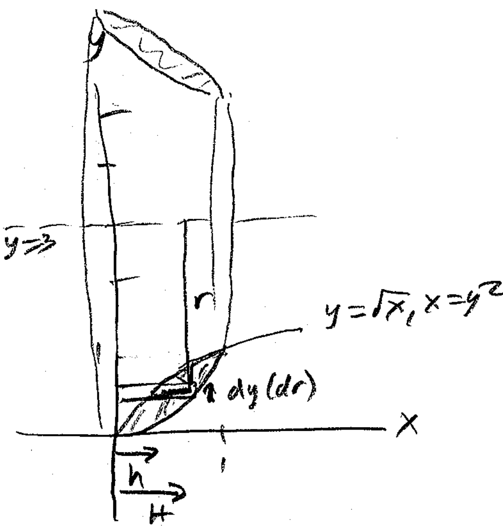
$r = y$

#9b. $y = x^2$, $y = \sqrt{x}$ *around* $y = 3$

$x = \pm\sqrt{y}$ $x = y^2$

$$V = \int_a^b 2\pi r H\,dr - \int_a^b 2\pi r h\,dr$$

$$\boxed{\int_0^1 2\pi(3-y)(\sqrt{y})\,dy - \int_0^1 2\pi(3-y)(y^2)\,dy = 5.341}$$

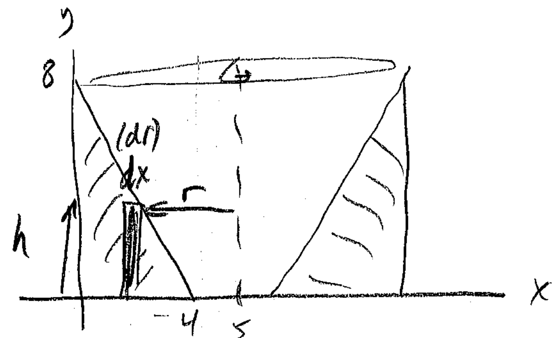


$h = y^2$
$H = \sqrt{y}$
$r + y = 3$
so $r = 3 - y$

#10b. $y = -2x + 8$, $y = 0$, $x = 0$ *around* $x = 5$

$$V = \int_a^b 2\pi r h\,dr$$

$$\boxed{\int_0^4 2\pi(5-x)(-2x+8)\,dx = 368.614}$$



$h = -2x + 8$
$r + x = 5$
so $r = 5 - x$

#11b. $y = x^3, \quad y = 0, \quad x = 2 \qquad around\ x = 4$

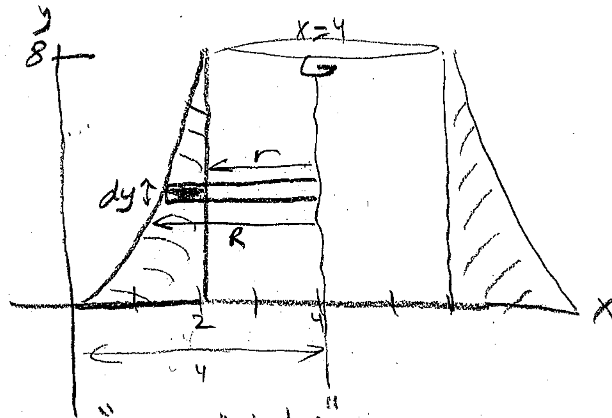
a) using Disk method...

$y = x^3 \Rightarrow x = \sqrt[3]{y}$

$$V = \int_a^b \pi R^2 dh - \int_a^b \pi r^2 dh$$

$$\boxed{\int_0^8 \pi(4 - \sqrt[3]{y})^2 dy - \int_0^8 \pi(2)^2 dy = 60.319}$$

$160.8495433 - 100.5309649$



"peperndiskular,"
growing from rotation axis

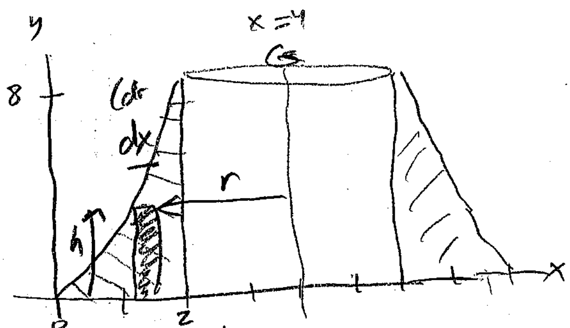
$r+2=4$ so $r=4-2=2$

$R+\sqrt[3]{y}=4$ so $R=4-\sqrt[3]{y}$

b) using Shell method...

$$V = \int_a^b 2\pi r h\, dr$$

$$\boxed{\int_0^2 2\pi(4-x)(x^3)\, dx = 60.319}$$



"parashell"
growing from a coordinate axis

$h = x^3$

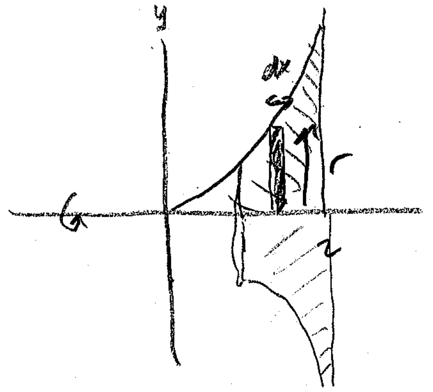
$r+x=4$ so $r=4-x$

#12b. $y=x^2$, $y=0$, $x=1$, $x=2$ *around the x-axis*

a) using Disk method...

$$V=\int_a^b \pi r^2\,dh$$

$$\boxed{\int_1^2 \pi(x^2)^2\,dx = 19.478}$$



"perpendiskular" rectangle grows from rotation axis

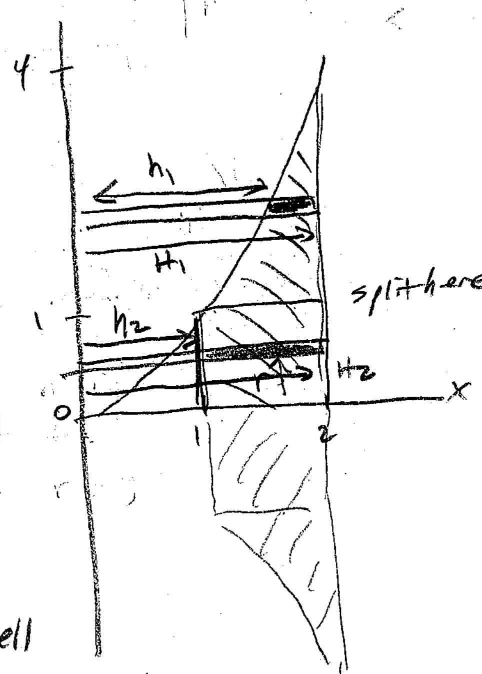
b) using Shell method...

"parashell" rectangle grows from a coordinate axis

$$y=x^2 \to x=\pm\sqrt{y}$$

$$V=\left[\int_a^b 2\pi r H_1\,dr-\int_a^b 2\pi r h_1\,dr\right]+\left[\int_a^b 2\pi r H_2\,dr-\int_a^b 2\pi r h_2\,dr\right]$$

$$\boxed{\begin{gathered}\left[\int_1^4 2\pi(y)(2)\,dy-\int_1^4 2\pi(y)(\sqrt{y})\,dy\right]+\left[\int_0^1 2\pi(y)(2)\,dy-\int_0^1 2\pi(y)(1)\,dy\right]\\ \left[94.24777961-77.91149781\right]+\left[6.283185307-3.141592654\right]\\ =19.478\end{gathered}}$$



$r=y$

$h_1=\sqrt{y}$, $H_1=2$

$h_2=1$, $H_2=2$

(only disk method appears on the Alexam) but our unit test includes disk and shell