AP Calculus BC – Unit 6Extra Practice

6.1 – Extra Practice

Determine whether or not the given solution is a solution to the differential equation.

#6b. *Differential Equation*: $y' = \dfrac{2xy}{x^2 - y^2}$

Solution: $x^2 + y^2 = Cy$

$x^2+y^2=Cy$
(implicit differentiation)

$2x+2y\frac{dy}{dx}=C\frac{dy}{dx}$

$(2y-C)\frac{dy}{dx}=-2x$

$y'=\frac{dy}{dx}=\frac{-2x}{2y-C}$

also: $x^2+y^2=Cy$
so $C=\frac{x^2+y^2}{y}$
substitute into

$y'=\frac{-2x}{2y-\left(\frac{x^2+y^2}{y}\right)}$

$y'=\frac{-2xy}{2y^2-(x^2+y^2)}$

$y'=\frac{-2xy}{2y^2-x^2-y^2}=\frac{-2xy}{y^2-x^2}$

$y'=\frac{-2xy}{-(x^2-y^2)}=\frac{2xy}{x^2-y^2}$

matches RHS

$\therefore$ yes, $x^2+y^2=Cy$ is a solution to $y'=\frac{2xy}{x^2-y^2}$

#7b. *Differential Equation*: $y' = -12xy$

Solution: $y = 4e^{-6x^2}$

$y'=4(e^{-6x^2})(-12x)=-48xe^{-6x^2}$

$y'=-12xy$

$[-48xe^{-6x^2}]\overset{?}{=}-12x[4e^{-6x^2}]$

$-48xe^{-6x^2}=-48xe^{-6x^2}$ ✓

yes, $y=4e^{-6x^2}$ is a solution to $y'=-12xy$

#8b. *Differential Equation*: $xy' - 2y = x^3e^x$

Solution: $y = x^2$

$y'=2x$

$xy'-2y=x^3e^x$

$x[2x]-2[x^2]\overset{?}{=}x^3e^x$

$2x^2-2x^2\overset{?}{=}x^3e^x$

$0\neq x^3e^x$

No, $y=x^2$ is not a solution to $xy'-2y=x^3e^x$

#9b. Verify the given solution is a solution to the differential equation. Then use the initial condition to find the particular solution.

Differential Equation: $3x+2yy'=0$

Solution: $3x^2+2y^2=C$

Initial condition: $y(1)=3$

$3x^2+2y^2=c$
(implicit)

$6x+4y\frac{dy}{dx}=0$

$\frac{dy}{dx}=y'=\frac{-6x}{4y}$

$3x+2yy'\overset{?}{=}0$

$3x+2y\left[\frac{-6x}{4y}\right]\overset{?}{=}0$

$3x-\frac{12x}{4}\overset{?}{=}0$

$3x-3x\overset{?}{=}0$

$0=0$ ✓

(verified)

$3x^2+2y^2=c$

$3(1)^2+2(3)^2=c$

$3+18=c$

$c=21$

$\boxed{3x^2+2y^2=21}$

Use integration to find the general solution of the differential equation.

#10b. $\frac{dy}{dx}=10x^4-2x^3$

$y=\int(10x^4-2x^3)dx$

$\boxed{y=2x^5-\frac{1}{2}x^4+c}$

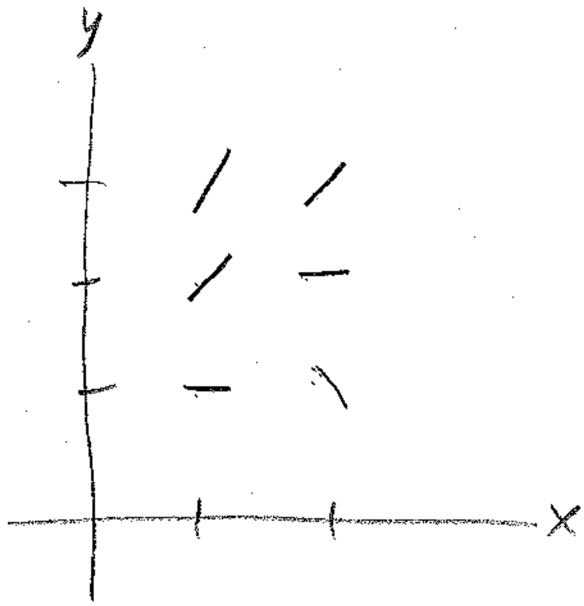
#11b. $y'=4\sec^2(x)$

$y=4\int\sec^2 x\,dx$

$\boxed{y=4\tan(x)+c}$

#12b. Sketch a slope field for the differential equation $\frac{dy}{dx} = y - x$. Include at least 6 points in the first quadrant.

(x,y)	$\frac{dy}{dx} = y - x$
$(1,1)$	0
$(1,2)$	$2-1=1$
$(1,3)$	$3-1=2$
$(2,1)$	$1-2=-1$
$(2,2)$	$2-2=0$
$(2,3)$	$3-2=1$



#13 (hint): Integrate to find the general solution for each of the differential equations, then match the curve shapes of those solutions to the slope fields.

6.2 – Extra Practice

#3b. Use Euler's method to approximate $y(0.4)$ where $y(x)$ is the solution to $y' = e^{xy}$ if $y(0) = 1$.
(use 4 equal size steps)

(x,y)	$y_{n+1} = y_n + [e^{xy}]0.1$
$(0,1)$	$y = 1 + [e^{(0)(1)}]0.1 = 1.1$
$(0.1, 1.1)$	$y = 1.1 + [e^{(0.1)(1.1)}]0.1 = 1.211627807$
$(0.2, 1.2116...)$	$y = 1.2162... + [e^{(0.2)(1.2116...)}]0.1 = 1.339048703$
$(0.3, 1.339...)$	$y = 1.339... + [e^{(0.3)(1.339...)}]0.1 = 1.488487182$
$(0.4, 1.4884...)$	

$\boxed{y(0.4) \approx 1.488}$

#4b. The table gives values of $f'(x)$, the derivative of a function $f(x)$. If $f(3) = 7$ what is the approximation of $f(5)$ obtained by using Euler's method with a step size of 0.5?

x	$f'(x)$
3	-0.8
3.5	-0.6
4	-0.4
4.5	-0.3

(x,y)	$y_{n+1} = y_n + [f'(x)]0.5$
$(3,7)$	$y = 7 + [-0.8]0.5 = 6.6$
$(3.5, 6.6)$	$y = 6.6 + [-0.6]0.5 = 6.3$
$(4, 6.3)$	$y = 6.3 + [-0.4]0.5 = 6.1$
$(4.5, 6.1)$	$y = 6.1 + [-0.3]0.5 = 5.95$
$(5, 5.95)$	

$\boxed{f(5) \approx 5.950}$

6.3 – Extra Practice

Find the general solution of the differential equation. If an initial condition is given also find the particular solution.

#6b. $\frac{dr}{ds} = 3.7rs$

$\frac{1}{r}dr = 3.7s\,ds$

$\int \frac{1}{r}dr = \int 3.7s\,ds$

$\ln|r| = \frac{3.7}{2}s^2 + c_1$

$|r| = e^{(\frac{3.7}{2}s^2 + c_1)} = e^{\frac{3.7}{2}s^2} e^{c_1}$

also a constant

$|r| = Ce^{\frac{3.7}{2}s^2}$

$\boxed{r = Ce^{\frac{3.7}{2}s^2}}$

#7b. $xy' = y$

$x\frac{dy}{dx} = y$

$\frac{1}{y}dy = \frac{1}{x}dx$

$\int \frac{1}{y}dy = \int \frac{1}{x}dx$

$\ln|y| = \ln|x| + c_1$

$e^{\ln|y|} = e^{(\ln|x| + c_1)} = e^{\ln|x|}e^{c_1}$

$|y| = C|x|$

$\boxed{y = C|x|}$

#8b. $12yy' - 7e^x = 0$ (constants can be on either side, might as well leave them so no fractions)

$12y\frac{dy}{dx} = 7e^x$

$12y\,dy = 7e^x dx$

$\int 12y\,dy = \int 7e^x dx$

$6y^2 = 7e^x + c$ ← this constant is the first constant ÷6 but no need to keep renaming it :)

$\boxed{y^2 = \frac{7}{6}e^x + c}$

implicit, general solution

continuing:

$\boxed{y = \pm\sqrt{\frac{7}{6}e^x + c}}$

this is the explicit, general solution (but the ± means it isn't a function, so in this case we usually leave the answer in implicit form)

#9b. $\sqrt{x} + \sqrt{y}y' = 0,\ y(1) = 9$ plug in to find a particular solution

$\sqrt{y}\frac{dy}{dx} = -\sqrt{x}$

$y^{1/2}dy = -x^{1/2}dx$

$\int y^{1/2}dy = -\int x^{1/2}dx$

$\frac{2}{3}y^{3/2} = -\frac{2}{3}x^{3/2} + c$ ← use $y(1)=9$ now

$\frac{2}{3}(9)^{3/2} = -\frac{2}{3}(1)^{3/2} + c$

$18 = -\frac{2}{3} + c \rightarrow c = \frac{56}{3}$

$\frac{2}{3}y^{3/2} = -\frac{2}{3}x^{3/2} + \frac{56}{3}$ (multiply by $\frac{3}{2}$)

$y^{3/2} = -x^{3/2} + 28$

$\boxed{y = (-x^{3/2} + 28)^{2/3}}$

#10b. $\frac{dr}{ds}=e^{r-2s}, \quad r(0)=0$

$dr=e^{r}e^{-2s}ds$

$\frac{1}{e^{r}}dr=e^{-2s}ds$

$e^{-r}dr=e^{-2s}ds$

$\int e^{-r}dr=\int e^{-2s}ds$

$-e^{-r}=-\frac{1}{2}e^{-2s}+c$

$e^{-r}=\frac{1}{2}e^{-2s}+c$

$-r=\ln(\frac{1}{2}e^{-2s}+c)$

$r=-\ln(\frac{1}{2}e^{-2s}+c)$

technically, $r=-\ln|\frac{1}{2}e^{-2s}+c|$ to avoid $\ln(-)$

$r=-\ln|\frac{1}{2}e^{-2s}+c| \quad r(0)=0$

$0=-\ln|\frac{1}{2}e^{0}+c|$

$0=-\ln|\frac{1}{2}+c|$

$0=\ln|\frac{1}{2}+c|$

$e \quad e$

$1=\frac{1}{2}+c \rightarrow c=\frac{1}{2}$

$\boxed{r=-\ln|\frac{1}{2}e^{-2s}+\frac{1}{2}|}$

#11b. $dP-kPdt=0, \quad P(0)=P_o$ (*a constant*)

$dP=kPdt$

$\frac{1}{P}dP=kdt$

$\int\frac{1}{P}dP=\int kdt$

$\ln|P|=kt+c_1$

$e^{\ln|P|}=e^{(kt+c_1)}=e^{kt}\cdot e^{c_1}=Ce^{kt}$

$|P|=Ce^{kt}$

$P=Ce^{kt}$

$P=Ce^{kt} \quad P(0)=P_0$

$P_0=Ce^{k(0)}=C \rightarrow C=P_0$

$\boxed{P=P_0e^{kt}}$

(the model for unrestricted exponential growth)

#12 (hints): *k* is a constant in this problem – first, solve the differential equation by finding the general equation (which should contain the integration constant *C* as well as the problem's constant *k*. Then use the initial condition $w(0)=60$ to solve for *C* (you'll still have a *k* in your particular solution equation). In (d) to find maximum weight...make sure that your graphs go far enough out in time to see the end behavior of the curves...they should be approaching a horizontal asymptote.

6.4 – Extra Practice

#3b. Write and solve a differential equation for the statement "the rate of change of P with respect to t is inversely proportional to the square root of $25-t$."

$$\boxed{\frac{dP}{dt} = k\frac{1}{\sqrt{25-t}}}$$

$$\int dP = \int k \frac{1}{\sqrt{25-t}}\,dt$$

$u = 25-t$

$\frac{du}{dt} = -1$

$du = -dt$

$$P = -k\int u^{-1/2}\,du$$

$$P = -k\,2u^{1/2} + c$$

$$\boxed{P = -2k\sqrt{25-t} + C}$$

#4b. a) Write and solve a differential equation for the statement "the rate of change of P is proportional to P."

b) If $P = 5000$ when $t = 0$ and $P = 4750$ when $t = 1$ what is the value of P when $t = 5$?

a) $\boxed{\frac{dP}{dt} = kP}$

$\int \frac{1}{P}\,dP = \int k\,dt$

$\ln|P| = kt + c$

$|P| = e^{(kt+c)} = e^{kt}e^{c}$

$\boxed{P = Ce^{kt}}$

b)

t	P
0	5000
1	4750
5	?

$P = Ce^{kt}$

$5000 = Ce^{k(0)} \rightarrow C = 5000$

$P = 5000e^{kt}$

$4750 = 5000e^{k(1)}$, $e^{k} = \frac{4750}{5000}$

$k = \ln\left(\frac{4750}{5000}\right)$

$$P = 5000e^{\left[\ln\left(\frac{4750}{5000}\right)\right]t}$$

$$P(5) = 5000e^{\left[\ln\left(\frac{4750}{5000}\right)\right](5)}$$

$$= \boxed{3868.905}$$

#5b. Radioactive element ^{226}Ra has a half-life of 1599 years. If the amount remaining after 10000 years is 0.262 grams, what was the initial quantity?

(yrs) t	Q (g)
0	Q_0
1599	$\frac{1}{2}Q_0$
10000	0.262

$$Q = Q_0 e^{kt}$$

$$\tfrac{1}{2}Q_0 = Q_0 e^{k(1599)}$$

$$\tfrac{1}{2} = e^{k(1590)}$$

$$\ln(\tfrac{1}{2}) = k(1590)$$

$$k = \frac{\ln(\frac{1}{2})}{1590}$$

$$Q = Q_0 e^{\left[\frac{\ln(\frac{1}{2})}{1599}\right]t}$$

$$0.262 = Q_0\, e^{\left[\frac{\ln(\frac{1}{2})}{1599}\right]\cdot 10000}$$

$$Q_0 = \frac{0.262}{e^{\left(\left[\frac{\ln(\frac{1}{2})}{1599}\right]\cdot 10000\right)}} = \boxed{19{,}995 \text{ grams}}$$

#6b. An investment has interest which compounds continuously with an annual interest rate of 8%. How long does it take for the amount invested to double?

$$A = Pe^{rt}$$

$$A = Pe^{.08t}$$

$$2P = Pe^{.08t}$$

$$2 = e^{.08t}$$

$$\ln(2) = .08t$$

$$t = \frac{\ln(2)}{.08} = \boxed{8.664 \text{ years}}$$

6.5 – Extra Practice

#5 (hints): For each given logistic equation, find $y(0)$. To separate the two curves with $y(0)=6$, remember that e^{-2t} transitions more rapidly than e^{-t}.

#6b. $P(t)=\dfrac{6000}{1+4999e^{-0.8t}}$ models the growth of a population.

a) What is the carrying capacity and constant *k*, for this model?
b) Find the initial population.
c) Determine when the population will reach 50% of its carrying capacity.
d) Write the differential equation for which the given $P(t)$ is the solution.

a) $\boxed{L=6000}$ $\boxed{k=0.8}$

b) $P(0)=\dfrac{6000}{1+4999e^{0}}=\boxed{1.2}$

c) $3000=\dfrac{6000}{1+4999e^{-0.8t}}$

$1+4999e^{-0.8t}=\dfrac{6000}{3000}=2$

$4999e^{-0.8t}=1$

$e^{-0.8t}=\dfrac{1}{4999}$

$-0.8t=\ln\left(\dfrac{1}{4999}\right)$

$t=\dfrac{\ln\left(\frac{1}{4999}\right)}{-0.8}=\boxed{10.646}$ (years?)

d) $\dfrac{dP}{dt}=kP\left(1-\dfrac{P}{L}\right)$

$$\boxed{\dfrac{dP}{dt}=0.8P\left(1-\dfrac{P}{6000}\right)}$$

#7b. $\frac{dP}{dt} = 0.1P - 0.0004P^2$ models the growth of a population.

a) What is the carrying capacity and constant *k*, for this model?

b) Solve the differential equation given $P(0) = 10$

c) Determine the value of *P* at which the population growth rate is the greatest.

a) $\frac{dP}{dt} = 0.1P - 0.0004P^2$

$\frac{dP}{dt} = 0.1P\left(1 - \frac{0.0004}{0.1}P\right)$

$\frac{dP}{dt} = 0.1P\left(1 - \frac{P}{\left(\frac{0.1}{0.0004}\right)}\right)$ $\boxed{L = 250}$

$\frac{dP}{dt} = 0.1P\left(1 - \frac{P}{250}\right)$ $\boxed{k = 0.1}$

b) standard solution: $P = \frac{L}{1 + Ce^{-kt}}$

t	P
0	10

$P = \frac{250}{1 + Ce^{-0.1t}}$

$10 = \frac{250}{1 + Ce^{0}} = \frac{250}{1 + C}$, $1 + C = \frac{250}{10} = 25$

$C = 24$

$$\boxed{P(t) = \frac{250}{1 + 24e^{-0.1t}}}$$

c) greatest growth rate when $P = \frac{1}{2}L = \frac{1}{2}(250) = \boxed{125}$

#8b. Managers of a wildlife preserve where grizzly bears have previous died out are re-introducing new bears to re-establish the bear population. At time $t = 0$ years, 40 bears are added to the wildlife preserve. After 4 years, there are 60 bears in the preserve. This wildlife preserve can support a maximum of 300 bears.

a) Write a logistic equation that models the population of bears in the preserve.

b) Find the population after 6 years.

c) When will the population reach 150?

d) Write a logistic differential equation that models the growth rate of the bear population. Then repeat part b using Euler's method of approximation with a step size of 1. Compare the approximation with the exact number.

e) How many years after the introduction of the bears is the bear population growing most rapidly? Explain.

a) $P = \frac{L}{1+Ce^{-kt}} = \frac{300}{1+Ce^{-kt}}$

t	P
0	40
4	60

$40 = \frac{300}{1+Ce^{0}} = \frac{300}{1+C}, \quad 1+C = \frac{300}{40} = \frac{15}{2}$

$C = \frac{13}{2}$

$P = \frac{300}{1+\frac{13}{2}e^{-kt}}$

$60 = \frac{300}{1+\frac{13}{2}e^{-k(4)}}, \quad 1+\frac{13}{2}e^{-4k} = \frac{300}{60} = 5$

$\frac{13}{2}e^{-4k} = 4$

$e^{-4k} = \frac{4}{\left(\frac{13}{2}\right)} = \frac{8}{13}$

$-4k = \ln\left(\frac{8}{13}\right)$

$k = \frac{\ln\left(\frac{8}{13}\right)}{-4} = 0.12138$

$\boxed{P(t) = \frac{300}{1+\frac{13}{2}e^{-0.12138t}}}$

b) $\boxed{P(6) = 72.502 \text{ bears}}$

c) $150 = \frac{300}{1+\frac{13}{2}e^{-.12138t}}$

$1+\frac{13}{2}e^{-.12138t} = \frac{300}{150} = 2$

$\frac{13}{2}e^{-.12138t} = 1$

$e^{-.12138t} = \frac{2}{13}$

$-.12138t = \ln\left(\frac{2}{13}\right)$

$t = \frac{\ln\left(\frac{2}{13}\right)}{-.12138} = \boxed{15.421 \text{ years}}$

d) $\frac{dP}{dt} = kP\left(1-\frac{P}{L}\right)$

$\boxed{\frac{dP}{dt} = 0.12138\,P\left(1-\frac{P}{300}\right)}$

See next page for Euler's →

#8b Euler's

(t,P)	$P_{next} = P_{current} + \underbrace{\left[0.12138P\left(1-\frac{P}{300}\right)\right]}_{\frac{dP}{dt}}\underbrace{(1)}_{\text{Step Size}}$
$(0,40)$	$P=40+\left[.12138(40)\left(1-\frac{40}{300}\right)\right](1) = 44.20784$
$(1,44.20784)$	$P=44.20784+\left[.12138(44.20784)\left(1-\frac{44.20784}{300}\right)\right](1) = 48.78306444$
$(2,48.78306444)$	$P=48.78306444+\left[.12138(48.78306444)\left(1-\frac{48.78306444}{300}\right)\right](1) = 53.74149083$
$(3,53.74149083)$	$P=53.74149083+\left[.12138(53.74149083)\left(1-\frac{53.74149083}{300}\right)\right](1) = 59.09608837$
$(4,59.09608837)$	$P=59.09608837+\left[.12138(59.09608837)\left(1-\frac{59.09608837}{300}\right)\right](1) = 64.85616771$
$(5,64.85616771)$	$P=64.85616771+\left[.12138(64.85616771)\left(1-\frac{64.85616771}{300}\right)\right](1) = 71.02653127$
$(6,71.02653127)$	

$$\boxed{P(6) \approx 71.027 \text{ bears}}$$

Obviously, the Euler's method question on our test will be much easier!

... but this gives you an appreciation for how difficult accurate calculations were before computers and calculators.

Recommendation! Watch the movie "Hidden Figures" about the black women who did all the hand calculations for NASA's Mercury and Apollo space missions ☺

#9b. (extra problem for challenge :)

Show that, for any logistic growth curve, the point of inflection occurs at $y = \frac{1}{2}L$ when the solution begins below the carrying capacity, L.

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{L}\right) = kP - \frac{k}{L}P^2$$ inflection when $\frac{d^2P}{dt^2} = 0$

$$\frac{d^2P}{dt^2} = \left(k - \frac{2k}{L}P\right)\frac{dP}{dt}$$ (chain rule because derivative was with respect to P)

$$= \left(k - \frac{2k}{L}P\right)\left(kP - \frac{k}{L}P^2\right)$$

$$= k^2P - \frac{k^2}{L}P^2 - \frac{2k^2}{L}P^2 + \frac{2k^2}{L^2}P^3$$

$$= \frac{2k^2}{L^2}P^3 - \frac{3k^2}{L}P^2 + k^2P$$

$$= k^2P\left(\frac{2}{L^2}P^2 - \frac{3}{L}P + 1\right)$$

$\frac{d^2P}{dt^2} = 0$ when $k^2P = 0$ (population zero)

or when $\frac{2}{L^2}P^2 - \frac{3}{L}P + 1 = 0$ (quadratic formula)

$$P = \frac{\frac{3}{L} \pm \sqrt{\frac{9}{L^2} - 4\left(\frac{2}{L^2}\right)(1)}}{2\left(\frac{2}{L^2}\right)} = \frac{\frac{3}{L} \pm \sqrt{\frac{9}{L^2} - \frac{8}{L^2}}}{\left(\frac{4}{L^2}\right)}$$

$$P = \frac{\frac{3}{L} \pm \sqrt{\frac{1}{L^2}}}{\left(\frac{4}{L^2}\right)} = \frac{\frac{3}{L} \pm \frac{1}{L}}{\left(\frac{4}{L^2}\right)} \left(\frac{L^2}{L^2}\right)$$

$$P = \frac{3L \pm L}{4} = \frac{L(3 \pm 1)}{4}$$

at $P = L$ and $P = L\left(\frac{1}{2}\right)$

curve approaches linear (concavity zero) as P approaches carrying capacity

$\boxed{\text{inflection point at } P = \frac{1}{2}L}$