AP Calculus BC – Unit 7 Extra Practice SOLUTIONS

7.1 – Extra Practice

Write the first 5 terms of the sequence:

#8a. $a_n = \sin\left(\frac{n\pi}{2}\right)$

$\boxed{1, 0, -1, 0, 1}$

#9b. $a_n = 2 + \frac{2}{n} - \frac{1}{n^2}$

$\boxed{3, \frac{11}{4}, \frac{23}{9}, \frac{39}{16}, \frac{59}{25}}$

#10b. Write the next two apparent terms of the sequence: $5, 10, 20, 40, \ldots$ $\boxed{80, 160}$

$\times 2 \quad \times 2 \quad \times 2$

#11b. Simplify: $\frac{(3n+2)!}{(3n-1)!}$

$= \frac{(3n+2)(3n+1)(3n)\cancel{(3n-1)!}}{\cancel{(3n-1)!}}$

$= \boxed{(3n+2)(3n+1)(3n)}$

#12b. Find the limit (if possible) of the sequence: $a_n = \frac{2n}{\sqrt{n^2+1}}$

$\lim_{n\to\infty} \frac{2n}{(n^2+1)^{1/2}}$ $\quad \lim_{n\to\infty} 2n = \infty$ $\quad \lim_{n\to\infty} (n^2+1)^{1/2} = \infty$ $\quad \left(\frac{\infty}{\infty}\right)$ use L'Hopital's rule

$= \lim_{n\to\infty} \frac{2}{\frac{1}{2}(n^2+1)^{-1/2}(2n)} = \lim_{n\to\infty} \frac{2(n^2+1)^{1/2}}{n}$ not getting easier...

instead: $\lim_{n\to\infty} \frac{2n}{\sqrt{n^2+1}}\left(\frac{\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)}\right) = \lim_{n\to\infty} \frac{2}{\sqrt{n^2+1}\sqrt{\frac{1}{n^2}}} = \lim_{n\to\infty} \frac{2}{\sqrt{\frac{n^2+1}{n^2}}}$

$= \lim_{n\to\infty} \frac{2}{\sqrt{\frac{n^2}{n^2}+\frac{1}{n^2}}} = \lim_{n\to\infty} \frac{2}{\sqrt{1+\frac{1}{n^2}}} = \frac{2}{\sqrt{1+0}} = \boxed{2}$

Determine the convergence or divergence of the sequence with the given nth term. If the sequence converges, find its limit:

#13b. $a_n = 8 + \frac{5}{n}$

$\lim_{n\to\infty} \left(8 + \frac{5}{n}\right) = 8$

$\boxed{\text{converges to } 8}$

#14b. $a_n = \frac{1+(-1)^n}{n^2}$

$\underbrace{\lim_{n\to\infty} \left(\frac{1}{n^2}\right)}_{0} + \lim_{n\to\infty} (-1)^n \frac{1}{n^2}$

$0 \quad +$ alternating sequence whose terms approach 0 also

$\boxed{\text{converges to } 0}$

#15b. $a_n = \frac{(n-2)!}{n!}$

$\lim_{n\to\infty} \frac{(n-2)!}{n!}$

$= \lim_{n\to\infty} \frac{(n-2)!}{n(n-1)(n-2)!}$

$= \lim_{n\to\infty} \frac{1}{n(n-1)} = 0$

$\boxed{\text{converges to } 0}$

#16b. $a_n = \cos\left(\frac{\pi n}{n^2}\right)$

$\lim_{n\to\infty} \cos\left(\frac{\pi n}{n^2}\right)$

$= \cos\left[\lim_{n\to\infty} \left(\pi \frac{n}{n^2}\right)\right]$

$= \cos\left[\lim_{n\to\infty} \left(\pi \frac{1}{n}\right)\right]$

$= \cos(0)$

$= 1 \quad \boxed{\text{converges to } 1}$

#17b. Determine whether the sequence is monotonic and whether it is bounded.

$a_n = \frac{3n}{n+2}$

$f(n) = \frac{3n}{n+2}$

$f'(n) = \frac{(n+2)(3) - (3n)(1)}{(n+2)^2}$

$f'(n) = \frac{3n+6-3n}{(n+2)^2} = \frac{6}{(n+2)^2} \quad \frac{(+)}{(+)}$

$f'(n) > 0$

$\therefore \boxed{\text{monotonic}}$ (increasing)

b/c increasing, lower bound: at $n=1$: $\frac{3(1)}{1+2} = 1$

upper bound: $\lim_{n\to\infty} \frac{3n}{n+2} = 3$

$\therefore$ also $\boxed{\text{bounded}}$

#18b. Find an expression for the nth term:

$1, -\frac{1}{4}, \frac{1}{9}, -\frac{1}{16}, \ldots$

n:	1	2	3	4
n^2	1	4	9	16

$\boxed{a_n = (-1)^{n-1} \frac{1}{n^2}}$

7.2 – Extra Practice

Find the sequence of partial sums S_1, S_2, S_3, and S_4.

#11b. $\frac{1}{2\bullet 3}+\frac{2}{3\bullet 4}+\frac{3}{4\bullet 5}+\frac{4}{5\bullet 6}+\frac{5}{6\bullet 7}+\ldots$

$$\boxed{\begin{aligned} S_1 &= \tfrac{1}{6} \\ S_2 &= \tfrac{1}{6}+\tfrac{2}{12} = \tfrac{1}{3} \\ S_3 &= \tfrac{1}{6}+\tfrac{2}{12}+\tfrac{3}{20} = \tfrac{29}{60} \\ S_4 &= \tfrac{1}{6}+\tfrac{2}{12}+\tfrac{3}{20}+\tfrac{4}{30} = \tfrac{37}{60} \end{aligned}}$$

#12b. $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\ldots$

$$\boxed{\begin{aligned} S_1 &= 1 \\ S_2 &= 1+\tfrac{1}{2} = \tfrac{3}{2} \\ S_3 &= 1+\tfrac{1}{2}+\tfrac{1}{4} = \tfrac{7}{4} \\ S_4 &= 1+\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{6} = \tfrac{23}{12} \end{aligned}}$$

Verify that the infinite series diverges:

#13b. $\sum_{n=0}^{\infty} 4(-1.05)^n$

Geometric, w/ $r=-1.05$

$|r| = |-1.05| = 1.05 > 1$

$\therefore \sum_{n=0}^{\infty} 4(-1.05)^n$ $\boxed{\text{diverges}}$

#14b. $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2+1}}$

n^{th} term test:

$\lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}}$ (can use L'Hopital's but will not simplify)

instead...

as $n \to \infty$, $n^2+1 \to n^2$

$$\therefore \lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2}}$$

$$= \lim_{n\to\infty} \frac{n}{n}$$

$$= \lim_{n\to\infty} 1$$

$$= 1 \neq 0$$

$\therefore \sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2+1}}$ $\boxed{\text{diverges}}$

by the n^{th} term test

Verify that the infinite series converges:

#15b. $\sum_{n=0}^{\infty} 2\left(-\frac{1}{2}\right)^n$

Geometric, w/ $r = -\frac{1}{2}$

$|r| = |-\frac{1}{2}| = \frac{1}{2} < 1$

$\therefore \sum_{n=0}^{\infty} 2(-\frac{1}{2})^n$ $\boxed{\text{Converges}}$

#16b. $\sum_{n=1}^{\infty} \frac{1}{2^n}$

could use Geometric:

$\sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \frac{1^n}{2^n} = \sum_{n=1}^{\infty} (\frac{1}{2})^n$

$\boxed{\text{converges}}$ w/ $|r| < 1$

—or— could find a pattern in the partial sums:

$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$

$S_1 = \frac{1}{2}$

$S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

$S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$

$S_4 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}$

$\vdots$

$S_n = \frac{2^n - 1}{2^n}$

n	2^n
1	2
2	4
3	8
4	16

$\therefore$ the limit for the sum (not a_n) is:

$\lim_{n \to \infty} \frac{2^n - 1}{2^n} = 1 \quad \therefore \sum_{n=1}^{\infty} \frac{1}{2^n} = 1$

($\boxed{\text{converges}}$ to the sum 1)

#17b. Find the sum of the convergent series

$\sum_{n=2}^{\infty} 5\left(\frac{2}{3}\right)^n$

Geometric, w/ $r = 2/3$

$|\frac{2}{3}| < 1$

(converges) to

$S = \frac{a}{1-r}$

$a = 5(\frac{2}{3})^2 = 5 \cdot \frac{4}{9} = \frac{20}{9}$

$r = \frac{2}{3}$

$S = \boxed{\frac{(\frac{20}{9})}{1 - (\frac{2}{3})}} = \frac{(\frac{20}{9})}{1 - \frac{2}{3}} \left(\frac{9}{9}\right)$

$= \frac{20}{9 - 2}$

$= \boxed{\frac{20}{7}}$

#18b. Write the repeating decimal as a geometric
And write the sum of the series as a fraction:

$0.\overline{49} = .49 + .0049 + .000049 + \ldots$

$= .49 + .49(\frac{1}{100}) + .49(\frac{1}{100})^2 + \ldots$

$\sum_{n=0}^{\infty} .49(\frac{1}{100})^n$ geometric

w/ $r = \frac{1}{100} < 1$

Converges

to sum $S = \frac{a}{1-r}$

$a = .49$

$r = \frac{1}{100}$

$S = \frac{.49}{1 - \frac{1}{100}} = \frac{.49}{(\frac{99}{100})} = \frac{.49(100)}{99}$

$= \boxed{\frac{49}{99}}$

Determine if the series is convergent or divergent:

#19b. $\sum_{n=0}^{\infty}(1.075)^n$ Geometric, w/ $r=1.075>1$

$\therefore \sum_{n=0}^{\infty}(1.075)^n$ $\boxed{\text{diverges}}$

#20b. $\sum_{n=1}^{\infty}\ln\left(\frac{1}{n}\right)$ nth term test:

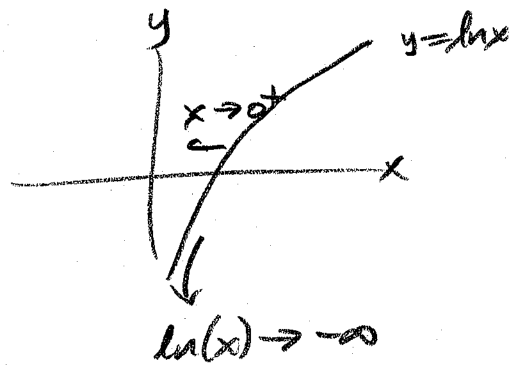
$$\lim_{n\to\infty}\ln\left(\frac{1}{n}\right)$$

$$=\ln\left[\lim_{n\to\infty}\frac{1}{n}\right]$$

$$=\ln\left[\underset{\text{from }+}{\to 0}\right]$$

$$=-\infty\neq 0$$

$\therefore \sum_{n=1}^{\infty}\ln\left(\frac{1}{n}\right)$ $\boxed{\text{diverges}}$ by the nth term test



#21b. $\sum_{n=1}^{\infty}\left(1+\frac{k}{n}\right)^n$ nth term test?

$$y=\lim_{n\to\infty}\left(1+\frac{k}{n}\right)^n \quad \text{(use log)}$$

$$\ln y=\ln\left(\lim_{n\to\infty}\left(1+\frac{k}{n}\right)^n\right)$$

$$\ln y=\lim_{n\to\infty}\left[n\ln\left(1+\frac{k}{n}\right)\right]$$

$\lim_{n\to\infty} n=\infty$, $\lim_{n\to\infty}\ln\left(1+\frac{k}{n}\right)$

$\ln\left[\lim_{n\to\infty}\left(1+\frac{k}{n}\right)\right]$

$\ln[1]=0$

$(\infty)(0)$ change to indeterminate form

$$\ln y=\lim_{n\to\infty}\frac{\ln\left(1+\frac{k}{n}\right)}{\left(\frac{1}{n}\right)}\to\frac{\infty}{\infty} \quad \text{can use L'Hôpital's}$$

$$\ln y=\lim_{n\to\infty}\frac{\left(\frac{1}{1+\frac{k}{n}}\right)\frac{d}{dn}\left[1+kn^{-1}\right]}{\frac{d}{dn}\left[n^{-1}\right]}=\lim_{n\to\infty}\frac{\left(\frac{1}{1+\frac{k}{n}}\right)\left(-kn^{-2}\right)}{-n^{-2}}=\lim_{n\to\infty}\frac{k}{1+\frac{k}{n}}=k$$

$$\ln y=k$$

$$\therefore y=\lim_{n\to\infty}\left(1+\frac{k}{n}\right)^n=e^k\neq 0$$

$\therefore \sum_{n=1}^{\infty}\left(1+\frac{k}{n}\right)^n$ $\boxed{\text{diverges}}$ by the nth term test

7.3 – Extra Practice

Confirm that the integral test applies, then use it to determine if the series converges or diverges.

#9b. $\sum_{n=1}^{\infty} \frac{2}{3n+5}$ ✓ a_n positive for $n \geq 1$

✓ $f(x) = \frac{2}{3x+5}$ continuous for $x \geq 1$

✓ $f'(x) = \frac{(3x+5)(0) - 2(3)}{(3x+5)^2}$

$= \frac{-6}{(3x+5)^2} < 0$ for $x \geq 1$

f is decreasing

$\therefore$ Integral test applies

$\int_1^{\infty} \frac{2}{3x+5}\,dx$ $\quad u = 3x+5$, $du = 3dx$, $\frac{1}{3}du = dx$

$\lim_{b\to\infty} \int_8^b \frac{1}{u}\,du$

$\lim_{b\to\infty} [\ln|u|]_8^b$

$\lim_{b\to\infty} [\ln(b)] - \ln(8)$

$\underline{\underline{\infty}} - \ln(8)$

integral diverges

$\therefore \sum_{n=1}^{\infty} \frac{2}{3n+5}$ also $\boxed{\text{diverges}}$

by the integral test

#10b. $\sum_{n=1}^{\infty} ne^{-\frac{1}{2}n}$ ✓ a_n positive for $n \geq 1$

✓ $f(x) = xe^{-\frac{1}{2}x}$ continuous for $x \geq 1$

✓ $f'(x) = x\left(-\frac{1}{2}e^{-\frac{1}{2}x}\right) + e^{-\frac{1}{2}x}(1)$

$= \underbrace{e^{-\frac{1}{2}x}}_{+}\underbrace{\left(-\frac{1}{2}x+1\right)}_{+ \text{ for } \underline{\underline{x>2}}}$

f decreasing for $x>2$

$\therefore$ integral test applies for $x>2$

$\int_2^{\infty} xe^{-\frac{1}{2}x}\,dx$ $\quad$ by parts: $u = x$, $dv = e^{-\frac{1}{2}x}dx$, $du = dx$, $v = -2e^{-\frac{1}{2}x}$

$\lim_{b\to\infty} \int_2^b xe^{-\frac{1}{2}x}dx$

$\lim_{b\to\infty} [uv - \int v\,du]_2^b$

$\lim_{b\to\infty} \left[-2xe^{-\frac{1}{2}x} - \int(-2e^{-\frac{1}{2}x}dx)\right]_2^b$

$\lim_{b\to\infty} \left[-2xe^{-\frac{1}{2}x} + 2\int e^{-\frac{1}{2}x}dx\right]_2^b$

$\lim_{b\to\infty} \left[-2xe^{-\frac{1}{2}x} - 4e^{-\frac{1}{2}x}\right]_2^b$

$\lim_{b\to\infty} \left[-\frac{2b}{e^{\frac{1}{2}b}} - \frac{4}{e^{\frac{1}{2}b}}\right] - \left[-2(2)e^{-\frac{1}{2}(2)} - 4e^{-\frac{1}{2}(2)}\right]$

$\lim_{b\to\infty} \frac{-2b}{e^{\frac{1}{2}b}}$ $\quad \lim_{b\to\infty} -2b = -\infty$, $\lim_{b\to\infty} e^{\frac{1}{2}b} = \infty$

$\left(\frac{\infty}{\infty}\right)$ use L'Hopital's

$= \lim_{b\to\infty} \frac{-2}{\frac{1}{2}e^{\frac{1}{2}b}} = 0$

$0 - 0 + 4e^{-1} + 4e^{-1}$

integral converges

$\therefore \sum_{n=1}^{\infty} ne^{-\frac{1}{2}n}$ also $\boxed{\text{converges}}$

by the integral test

Confirm that the integral test applies, then use it to determine if the series converges or diverges.

#11b. $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln(n)}}$

✓ a_n positive for $n \geq 2$

✓ $f(x) = \frac{1}{x\sqrt{\ln x}}$ continuous for $x \geq 2$

$= x^{-1}(\ln x)^{-1/2}$

✓ $f'(x) = x^{-1}\left(-\frac{1}{2}(\ln x)^{-3/2}\frac{1}{x}\right) + (\ln x)^{-1/2}(-x^{-2})$

$= \frac{-1}{2x^2(\ln x)^{3/2}} - \frac{1}{x^2(\ln x)^{1/2}}\frac{(2\ln x)}{(2\ln x)}$

$= \frac{-1-2\ln x}{2x^2(\ln x)^{3/2}} < 0$ for $x \geq 2$

f decreasing

∴ integral test applies

$\int_2^{\infty} \frac{1}{x\sqrt{\ln x}}\,dx$

$= \lim_{b\to\infty} \int_2^b \frac{1}{x}(\ln x)^{-1/2}\,dx$ $u = \ln x$, $du = \frac{1}{x}dx$

$= \lim_{b\to\infty} \int_{\ln 2}^b u^{-1/2}\,du$

$= \lim_{b\to\infty} \left[2u^{1/2}\right]_{\ln 2}^b$

$= \lim_{b\to\infty} \left[2\sqrt{b}\right] - 2\sqrt{\ln 2}$

∞

integral diverges

∴ $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln(n)}}$ also $\boxed{\text{diverges}}$ by the integral test

#12b. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+2}}$ ✓ a_n is positive for $n \geq 1$

✓ $f(x) = \frac{1}{\sqrt{x+2}}$ continuous for $x \geq 1$

$= (x+2)^{-1/2}$

✓ $f'(x) = -\frac{1}{2}(x+2)^{-3/2}(1)$

$= \frac{-1}{2(x+2)^{3/2}} < 0$ for $x \geq 1$

f decreasing

∴ integral test applies

$\int_1^{\infty} \frac{1}{\sqrt{x+2}}\,dx$

$= \lim_{b\to\infty} \int_1^b (x+2)^{-1/2}\,dx$ $u = x+2$, $du = dx$

$= \lim_{b\to\infty} \int_3^b u^{-1/2}\,du$

$= \lim_{b\to\infty} \left[2u^{1/2}\right]_3^b$

$= \lim_{b\to\infty} \left[2\sqrt{b}\right] - 2\sqrt{3}$

∞

integral diverges

∴ $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+2}}$ also $\boxed{\text{diverges}}$ by the integral test

Explain why the integral test does not apply to the series.

#13b. $\sum_{n=1}^{\infty} e^{-n}\cos(n)$

e^{-n} is always positive

but $\cos(n)$ changes signs

for different values of n

$\boxed{\therefore \text{ not all } a_n \text{ are positive}}$

#14b. $\sum_{n=1}^{\infty}\left(\frac{\sin(n)}{n}\right)^2$

✓ a_n positive for $n \geq 1$

✓ $f(x) = \left(\frac{\sin x}{x}\right)^2$ continuous for $x \geq 1$

$f'(x) = 2\left(\frac{\sin x}{x}\right)\left[\frac{x(\cos x) - \sin x(1)}{x^2}\right]$

$= \frac{2\sin x}{x^3}\left[x\cos x - \sin x\right]$

not always negative

So $\boxed{f \text{ is not always decreasing}}$

Use the p-series test to determine the convergence or divergence of the series.

#15b. $\sum_{n=1}^{\infty}\frac{1}{n^{1/2}}$

p-series, w/ $p = 1/2$

$\boxed{\text{diverges}}$

by p-series test

#16b. $\sum_{n=1}^{\infty}\frac{1}{n^5}$

p-series, w/ $p = 5$

$\boxed{\text{converges}}$

by p-series test

#17b. $1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \ldots$

$= \sum_{n=1}^{\infty}\frac{1}{n\sqrt{n}} = \sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$

p-series, w/ $p = 3/2$

$\boxed{\text{converges}}$

by p-series test

7.4 – Extra Practice

Use the Direct Comparison Test to determine the convergence or divergence of the series.

#10b. $\sum_{n=1}^{\infty} \frac{1}{3n^2+2}$ compare with $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{3n^2} = \frac{1}{3}\sum_{n=1}^{\infty} \frac{1}{n^2}$ p-series, w/ $p=2$ converges

$\frac{1}{3n^2+2} < \frac{1}{3n^2}$

pushes this side down

Since $\frac{1}{3n^2+2} < \frac{1}{3n^2}$

and $\sum_{n=1}^{\infty} \frac{1}{3n^2}$ converges by p-series ~~test~~

$\sum_{n=1}^{\infty} \frac{1}{3n^2+2}$ also $\boxed{\text{converges}}$

by the Direct Comparison Test

#11b. $\sum_{n=1}^{\infty} \frac{4^n}{5^n+3}$ compare with $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{4^n}{5^n} = \sum_{n=1}^{\infty} \left(\frac{4}{5}\right)^n$ Geometric w/ $|r|<1$ converges

$\frac{4^n}{5^n+3} < \frac{4^n}{5^n}$

Since $\frac{4^n}{5^n+3} < \frac{4^n}{5^n}$

and $\sum_{n=1}^{\infty} \frac{4^n}{5^n}$ converges by Geometric Series ~~test~~

$\sum_{n=1}^{\infty} \frac{4^n}{5^n+3}$ also $\boxed{\text{converges}}$

by Direct Comparison Test

#12b. $\sum_{n=1}^{\infty} \frac{1}{n!}$ compare to ??

n:	1	2	3	4	5	6
$n!$:	1	2	6	24	120	720
n^2:	1	4	9	16	25	36

for $n>3$: $n^2 < n!$

compare to $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$

p-series, w/ $p=2$

converges

$n^2 < n!$

so $\frac{1}{n!} < \frac{1}{n^2}$

(correct side)

Since $\frac{1}{n!} < \frac{1}{n^2}$

and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by p-series test

$\sum_{n=1}^{\infty} \frac{1}{n!}$ also $\boxed{\text{converges}}$

by Direct Comparison Test

Use the Limit Comparison Test to determine the convergence or divergence of the series.

#13b. $\sum_{n=1}^{\infty} \frac{5}{4^n+1}$ compare to $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{5}{4^n} = \sum_{n=1}^{\infty} 5\left(\frac{1}{4}\right)^n$ Geometric w/ $|r|<1$ converges

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\left(\frac{5}{4^n+1}\right)}{\left(\frac{5}{4^n}\right)}$

$= \lim_{n\to\infty} \frac{4^n}{4^n+1} = 1$

(finite, positive)
So series are "linked"

Since $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$

and $\sum_{n=1}^{\infty} \frac{5}{4^n}$ converges by Geometric Series test

$\sum_{n=1}^{\infty} \frac{5}{4^n+1}$ also $\boxed{\text{converges}}$
by Limit Comparison Test

#14b. $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n^2+1}}$ compare with $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n^2}} = \sum_{n=1}^{\infty} \frac{1}{n^2}$. p-series w/ $p=2$ converges

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\left(\frac{1}{n\sqrt{n^2+1}}\right)}{\left(\frac{1}{n\sqrt{n^2}}\right)}$

$= \lim_{n\to\infty} \frac{\sqrt{n^2}}{\sqrt{n^2+1}}$ (L'Hopital's doesn't work well...)

for $n\to\infty$, $n^2+1 \to n^2$

so $\lim_{n\to\infty} \frac{\sqrt{n^2}}{\sqrt{n^2+1}} = \lim_{n\to\infty} \frac{\sqrt{n^2}}{\sqrt{n^2}} = \lim_{n\to\infty} 1 = 1$

(finite, positive)
(series "linked")

Since $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$

and $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n^2}}$ converges by p-series test

$\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n^2+1}}$ also $\boxed{\text{converges}}$
by Limit Comparison Test

#15b. $\sum_{n=1}^{\infty} \frac{n}{(n+1)2^{n-1}}$ compare with $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{n}{n2^n} = \sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ Geometric w/ $|r|<1$ converges

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\left(\frac{n}{(n+1)2^{n-1}}\right)}{\left(\frac{n}{n2^n}\right)}$

$= \lim_{n\to\infty} \frac{n2^n}{(n+1)2^{n-1}} = \lim_{n\to\infty} \frac{n2\cdot 2^{n-1}}{(n+1)2^{n-1}}$

$= \lim_{n\to\infty} \frac{2n}{n+1} = 2$ (finite, positive)

Since $\lim_{n\to\infty} \frac{a_n}{b_n} = 2$

and $\sum_{n=1}^{\infty} \frac{n}{n2^n}$ converges by Geometric Series test

$\sum_{n=1}^{\infty} \frac{n}{(n+1)2^{n-1}}$ also $\boxed{\text{converges}}$
by Limit Comparison Test

7.5 – Extra Practice

Determine the convergence or divergence of the series.

#6b. $\sum_{n=1}^{\infty} \frac{n(-1)^{n+1}}{3n+2}$

- $\lim_{n \to \infty} \frac{n}{3n+2} = \frac{1}{3}$

$\neq 0$

cannot use Alternating Series Test

but... since $\lim_{n \to \infty} a_n = \frac{1}{3} \neq 0$

$\therefore \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{3n+2}$ $\boxed{\text{diverges}}$

by the nth term test

#7b. $\sum_{n=1}^{\infty} (-1)^n \frac{1}{e^n}$

- $\lim_{n \to \infty} \frac{1}{e^n} = 0$ ✓

- $a_{n+1} \overset{?}{\leq} a_n$

$\frac{1}{e^{n+1}} \leq \frac{1}{e^n}$ ✓

Since $\lim_{n \to \infty} a_n = 0$

and $a_{n+1} \leq a_n$

the Alternating Series

$\sum_{n=1}^{\infty} (-1)^n \frac{1}{e^n}$ $\boxed{\text{converges}}$

by the Alternating Series Test

#8b. $\sum_{n=1}^{\infty} (-1)^n \frac{n}{\ln(n+1)}$

- $\lim_{n \to \infty} \frac{n}{\ln(n+1)}$ $\quad \lim_{n \to \infty} n = \infty$ $\quad \lim_{n \to \infty} \ln(n+1) = \infty$

$\left(\frac{\infty}{\infty}\right)$ use L'Hôpital's:

$= \lim_{n \to \infty} \frac{1}{\left(\frac{1}{n+1}\right)} = \lim_{n \to \infty} (n+1) = \infty$

(alt. series test does not apply)

but since $\lim_{n \to \infty} a_n = \infty \neq 0$

$\sum_{n=1}^{\infty} (-1)^n \frac{n}{\ln(n+1)}$ $\boxed{\text{diverges}}$

by the nth term test

Determine the convergence or divergence of the series.

#9b. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{n^2+5}$

$\lim_{n\to\infty}\frac{n}{n^2+5}=0$ ✓

$a_{n+1}\overset{?}{\leq}a_n$

$f(x)=\frac{x}{x^2+5}$

$f'(x)=\frac{(x^2+5)(1)-x(2x)}{(x^2+5)^2}$

$=\frac{x^2-5-2x^2}{(x^2+5)^2}=\frac{-x^2-5}{(x^2+5)^2}<0$ for $x\geq 1$

decreasing, so

$a_{n+1}\leq a_n$ ✓

Since $\lim_{n\to\infty}a_n=0$

and $a_{n+1}\leq a_n$

the alternating series

$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{n^2+5}$ $\boxed{\text{converges}}$

by the Alternating Series Test

#10b. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2}{n^2+4}$

$\lim_{n\to\infty}\frac{n^2}{n^2+4}=1\neq 0$

(alt. series test does not apply)

but Since $\lim_{n\to\infty}a_n=1\neq 0$

$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2}{n^2+4}$ $\boxed{\text{diverges}}$

by the nth term test

#11b. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\ln(n+1)}{n+1}$

$\lim_{n\to\infty}\frac{\ln(n+1)}{n+1}$ $\quad \lim_{n\to\infty}\ln(n+1)=\infty$ $\quad \lim_{n\to\infty}(n+1)=\infty$

$\left(\frac{\infty}{\infty}\right)$ use L'Hop.

$=\lim_{n\to\infty}\frac{\left(\frac{1}{n+1}\right)}{1}=0$ ✓

$a_{n+1}\overset{?}{\leq}a_n$

$f(x)=\frac{\ln(x+1)}{x+1}$

$f'(x)=\frac{(x+1)\frac{1}{x+1}-\ln(x+1)(1)}{(x+1)^2}$

$=\frac{1-\ln(x+1)}{(x+1)^2}<0$ for $x\geq 0$

decreasing,

$\therefore a_{n+1}\leq a_n$ ✓

Since $\lim_{n\to\infty}a_n=0$

and $a_{n+1}\leq a_n$

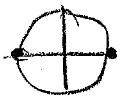
the alternating series

$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\ln(n+1)}{n+1}$

$\boxed{\text{converges}}$

by the Alternating Series Test

Determine the convergence or divergence of the series.

#12b. $\sum_{n=1}^{\infty} \frac{1}{n}\cos(n\pi)$ 

$= \frac{1}{1}\cos(\pi) + \frac{1}{2}\cos(2\pi) + \frac{1}{3}\cos(3\pi) + \frac{1}{4}\cos(4\pi) \cdots$

$= 1(-1) + \frac{1}{2}(1) + \frac{1}{3}(-1) + \frac{1}{4}(1) + \cdots$

alternating series: $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$

- $\lim_{n\to\infty} \frac{1}{n} = 0$ ✓

- $a_{n+1} \le a_n$

  $\frac{1}{n+1} \le \frac{1}{n}$ ✓

Since $\lim_{n\to\infty} a_n = 0$ and $a_{n+1} \le a_n$

alternating series

$\sum_{n=1}^{\infty} \frac{1}{n}\cos(n\pi) = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ $\boxed{\text{Converges}}$

by the Alternating Series Test

#13b. $\sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!}$

- $\lim_{n\to\infty} \frac{1}{(2n+1)!} = 0$ ✓

- $a_{n+1} \overset{?}{\le} a_n$

  $\frac{1}{(2(n+1)+1)!} \overset{?}{\le} \frac{1}{(2n+1)!}$

  $\frac{1}{(2n+3)!} < \frac{1}{(2n+1)!}$ ✓

Since $\lim_{n\to\infty} a_n = 0$

and $a_{n+1} \le a_n$

the alternating series

$\sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!}$ $\boxed{\text{Converges}}$

by the Alternating Series test

#14b. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sqrt{n}}{\sqrt[3]{n}}$

- $\lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt[3]{n}}$

$= \lim_{n\to\infty} \frac{n^{1/2}}{n^{1/3}} = \lim_{n\to\infty} n^{1/6} = \infty$

$\ne 0$

So alt. Series test does not apply

but since $\lim_{n\to\infty} a_n = \infty \ne 0$

$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sqrt{n}}{\sqrt[3]{n}}$ $\boxed{\text{diverges}}$

by the nth term test

Determine the convergence or divergence of the series.

#15b. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n(n+1)} = \sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)$ (telescoping series) write terms out and see cancelling

partial fraction expansion:

$$\frac{1}{n(n+1)} = \frac{A}{n}+\frac{B}{n+1}$$

$$\frac{1}{n(n+1)} = \frac{A(n+1)}{n(n+1)}+\frac{Bn}{n(n+1)}$$

$$A(n+1)+Bn = 1$$

$$An+A+Bn=1$$

$$(A+B)n+(A) = (0)n+(1)$$

$$\begin{cases} A+B=0 \\ A=1 \end{cases} \quad A=1, \quad B=-1$$

$$\frac{1}{n(n+1)} = \frac{1}{n}-\frac{1}{n+1}$$

$$= \underbrace{\left(\frac{1}{1}\right.}_{}\left.-\cancel{\frac{1}{2}}\right)+\left(\cancel{\frac{1}{2}}-\cancel{\frac{1}{3}}\right)+\left(\cancel{\frac{1}{3}}-\cancel{\frac{1}{4}}\right)+\cdots$$

only 1 term at beginning, leaves one term at end ...

$$+\cdots+\left(\cancel{\frac{1}{n}}-\frac{1}{n+1}\right)$$

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)} = \frac{1}{1}-\lim_{n\to\infty}\left(\frac{1}{n+1}\right)$$

$$=1-0$$

$$=1$$

Telescoping Series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$ $\boxed{\text{Converges}}$

(to the sum 1)

#15c. $\displaystyle\sum_{n=1}^{\infty}\frac{6}{n(n+3)} = \sum_{n=1}^{\infty}\left(\frac{2}{n}-\frac{2}{n+3}\right)$ (telescoping series)

partial fraction expansion:

$$\frac{6}{n(n+3)} = \frac{A}{n}+\frac{B}{n+3}$$

$$A(n+3)+Bn=6$$

$$An+3A+Bn=6$$

$$(A+B)n+(3A) = (0)n+(6)$$

$$\begin{cases} A+B=0 \\ 3A=6 \end{cases} \quad A=2, \quad B=-2$$

$$=\left(\frac{2}{1}-\cancel{\frac{2}{4}}\right)+\left(\frac{2}{2}-\cancel{\frac{2}{5}}\right)+\left(\frac{2}{3}-\cancel{\frac{2}{6}}\right)+\left(\cancel{\frac{2}{4}}-\cancel{\frac{2}{7}}\right)+\left(\cancel{\frac{2}{5}}-\frac{2}{8}\right)+\cdots$$

3 uncancelled

$$\cdots+\left(\cancel{\frac{2}{n-2}}-\frac{2}{n+1}\right)+\left(\cancel{\frac{2}{n-1}}-\frac{2}{n+2}\right)+\left(\cancel{\frac{2}{n}}-\frac{2}{n+3}\right)$$

$$\sum_{n=1}^{\infty}\frac{6}{n(n+3)} = \frac{2}{1}+\frac{2}{2}+\frac{2}{3}-\lim_{n\to\infty}\left[\frac{2}{n+1}+\frac{2}{n+2}+\frac{2}{n+3}\right]$$

$$=2+1+\frac{2}{3}-(0+0+0+0)$$

Telescoping series $\displaystyle\sum_{n=1}^{\infty}\frac{6}{n(n+3)}$ $\boxed{\text{Converges}}$

(to the sum $2+1+\frac{2}{3}=\frac{11}{3}$)

7.6 – Extra Practice

Determine whether the series converges absolutely, conditionally, or diverges.

#4b. $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n!}$

check $\sum |a_n| = \sum_{n=1}^{\infty} \frac{1}{n!}$

n:	1	2	3	4	5
$n!$	1	2	6	24	120
n^2	1	4	9	16	25

→ for $n>3$

$n^2 < n!$

so $\frac{1}{n!} < \frac{1}{n^2}$

Direct compare w/ $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ p-series $p=2$, converges

$\frac{1}{n!} < \frac{1}{n^2}$ ← correct side for direct

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges and $\frac{1}{n!} < \frac{1}{n^2}$, $\sum_{n=1}^{\infty} \frac{1}{n!}$ also converges by Direct Comparison test

Since $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges,

$\boxed{\sum_{n=1}^{\infty} (-1)^n \frac{1}{n!} \text{ converges absolutely}}$

#5b. $\sum_{n=1}^{\infty}(-1)^n \frac{1}{\sqrt{n+4}}$

check $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+4}}$

compare with $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$

p-series, $p = 1/2$, diverges

$\frac{1}{\sqrt{n+4}} < \frac{1}{\sqrt{n}}$ wrong side for direct

Limit Comparison: $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n+4}}}{\frac{1}{\sqrt{n}}}$

$= \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+4}}$ for $n\to\infty$, $n+4 \to n$

$= \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}} = \lim_{n\to\infty} 1 = 1$ finite, positive series "linked"

$\therefore \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+4}}$ also diverges by Limit Comparison test

now, alternating series test: $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n+4}}$

✓. $\lim_{n\to\infty} \frac{1}{\sqrt{n+4}} = 0$

✓. $a_{n+1} \overset{?}{\leq} a_n$

$\frac{1}{\sqrt{n+5}} < \frac{1}{\sqrt{n+4}}$

Since $\lim_{n\to\infty} a_n = 0$ and $a_{n+1} \leq a_n$, $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n+4}}$ converges by the Alternating Series Test

then... Since $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+4}}$ diverges but $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n+4}}$ converges

$\boxed{\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n+4}} \text{ converges conditionally}}$

#6b. $\sum_{n=2}^{\infty}(-1)^n \frac{1}{n\ln(n)}$

check $\sum |a_n| = \sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$

integral test?

✓ • a_n positive for $n \geq 2$

✓ • $f(x) = \frac{1}{x\ln(x)}$ continuous for $x \geq 2$

$= (x\ln(x))^{-1}$

✓ • $f'(x) = -(x\ln(x))^{-2}(x\frac{1}{x} + \ln(x)(1))$

$= \frac{-1-\ln x}{(x\ln x)^2} < 0$ for $x \geq 2$

decreasing

∴ integral test applies

$\int_2^{\infty} \frac{1}{x\ln x}dx$ $\quad u = \ln x$, $du = \frac{1}{x}dx$

$= \lim_{b\to\infty} \int_{\ln 2}^{b} \frac{1}{u}du$

$= \lim_{b\to\infty} [\ln|u|]_{\ln 2}^{b}$

$= \lim_{b\to\infty} (\underset{\infty}{\ln b} - \ln 2)$

integral diverges

∴ $\sum_{n=1}^{\infty} \frac{1}{n\ln(n)}$ also diverges

by integral test

now, alternating series test on

$\sum_{n=2}^{\infty} (-1)^n \frac{1}{n\ln(n)}$

✓ • $\lim_{n\to\infty} \frac{1}{n\ln(n)} = 0$

✓ • $a_{n+1} \overset{?}{\leq} a_n$ $\quad$ showed $f'(x) < 0$ decreasing

∴ $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n\ln n}$ converges

by Alternating Series Test

Since $\sum_{n=1}^{\infty} \frac{1}{n\ln(n)}$ diverges

but $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n\ln(n)}$ converges

$\boxed{\sum_{n=1}^{\infty} (-1)^n \frac{1}{n\ln(n)} \text{ converges conditionally}}$

#7b. $\sum_{n=1}^{\infty}(-1)^n \frac{1}{(2n+1)!}$

check $\sum |a_n| = \sum_{n=1}^{\infty} \frac{1}{(2n+1)!}$

Direct comparison

w/ $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ $\quad$ p-series w/ $p=2$ converges

$\frac{1}{(2n+1)!} < \frac{1}{n^2}$

correct side for direct

n:	1	2	3	4
$2n+1$:	3	5	7	9
$(2n+1)!$	6	120	5040	362880
n^2	1	4	9	16

$n^2 < (2n+1)!$

so

$\frac{1}{(2n+1)!} < \frac{1}{n^2}$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges

and $\frac{1}{(2n+1)!} < \frac{1}{n^2}$

$\sum_{n=1}^{\infty} \frac{1}{(2n+1)!}$ also converges by Direct Comparison Test

Then, since $\sum_{n=1}^{\infty} \frac{1}{(2n+1)!}$ converges,

$\boxed{\sum_{n=1}^{\infty} (-1)^n \frac{1}{(2n+1)!} \text{ converges absolutely}}$

7.7 – Extra Practice

Simplify the expression fully.

#6b. $\dfrac{(n+1)4^n}{n4^{n+1}}$

$\dfrac{(n+1)4^n}{n\,4\,4^n}$

$\boxed{\dfrac{n+1}{4n}}$

#7b. $\dfrac{(2k-2)!}{(2k)!}$

$\dfrac{(2k-2)!}{(2k)(2k-1)(2k-2)!}$

$\boxed{\dfrac{1}{2k(2k-1)}}$

Use the Ratio Test to determine the convergence or divergence of the series.

#8b. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n!}$ $\displaystyle\lim_{n\to\infty}\left|\frac{\left(\frac{1}{(n+1)!}\right)}{\left(\frac{1}{n!}\right)}\right|$

$\displaystyle\lim_{n\to\infty}\left|\frac{1}{(n+1)!}\cdot\frac{n!}{(1)}\right|$

$\displaystyle\lim_{n\to\infty}\left|\frac{n!}{(n+1)n!}\right|$

$\displaystyle\lim_{n\to\infty}\left|\frac{1}{n+1}\right| = 0 < 1$

$\therefore \boxed{\displaystyle\sum_{n=1}^{\infty}\frac{1}{n!}\text{ converges}}$ by the Ratio Test

#9b. $\displaystyle\sum_{n=1}^{\infty}\frac{6^n}{n!}$ $\displaystyle\lim_{n\to\infty}\left|\frac{\left(\frac{6^{n+1}}{(n+1)!}\right)}{\left(\frac{6^n}{n!}\right)}\right|$

$\displaystyle\lim_{n\to\infty}\left|\frac{6^{n+1}}{(n+1)!}\cdot\frac{n!}{6^n}\right|$

$\displaystyle\lim_{n\to\infty}\left|\frac{6\cdot 6^n\, n!}{(n+1)n!\cdot 6^n}\right|$

$\displaystyle\lim_{n\to\infty}\left|\frac{6}{n+1}\right| = 0 < 1$

$\therefore \boxed{\displaystyle\sum_{n=1}^{\infty}\frac{6^n}{n!}\text{ converges}}$ by the Ratio Test

#10b. $\displaystyle\sum_{n=1}^{\infty}\frac{n}{4^n}$

$\displaystyle\lim_{n\to\infty}\left|\frac{(n+1)}{4^{n+1}}\cdot\frac{4^n}{n}\right|$

$\displaystyle\lim_{n\to\infty}\left|\frac{(n+1)4^n}{4\,4^n\,n}\right|$

$\displaystyle\lim_{n\to\infty}\left|\frac{n+1}{4n}\right| = \frac{1}{4} < 1$

$\therefore \boxed{\displaystyle\sum_{n=1}^{\infty}\frac{n}{4^n}\text{ converges}}$ by the Ratio Test

#11b. $\displaystyle\sum_{n=0}^{\infty}\frac{(n!)^2}{(3n)!}$

$\displaystyle\lim_{n\to\infty}\left|\frac{((n+1)!)^2}{(3n+3)!}\cdot\frac{(3n)!}{(n!)^2}\right|$

$[(n+1)n!]^2 = (n+1)^2(n!)^2$

$\displaystyle\lim_{n\to\infty}\left|\frac{[(n+1)n!]^2(3n)!}{(3n+3)(3n+2)(3n+1)(3n)!\,(n!)^2}\right|$

$\displaystyle\lim_{n\to\infty}\left|\frac{(n+1)^2}{(3n+3)(3n+2)(3n+1)}\right|$

$\displaystyle\lim_{n\to\infty}\left|\frac{n^2+\cdots}{9n^3+\cdots}\right| = 0 < 1$

$\therefore \boxed{\displaystyle\sum_{n=0}^{\infty}\frac{(n!)^2}{(3n)!}\text{ converges}}$ by the Ratio Test

Use the Ratio Test to determine the convergence or divergence of the series.

#12b. $\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{n2^n}$

$\lim_{n\to\infty} \left| \frac{3^{n+1}}{(n+1)2^{n+1}} \frac{n2^n}{3^n} \right|$

$\lim_{n\to\infty} \left| \frac{3\, 3^n\, n 2^n}{(n+1)\, 2 \cdot 2^n\, 3^n} \right|$

$\lim_{n\to\infty} \left| \frac{3n}{(n+1)2} \right|$

$\lim_{n\to\infty} \left| \frac{3n}{2n+1} \right| = \frac{3}{2} > 1$

$\therefore \boxed{\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{n2^n} \text{ diverges}}$

by the Ratio Test

w/ Ratio Test, if the alternating series diverges, there is no need to separately check with alternating series test

#13b. $\sum_{n=0}^{\infty} \frac{6^n}{(n+1)^n}$

$\lim_{n\to\infty} \left| \frac{6^{n+1}}{(n+2)^{n+1}} \frac{(n+1)^n}{6^n} \right|$

$\lim_{n\to\infty} \left| \frac{6 \cdot 6^n (n+1)^n}{(n+2)(n+2)^n 6^n} \right| = \lim_{n\to\infty} \left| \frac{6(n+1)^n}{(n+2)(n+2)^n} \right|$

$\left(\lim_{n\to\infty} \frac{6}{n+2}\right)\left(\lim_{n\to\infty} \frac{(n+1)^n}{(n+2)^n}\right)$

$(0) \left(\lim_{n\to\infty} \frac{n+1}{n+2}\right)^n$

$(0)\ (1)^n$

$(0)(1)$

$0 < 1$

$\therefore \boxed{\sum_{n=0}^{\infty} \frac{6^n}{(n+1)^n} \text{ converges}}$

by the Ratio Test

Use the Root Test to determine the convergence or divergence of the series.

#14b. $\sum_{n=1}^{\infty}\left(\frac{2n}{n+1}\right)^n$

$\lim_{n\to\infty} \sqrt[n]{\left(\frac{2n}{n+1}\right)^n}$

$\lim_{n\to\infty} \frac{2n}{n+1} = 2 > 1$

$\therefore \boxed{\sum_{n=1}^{\infty}\left(\frac{2n}{n+1}\right)^n \text{ diverges}}$

by the Root Test

#15b. $\sum_{n=1}^{\infty}\left(-\frac{3n}{2n+1}\right)^n$

$\lim_{n\to\infty} \sqrt[n]{\left|\left(-\frac{3n}{2n+1}\right)^n\right|}$

$\lim_{n\to\infty} \left|\left(-\frac{3n}{2n+1}\right)\right| = \frac{3}{2} > 1$

$\therefore \boxed{\sum_{n=1}^{\infty}\left(-\frac{3n}{2n+1}\right)^n \text{ diverges}}$

by the Root Test

#16b. $\sum_{n=1}^{\infty}\frac{n}{3^n}$

$= \sum_{n=1}^{\infty}\left(\frac{(n^{1/n})^n}{3^n}\right)$

$= \sum_{n=1}^{\infty}\left(\frac{n^{1/n}}{3^n}\right)^n$

$\lim_{n\to\infty} \sqrt[n]{\left(\frac{n^{1/n}}{3^n}\right)^n}$

$\lim_{n\to\infty} \frac{n^{1/n}}{3^n}$

$y = \lim_{n\to\infty} n^{1/n}$

$\ln y = \lim_{n\to\infty} \frac{1}{n}\ln(n)$

$\ln y = \lim_{n\to\infty} \frac{\ln(n)}{n} \quad \frac{\infty}{\infty}$

$\ln y = \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = 0$

$y = e^0 = 1$

$\lim_{n\to\infty} 3^n = \infty$

$\therefore \lim_{n\to\infty} \frac{n^{1/n}}{3^n} = 0 < 1$

$\boxed{\therefore \sum_{n=1}^{\infty}\frac{n}{3^n} \text{ converges}}$

by the Root Test

#17b. $\sum_{n=1}^{\infty}\left(\frac{n}{500}\right)^n$

$\lim_{n\to\infty} \sqrt[n]{\left(\frac{n}{500}\right)^n}$

$\lim_{n\to\infty} \frac{n}{500} = \infty$

$\therefore \boxed{\sum_{n=1}^{\infty}\left(\frac{n}{500}\right)^n \text{ diverges}}$

by the Root Test

7.8 – Extra Practice

Approximate the sum of the series using the first 6 terms.

#4b. $\sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{\ln(n+1)}$

$\frac{4}{\ln(2)} - \frac{4}{\ln(3)} + \frac{4}{\ln(4)} - \frac{4}{\ln(5)} + \frac{4}{\ln(6)} - \frac{4}{\ln(7)}$

$= \boxed{2.7067}$

#5b. $\sum_{n=1}^{\infty} \frac{n(-1)^{n+1}}{3^n}$

$\approx \frac{1}{3} - \frac{2}{9} + \frac{3}{27} - \frac{4}{81} + \frac{5}{243} - \frac{6}{729}$

$= \boxed{\frac{5}{27}}$

How many terms are required to approximate the series with an error of less than 0.001?

#6b. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n^3 - 1}$

$|\text{error}| < |1^{st} \text{ neglected term}|$

$< |(N+1) \text{ term}| < 0.001$

$\frac{1}{2(N+1)^3 - 1} < 0.001$

$2(N+1)^3 - 1 > 1000$

$2(N+1)^3 > 1001$

$(N+1)^3 > \frac{1001}{2}$

$N+1 > \sqrt[3]{\frac{1001}{2}}$

$N > \sqrt[3]{\frac{1001}{2}} - 1 = 6.939$

$\boxed{\text{include } N = 7 \text{ terms}}$