AP Calculus BC – Unit 7 Required Practice Name: answers only

7.1 – Required Practice

#1. Write out the first 4 terms of the sequence defined by $a_n = (-1)^{n-1}\frac{n}{n+1}$

$\boxed{\frac{1}{2}, \frac{-2}{3}, \frac{3}{4}, \frac{-4}{5}}$

#2. $a_n = \frac{5n^2+3n}{2n^2-n+4}$

$\boxed{\text{converges to } \frac{10}{4}}$

#3. $a_n = \frac{n!}{(n+1)!}$

$\boxed{\text{converges to } 0}$

#4. List the first 4 terms of the sequence:

$a_n = 1-(0.2)^n$

$\boxed{0.8, 0.96, 0.992, 0.9984}$

#5. Find an expression for the nth term:

$\left\{1, -\frac{2}{3}, \frac{4}{9}, -\frac{8}{27}, \ldots\right\}$

$a_n = (-1)^n \frac{2^n}{3^n}$ (n starts at 0)

or

$a_n = (-1)^{n-1}\left(\frac{2}{3}\right)^{n-1}$ (n starts at 1)

#6. Determine if the sequence converges and if so find the limiting value:

$a_n = \frac{3+5n^2}{n+n^2}$

$\boxed{\text{Converges to } 5}$

$a_n = \frac{n!}{2^n}$

$\boxed{\text{diverges}}$

#7. Determine if the sequence is monotonic, bounded, and if it converges:

$a_n = \frac{n}{n^2+1}$

$\boxed{\text{The sequence is monotonic and bounded, and therefore converges.}}$

Write the first 5 terms of the sequence:

#8. $a_n = 4n-3$

$\boxed{1, 5, 9, 13, 17}$

#9. $a_n = (-1)^{n+1}\left(\frac{2}{n}\right)$

$\boxed{2, -1, \frac{2}{3}, \frac{-1}{2}, \frac{2}{5}}$

#10. Write the next two apparent terms

of the sequence: $2, 5, 8, 11, \ldots$

$\boxed{14, 17}$

#11. Simplify: $\frac{(2n-1)!}{(2n+1)!}$

$\boxed{\frac{1}{(2n+1)(2n)}}$

#12. Find the limit (if possible) of the sequence: $a_n = \frac{3n+1}{5n-2}$

$\boxed{\frac{3}{5}}$

Determine the convergence or divergence of the sequence with the given nth term. If the sequence converges, find its limit:

#13. $a_n = \dfrac{5}{n+2}$

$\boxed{\text{converges to } 0}$

#14. $a_n = (-1)^n \left(\dfrac{n}{n+1}\right)$

$\boxed{\text{diverges}}$

#15. $a_n = \dfrac{(n+1)!}{n!}$

$\boxed{\text{diverges}}$

#16. $a_n = \dfrac{\sin(n)}{n}$

$\boxed{\text{converges to } 0}$

#17. Determine whether the sequence is monotonic and whether it is bounded.

$a_n = 4 - \dfrac{1}{n}$

$\boxed{\text{monotonic and bounded}}$

#18. Find an expression for the nth term:

$\dfrac{1}{(2)(3)}, \dfrac{2}{(3)(4)}, \dfrac{3}{(4)(5)}, \dfrac{4}{(5)(6)}, \ldots$

$\boxed{a_n = \dfrac{n}{(n+1)(n+2)}}$

7.2 – Required Practice

Examples: Determine if the series converges and if so to what sum.

#1. $\sum_{n=0}^{\infty} 3\left(\frac{1}{4}\right)^n$

$\boxed{\text{converges to } \frac{3}{1-\left(\frac{1}{4}\right)} = 4}$

#2. $\sum_{n=1}^{\infty} 3\left(\frac{1}{4}\right)^n$

$\boxed{\text{converges to } \frac{\left(\frac{3}{4}\right)}{1-\left(\frac{1}{4}\right)} = 1}$

#3. $\sum_{n=1}^{\infty} 3(1.1)^n$

$\boxed{\text{diverges}}$

#4. Write the repeating decimal as a geometric series and as a ratio of two integers: $0.\overline{36}$

$\boxed{\sum_{n=0}^{\infty} 0.36\left(\frac{1}{100}\right)^n} = \boxed{\frac{36}{99}}$

Determine if the series converges and if so to what sum.

#5. $\sum_{n=1}^{\infty} \frac{n+1}{2n-1}$

$\boxed{\text{diverges by the } n^{th} \text{ term test}}$

#6. $\sum_{n=1}^{\infty} \frac{1}{n}$

none of our tools
work on this one
(stay tuned... more to come...)

Examples

Determine if the series converges and if so to what sum.

#7. $-2+\frac{5}{2}-\frac{25}{8}+\frac{125}{32}-\dots$

This geometric series $\boxed{\text{diverges}}$

#8. $\sum_{n=1}^{\infty}\ln\left(\frac{n}{2n+5}\right)$

$\boxed{\sum_{n=1}^{\infty}\ln\left(\frac{n}{2n+5}\right) \text{ diverges by the } n^{th} \text{ term test}}$

Determine if the series converges and if so to what sum.

#9. $\sum_{n=1}^{\infty}\frac{n}{\sqrt{1+n^2}}$

$\sum_{n=1}^{\infty}\frac{n}{\sqrt{1+n^2}}$ $\boxed{\text{diverges}}$

by the n^{th} term test

Find the values of x for which the series converges, and find the sum of the series for those values of x.

#10. $\sum_{n=1}^{\infty}\frac{x^n}{3^n}$

$\boxed{\sum_{n=1}^{\infty}\frac{x^n}{3^n} \text{ converges for } -3<x<3}$

Find the sequence of partial sums S_1, S_2, S_3, and S_4.

#11. $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\ldots$

$$\boxed{\begin{aligned} S_1 &= 1 \\ S_2 &= \frac{5}{4} \\ S_3 &= \frac{49}{36} \\ S_4 &= \frac{205}{149} \end{aligned}}$$

#12. $3-\frac{9}{2}+\frac{27}{4}-\frac{81}{8}+\frac{243}{16}-\ldots$

$$\boxed{\begin{aligned} S_1 &= 3 \\ S_2 &= -\frac{3}{2} \\ S_3 &= \frac{21}{4} \\ S_4 &= -\frac{39}{8} \end{aligned}}$$

Verify that the infinite series diverges:

#13. $\sum_{n=0}^{\infty} 5\left(\frac{5}{2}\right)^n$

$\boxed{\text{diverges}}$

(geometric)

#14. $\sum_{n=1}^{\infty} \frac{n^2}{n^2+1}$

$\boxed{\text{diverges}}$

(by nth term test)

Verify that the infinite series converges:

#15. $\sum_{n=0}^{\infty}\left(\frac{5}{6}\right)^{n}$ $\boxed{\text{converges}}$ by Geometric Series w/ $|r|<1$

#16. $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$ $\boxed{\text{Converges}}$ to 1

- Not Geometric,
- nth term test inconclusive
- but can find a pattern in the partial sums and evaluate $\lim_{n\to\infty} S_n$

#17. Find the sum of the convergent series

$$\sum_{n=1}^{\infty}-\left(\frac{1}{5}\right)^{n}$$

$$\boxed{S=\frac{\left(-\frac{1}{5}\right)}{1-\left(\frac{1}{5}\right)}=\frac{-1}{4}}$$

#18. Write the repeating decimal as a geometric And write the sum of the series as a fraction:

$0.\overline{81} = \boxed{\frac{81}{99}}$

Determine if the series is convergent or divergent:

#19. $\sum_{n=0}^{\infty} \frac{3^n}{1000}$

$\boxed{\text{diverges}}$ (geometric)

#20. $\sum_{n=2}^{\infty} \frac{n}{\ln(n)}$

$\boxed{\text{diverges}}$ by nth term test

#21. $\sum_{n=1}^{\infty} e^{-n}$ $\boxed{\text{converges}}$ (geometric)

(nth term test $\to 0$, inconclusive)

7.3 – Required Practice

Use the Integral Test to determine if the series converges or diverges:

#1. $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ $\boxed{\text{diverges}}$ by the integral test

#2. $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ $\boxed{\text{converges}}$ by the integral test

Use the Integral Test to determine if the series converges or diverges:

#3. $\sum_{n=3}^{\infty} \frac{1}{n^2+1}$ $\boxed{\text{converges}}$ by the integral test

Determine if the series converges or diverges:

#4. $\sum_{n=1}^{\infty} \frac{1}{n}$

$\boxed{\text{diverges}}$

by the p-series test

#5. $\sum_{n=1}^{\infty} \frac{1}{n^2}$

$\boxed{\text{converges}}$

by the p-series test

Determine if the series converges or diverges:

#6. $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

$\boxed{\text{diverges}}$

by the integral test

#7. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

$\boxed{\text{diverges}}$

by the p-series test

(integral test can be used too)

#8. $\sum_{n=1}^{\infty} \frac{1}{3n+1}$

$\boxed{\text{diverges}}$

by the integral test

Confirm that the integral test applies, then use it to determine if the series converges or diverges.

(must be done to use integral test, even if the problem doesn't state it) **

#9. $\sum_{n=1}^{\infty} \frac{1}{n+3}$

$\boxed{\text{diverges}}$ by the integral test

** look at notes or extra packet problems to see required format for show conditions & work for full credit **

#10. $\sum_{n=1}^{\infty} e^{-n}$ $\boxed{\text{converges}}$ by the integral test

Confirm that the integral test applies, then use it to determine if the series converges or diverges.

#11. $\sum_{n=1}^{\infty} \frac{\ln(n)}{n^2}$ $\boxed{\text{converges}}$ by the integral test

#12. $\sum_{n=1}^{\infty} \frac{4n}{2n^2+1}$ $\boxed{\text{diverges}}$ by the integral test

Explain why the integral test does not apply to the series.

#13. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

Not all a_n are positive

#14. $\sum_{n=1}^{\infty} \frac{2+\sin(n)}{n}$

f is not always decreasing

Use the p-series test to determine the convergence or divergence of the series.

#15. $\sum_{n=1}^{\infty} \frac{1}{n^3}$

converges

by p-series test

#16. $\sum_{n=1}^{\infty} \frac{1}{n^{1/4}}$

diverges

by p-series test

#17. $\sum_{n=1}^{\infty} \frac{1}{n^{1.03}}$

converges

by p-series test

7.4 – Required Practice

#1. Example: Determine if the following series converges or diverges: $\sum_{n=1}^{\infty} \frac{3^n}{2^n - 1}$

$\boxed{\text{diverges}}$ by Direct Comparison Test with $\sum_{n=1}^{\infty} \frac{3^n}{2^n}$

#2. Example: Determine if the following series converges or diverges: $\sum_{n=1}^{\infty} \frac{1}{3n^2 - 4n + 5}$

$\boxed{\text{Converges}}$ by Limit Comparison Test with $\sum_{n=1}^{\infty} \frac{1}{n^2}$

#3. Example: Determine if the following series converges or diverges: $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$

$\boxed{\text{diverges}}$ by Limit Comparison Test with $\sum_{n=1}^{\infty} \frac{1}{n^2}$

#4. Example: Determine if the following series converges or diverges: $\sum_{n=1}^{\infty} \frac{1}{2+\sqrt{n}}$

(use limit comparison test)

$\boxed{\text{diverges}}$ by Limit Comparison Test with $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

Determine if the following series converge or diverge

#5. $\sum_{n=1}^{\infty} \frac{1}{n^3 - 1}$

$\boxed{\text{Converges}}$ by Limit Comparison Test w/ $\sum_{n=1}^{\infty} \frac{1}{n^3}$

(wrong side for direct — does not work)

#6. $\sum_{n=1}^{\infty} \frac{5}{2+3^n}$ $\boxed{\text{Converges}}$ by Direct Comparison Test w/ $\sum_{n=1}^{\infty} \frac{5}{3^n}$

#7. $\sum_{n=1}^{\infty} \frac{4+3^n}{2^n}$

$\boxed{\text{diverges}}$

by Direct Comparison Test

with $\sum_{n=1}^{\infty} \frac{3^n}{2^n}$

#9. $\sum_{n=1}^{\infty} \frac{4+3^n}{2^n}$ (use Limit comparison)

$\boxed{\text{diverges}}$

by Limit Comparison Test

with $\sum_{n=1}^{\infty} \frac{3^n}{2^n}$

#8. $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^5+4}}$

$\boxed{\text{Converges}}$ by Direct Comparison Test

w/ $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^5}}$

Use the Direct Comparison Test to determine the convergence or divergence of the series.

#10. $\sum_{n=1}^{\infty} \frac{1}{2n-1}$ $\boxed{\text{diverges}}$

by Direct Comparison Test with $\sum_{n=1}^{\infty} \frac{1}{2n}$

#11. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}-1}$ $\boxed{\text{diverges}}$

by Direct Comparison Test with $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

#12. $\sum_{n=1}^{\infty} \frac{1}{4\sqrt[3]{n}-1}$ $\boxed{\text{diverges}}$

by Direct Comparison Test with $\sum_{n=1}^{\infty} \frac{1}{4\sqrt[3]{n}}$

Use the Limit Comparison Test to determine the convergence or divergence of the series.

#13. $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ $\boxed{\text{diverges}}$

by Limit Comparison Test with $\sum_{n=1}^{\infty} \frac{n}{n^2}$

#14. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$ $\boxed{\text{diverges}}$

by Limit Comparison Test with $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2}}$

#15. $\sum_{n=1}^{\infty} \frac{2^n+1}{5^n+1}$ $\boxed{\text{converges}}$

by Limit Comparison Test with $\sum_{n=1}^{\infty} \frac{2^n}{5^n}$

#16. Which test (nth-Term, Geometric, p-Series, Integral, Direct Comparison, or Limit Comparison) would you use to determine the convergence of the series? (Don't actually perform the test, just state which test is best…each test must be used at least once.)

$$\sum_{n=1}^{\infty}\frac{\sqrt[3]{n}}{n}$$ p-series

$$\sum_{n=0}^{\infty}5\left(-\frac{4}{3}\right)^n$$ Geometric

$$\sum_{n=1}^{\infty}\frac{1}{5^n+1}$$ Direct Comparison w/ $\sum_{n=1}^{\infty}\frac{1}{5^n}$ (or Limit)

$$\sum_{n=1}^{\infty}2\frac{1}{n^3-8}$$ Limit Comparison w/ $\sum_{n=1}^{\infty}2\frac{1}{n^3}$ (direct won't work)

$$\sum_{n=1}^{\infty}\frac{2n}{3n-2}$$ nth term test

$$\sum_{n=1}^{\infty}\frac{n}{(n^2+1)^2}$$ Integral test

$$\sum_{n=1}^{\infty}\frac{3}{n(n+3)}$$ Direct Comparison w/ $\sum_{n=1}^{\infty}\frac{3}{n^2}$ (or Limit)

7.5 – Required Practice

Determine the convergence or divergence of the series.

#1. Example: $\sum_{n=1}^{\infty}(-1)^n \frac{n}{2^{n-1}}$

$\boxed{\text{Converges}}$ by the Alternating Series Test

** Look at the examples in the notes
or extra practice packet to see
how to format all the required supporting work **

#2. $\sum_{n=1}^{\infty}(-1)^{n+1} \frac{n^2}{n^2+4}$ $\boxed{\text{diverges}}$

by the nth term test

(cannot use Alternating Series Test)

Determine the convergence or divergence of the series.

#3. $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\ln(n)}{n}$

$\boxed{\text{converges}}$

by the Alternating Series Test

#4. $-\frac{1}{3}+\frac{2}{4}-\frac{3}{5}+\frac{4}{6}-\frac{5}{7}+\ldots$

$\boxed{\text{diverges}}$

by the nth term test

(Alt. series test does not apply)

Determine the convergence or divergence of the series.

#5. $\sum_{n=1}^{\infty} \frac{4}{n(n+4)}$

Telescoping series $\boxed{\text{converges}}$ (to the sum $\frac{25}{12}$)

#6. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1}$

$\boxed{\text{converges}}$

by the Alternating Series Test

Determine the convergence or divergence of the series.

#7. $\sum_{n=1}^{\infty}(-1)^n\frac{1}{3^n}$ $\boxed{\text{converges}}$

by the Alternating Series Test

#8. $\sum_{n=1}^{\infty}(-1)^n\frac{5n-1}{4n+1}$ $\boxed{\text{diverges}}$

by the nth term test

#9. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\ln(n)}{n}$ $\boxed{\text{converges}}$

by the Alternating Series Test

Determine the convergence or divergence of the series.

#10. $\sum_{n=1}^{\infty}(-1)^n\frac{1}{\sqrt{n}}$ $\boxed{\text{converges}}$

by the Alternating Series Test

#11. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n+1}{\ln(n+1)}$ $\boxed{\text{diverges}}$

by the nth term test

#12. $\sum_{n=1}^{\infty}\sin\left(\frac{(2n+1)\pi}{2}\right)$ $\boxed{\text{diverges}}$

by the nth term test

(term values oscillate between −1 and +1)

Determine the convergence or divergence of the series.

#13. $\sum_{n=0}^{\infty}(-1)^n \frac{1}{n!}$ $\boxed{\text{converges}}$

by the Alternating Series test

#14. $\sum_{n=1}^{\infty}(-1)^{n+1} \frac{\sqrt{n}}{n+2}$ $\boxed{\text{Converges}}$

by the Alternating Series Test

#15. $\sum_{n=1}^{\infty}\frac{6}{n(n+2)} = \sum_{n=1}^{\infty}\left(\frac{3}{n}-\frac{3}{n+2}\right)$ (by partial fraction expansion)

Telescoping Series $\boxed{\text{Converges}}$

(to the sum $\frac{9}{2}$)

7.6 – Required Practice

Determine whether the series coverges absolutely or conditionally, or diverges.

#1. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ $\boxed{\text{converges absolutely}}$

#2. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+3}$ $\boxed{\text{converges conditionally}}$

** As always with this unit, take a close look at all the supporting work required in notes or extra packet **

#3. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(2n+3)}{n+10}$ $\boxed{\text{diverges}}$

(by nth term test)

Determine whether the series converges absolutely, conditionally, or diverges.

#4. $\sum_{n=1}^{\infty}(-1)^n\frac{1}{2^n}$ $\boxed{\text{converges absolutely}}$

Determine whether the series converges absolutely, conditionally, or diverges.

#5. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{\sqrt{n}}$ $\boxed{\text{Converges conditionally}}$

#6. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2}{(n+1)^2}$ $\boxed{\text{diverges}}$

Determine whether the series converges absolutely, conditionally, or diverges.

7. $\sum_{n=2}^{\infty}(-1)^n \frac{n}{n^3-5}$

$\boxed{\text{Converges absolutely}}$

7.7 – Required Practice

Determine whether the series converges absolutely, conditionally, or diverges.

#1. $\sum_{n=1}^{\infty} \frac{2^n}{n!}$

converges absolutely

by the Ratio Test

#2. $a_1 = 2, \quad a_{n+1} = \frac{5n+1}{4n+3} a_n$

2, 2, 2.13333333, 2.357894....

diverges

by the Ratio Test

Determine whether the series converges absolutely, conditionally, or diverges.

#3. $\sum_{n=1}^{\infty}(-1)^n \frac{\sqrt{n}}{n+1}$

① Ratio Test

$$\lim_{n\to\infty}\left|\frac{\left(\frac{\sqrt{n+1}}{(n+1)+1}\right)}{\left(\frac{\sqrt{n}}{n+1}\right)}\right|$$

$$=\lim_{n\to\infty}\left|\frac{(n+1)\sqrt{n+1}}{(n+2)\sqrt{n}}\right|$$

$$=\left(\lim_{n\to\infty}\frac{n+1}{n+2}\right)\left(\lim_{n\to\infty}\frac{\sqrt{n+1}}{\sqrt{n}}\right)$$

$$=(1)\sqrt{\lim_{n\to\infty}\frac{n+1}{n}}$$

$$=(1)\sqrt{1}$$

$$=1$$

(inconclusive)

② check $\sum|a_n| = \sum_{n=1}^{\infty}\frac{\sqrt{n}}{n+1}$

Limit Comparison with $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}\frac{\sqrt{n}}{n} = \sum_{n=1}^{\infty}\frac{n^{1/2}}{n} = \sum_{n=1}^{\infty}\frac{1}{n^{1/2}}$

p-series, $p=1/2$, diverges

$$\lim_{n\to\infty}\frac{a_n}{b_n}$$

$$=\lim_{n\to\infty}\frac{\left(\frac{\sqrt{n}}{n+1}\right)}{\left(\frac{\sqrt{n}}{n}\right)}$$

$=\lim_{n\to\infty}\frac{n}{n+1}=1$ (finite, positive, series are "linked")

$\therefore \sum_{n=1}^{\infty}\frac{\sqrt{n}}{n+1}$ also diverges by Limit Comparison Test

③ Alternating Series Test on $\sum_{n=1}^{\infty}(-1)^n\frac{\sqrt{n}}{n+1}$

- $\lim_{n\to\infty}\frac{\sqrt{n}}{n+1}$ $\quad \lim_{n\to\infty}\sqrt{n}=\infty$, $\lim_{n\to\infty}(n+1)=\infty$ $\quad \left(\frac{\infty}{\infty}\right)$, use L'Hopital's Rule

$$=\lim_{n\to\infty}\frac{\frac{1}{2}n^{-1/2}}{1}=\lim_{n\to\infty}\frac{1}{2\sqrt{n}}=0 \checkmark$$

- $a_{n+1}\overset{?}{\leq}a_n$ $\quad f(x)=\frac{\sqrt{x}}{x+1}=\frac{x^{1/2}}{x+1}$

$$f'(x)=\frac{(x+1)\left(\frac{1}{2}x^{-1/2}\right)-x^{1/2}(1)}{(x+1)^2}$$

$$=\frac{\frac{x+1}{2\sqrt{x}}-\frac{\sqrt{x}}{1}}{(x+1)^2}=\frac{\frac{x+1}{2\sqrt{x}}-\frac{\sqrt{x}}{1}\left(\frac{2\sqrt{x}}{2\sqrt{x}}\right)}{(x+1)^2}$$

$$=\frac{x+1-2x}{2\sqrt{x}(x+1)^2}=\frac{-x+1}{2\sqrt{x}(x+1)^2}<0 \text{ for } x>1$$

decreasing, so $a_{n+1}\leq a_n$ ✓

$\therefore \sum_{n=1}^{\infty}(-1)^n\frac{\sqrt{n}}{n+1}$ converges by Alternating Series Test

④ Since $\sum_{n=1}^{\infty}\frac{\sqrt{n}}{n+1}$ diverges, but $\sum_{n=1}^{\infty}(-1)^n\frac{\sqrt{n}}{n+1}$ converges,

$\boxed{\sum_{n=1}^{\infty}(-1)^n\frac{\sqrt{n}}{n+1} \text{ is conditionally convergent}}$

Determine whether the series converges absolutely, conditionally, or diverges.

#4. $\sum_{n=1}^{\infty} \frac{e^{2n}}{n^n}$

converges absolutely

by the Root Test

#5. $\sum_{n=1}^{\infty} \left(\frac{2n}{n+1}\right)^n$

diverges

by the Root Test

Simplify the expression fully.

#6. $\dfrac{n5^{n+2}}{5^{n+1}} = \boxed{5n}$

#7. $\dfrac{(n+1)!}{(n-2)!} = \boxed{(n+1)(n)(n-1)}$

Use the Ratio Test to determine the convergence or divergence of the series.

#8. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{5^n}$ $\boxed{\text{converges}}$

by the Ratio Test

#9. $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{3^n}$ $\boxed{\text{diverges}}$

by the Ratio Test

#10. $\displaystyle\sum_{n=1}^{\infty} \frac{9^n}{n^5}$ $\boxed{\text{diverges}}$

by the Ratio Test

#11. $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{n^5}$ $\boxed{\text{diverges}}$

by the Ratio Test

Use the Ratio Test to determine the convergence or divergence of the series.

#12. $\sum_{n=0}^{\infty} \frac{(-1)^n 2^{4n}}{(2n+1)!}$

$\boxed{\text{Converges absolutely}}$

by the Ratio Test

#13. $\sum_{n=0}^{\infty} \frac{5^n}{2^n+1}$

$\boxed{\text{diverges}}$

by the Ratio Test

Use the Root Test to determine the convergence or divergence of the series.

#14. $\sum_{n=1}^{\infty}\left(\frac{n}{2n+1}\right)^{n}$

$\boxed{\text{converges}}$
by the Root Test

#15. $\sum_{n=2}^{\infty}\frac{(-1)^{n}}{(\ln(n))^{n}}$

$\boxed{\text{converges}}$
by the Root Test

#16. $\sum_{n=1}^{\infty}\left(2\sqrt[n]{n}+1\right)^{n}$

$\boxed{\text{diverges}}$
by the Root Test

#17. $\sum_{n=0}^{\infty}e^{-3n}$

$\boxed{\text{converges}}$
by the Root Test

7.8 – Required Practice

#1. Example: Approximate the sum of the series by its first six terms: $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n!} = S$

$$\boxed{\begin{array}{c} \frac{91}{144} - \frac{1}{5040} < S < \frac{91}{144} + \frac{1}{5040} \\ 0.631746 < S < 0.632143 \end{array}}$$

#2. Example: Determine the number of terms required to approximate the sum of the series with an error of less than 0.001: $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^4}$

$\boxed{\text{include } N=5 \text{ terms}}$

#3. How many terms of the series do we need to approximate the sum to an error of < 0.002? $\sum_{n=1}^{\infty}\frac{(-1)^n n}{4^n}$

$\boxed{\text{include } N=5 \text{ terms}}$

Approximate the sum of the series using the first 6 terms.

#4. $\sum_{n=1}^{\infty} \frac{5(-1)^n}{n!} \approx \boxed{\frac{-455}{144}}$

#5. $\sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n^3} \approx \boxed{1.76867}$

How many terms are required to approximate the series with an error of less than 0.001?

#6. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}$

$\boxed{\text{Include } N=10 \text{ terms}}$

Unit 7 Part 1 Test Review

Determine whether the series diverges or converges (you must show work to justify using a valid and identify the test).

#1) $\sum_{n=1}^{\infty}\left(\frac{2n^3+1}{n^3-1}\right)^n$ (root test)

$\lim_{n\to\infty} \sqrt[n]{\left(\frac{2n^3+1}{n^3-1}\right)^n}$

$\lim_{n\to\infty} \frac{2n^3+1}{n^3-1} = 2 > 1$

$\therefore$ $\boxed{\sum_{n=1}^{\infty}\left(\frac{2n^3+1}{n^3-1}\right)^n \text{ diverges by the root test}}$

#2) $\sum_{n=1}^{\infty}\frac{8^n}{5^n} = \sum_{n=1}^{\infty}\left(\frac{8}{5}\right)^n$

Geometric, w $|r| = \left|\frac{8}{5}\right| > 1$

$\boxed{\sum_{n=1}^{\infty}\frac{8^n}{5^n} \text{ diverges by Geometric Series Test}}$

#3) $\sum_{n=1}^{\infty}\frac{3^n}{4^n-1}$ (Limit Comparison)

w/ $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}\frac{3^n}{4^n} = \sum_{n=1}^{\infty}\left(\frac{3}{4}\right)^n$

Geometric, w/ $|r| = \left|\frac{3}{4}\right| < 1$ converges

$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\left(\frac{3^n}{4^n-1}\right)}{\left(\frac{3^n}{4^n}\right)} = \lim_{n\to\infty}\frac{4^n}{4^n-1} = 1$ (don't need to show L'Hôpital, but could)

finite, positive, so series "linked"

$\therefore$ Since $\sum_{n=1}^{\infty}\frac{3^4}{4^n}$ converges

$\boxed{\sum_{n=1}^{\infty}\frac{3^n}{4^n-1} \text{ also converges by Limit Comparison Test}}$

(Direct Compare doesn't work)

#4) $\sum_{n=1}^{\infty}\frac{n}{\ln(n)}$ (nth term)

$\lim_{n\to\infty}\frac{n}{\ln(n)}$ $\quad \lim_{n\to\infty} n = \infty$, $\lim_{n\to\infty}\ln(n) = \infty$ $\left(\frac{\infty}{\infty}\right)$ L'Hôp

$= \lim_{n\to\infty}\frac{1}{\left(\frac{1}{n}\right)} = \lim_{n\to\infty} n = \infty \neq 0$

$\therefore$ $\boxed{\sum_{n=1}^{\infty}\frac{n}{\ln(n)} \text{ diverges by nth term test}}$

Determine whether the series diverges or converges (you must show work to justify using a valid and identify the test).

#5) $\sum_{n=1}^{\infty} \frac{n!}{3^n}$ (ratio test)

$\lim_{n\to\infty} \left| \frac{(n+1)!}{3^{n+1}} \cdot \frac{3^n}{n!} \right|$

$\lim_{n\to\infty} \left| \frac{(n+1)n!\, 3^n}{3 \cdot 3^n\, n!} \right|$

$\lim_{n\to\infty} \left| \frac{n+1}{3} \right| = \infty$

$\therefore \sum_{n=1}^{\infty} \frac{n!}{3^n}$ diverges by the Ratio Test

#6) $\sum_{n=1}^{\infty} \frac{1}{5^n+1}$ (Direct Comparison)

compare w/ $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{5^n} = \sum_{n=1}^{\infty} \left(\frac{1}{5}\right)^n$

Geometric w/ $|r| = |\frac{1}{5}| < 1$

converges

$\frac{1}{5^n+1} < \frac{1}{5^n}$

(correct side)

$\therefore$ Since $\sum_{n=1}^{\infty} \frac{1}{5^n}$ converges

and $\frac{1}{5^n+1} < \frac{1}{5^n}$

$\sum_{n=1}^{\infty} \frac{1}{5^n+1}$ also converges by the Direct Comparison Test

#7) $\sum_{n=1}^{\infty} \frac{4n}{2n^2+1}$ (integral test)

✓ a_n positive for $n \geq 1$

✓ $f(x) = \frac{4x}{2x^2+1}$ continuous for $x \geq 1$

✓ $f'(x) = \frac{(2x^2+1)(4) - 4x(4x)}{(2x^2+1)^2}$

$= \frac{8x^2+4-16x^2}{(2x^2+1)^2} = \frac{-8x^2+4}{(2x^2+1)^2} < 0$ for $x \geq 1$

decreasing ✓

$\therefore$ integral test applies:

$\int_1^{\infty} \frac{4x}{2x^2+1}\,dx$ $\quad u = 2x^2+1$, $du = 4x\,dx$

$\lim_{b\to\infty} \int_3^b \frac{1}{u}\,du$

$\lim_{b\to\infty} \left[\ln|u| \right]_3^b$

$\lim_{b\to\infty} \ln b - \ln(3)$

∞

Integral diverges,

$\therefore \sum_{n=1}^{\infty} \frac{4n}{2n^2+1}$ diverges by Integral Test

#8) $\sum_{n=1}^{\infty} \frac{2}{n^2} = 2 \sum_{n=1}^{\infty} \frac{1}{n^2}$

p-series, w/ $p=2$

$\therefore \sum_{n=1}^{\infty} \frac{2}{n^2}$ converges by p-series Test

Determine whether the series diverges or converges (you must show work to justify using a valid and identify the test).

#9) $\sum_{n=1}^{\infty} \frac{2n^3}{n^3+4}$ (nth term)

$\lim_{n\to\infty} \frac{2n^3}{n^3+4} = 2 \neq 0$

$\boxed{\therefore \sum_{n=1}^{\infty} \frac{2n^3}{n^3+4} \text{ diverges by nth term test}}$

#10) $\sum_{n=1}^{\infty} (-1)^n \frac{5n-1}{4n+1}$ (nth term test)

$\lim_{n\to\infty} \frac{5n-1}{4n+1} = \frac{5}{4} \neq 0$

$\boxed{\therefore \sum_{n=1}^{\infty} (-1)^n \frac{5n-1}{4n+1} \text{ diverges by nth term test}}$

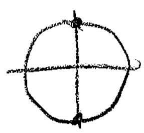
(don't need to do conditional vs absolute unless problem specifically asks for it)

#11) $\sum_{n=0}^{\infty} 5\frac{2^n}{3^n}$

Geometric

w/ $|r| = \left|\frac{2}{3}\right| < 1$

$\boxed{\therefore \sum_{n=0}^{\infty} 5\left(\frac{2}{3}\right)^n \text{ converges by Geometric Series Test}}$



#12) $\sum_{n=1}^{\infty} \sin\left(\frac{(2n-1)\pi}{2}\right)$ (write out some terms)

$\sin\left(\frac{\pi}{2}\right) + \sin\left(\frac{3\pi}{2}\right) + \sin\left(\frac{5\pi}{2}\right) + \sin\left(\frac{7\pi}{2}\right) + \cdots$

$(1) + (-1) + (1) + (-1) + \cdots$

oscillating

so $\lim_{n\to\infty} a_n \neq 0$

$\boxed{\therefore \sum_{n=1}^{\infty} \sin\left(\frac{(2n-1)\pi}{2}\right) \text{ diverges by nth term test}}$

Determine whether the series diverges or converges (you must show work to justify using a valid and identify the test).

#13) $\sum_{n=1}^{\infty} \frac{1}{5^n} = \sum_{n=1}^{\infty} \frac{1^n}{5^n} = \sum_{n=1}^{\infty} \left(\frac{1}{5}\right)^n$

Geometric

w/ $|r| = |\frac{1}{5}| < 1$

$\boxed{\therefore \sum_{n=1}^{\infty} \frac{1}{5^n} \text{ converges by Geometric Series Test}}$

#14) $\sum_{n=1}^{\infty} \frac{1}{2n-1}$ (Direct (or Limit) Comparison)

w/ $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n}$ p-series w/ $p=1$ diverges

$\frac{1}{2n} < \frac{1}{2n-1}$

✓ → correct side for direct

Since $\sum_{n=1}^{\infty} \frac{1}{2n}$ diverges

and $\frac{1}{2n} < \frac{1}{2n-1}$

$\boxed{\sum_{n=1}^{\infty} \frac{1}{2n-1} \text{ also diverges by Direct Comparison Test}}$

#15) $\sum_{n=1}^{\infty} \frac{\ln(n)}{n^2}$ (integral test)

✓ • a_n positive for $n > 1$

✓ • $f(x) = \frac{\ln(x)}{x^2}$ continuous for $n \geq 1$

✓ • $f'(x) = \frac{x^2 \frac{1}{x} - \ln(x)(2x)}{(x^2)^2}$

$= \frac{x - 2x\ln x}{x^4} = \frac{x(1-2\ln x)}{x^4} < 0$ for $x > 1$

(decreasing)

$\therefore$ integral test applies

$\int_1^{\infty} \frac{\ln x}{x^2}\,dx$ by parts: $u = \ln x$ $dv = x^{-2}dx$

$du = \frac{1}{x}dx$ $v = \frac{-1}{x}$

$\lim_{b\to\infty} \left[uv - \int v\,du\right]_1^b = \lim_{b\to\infty} \left[-\frac{1}{x}\ln x + \int x^{-2}dx\right]_1^b$

$\lim_{b\to\infty} \left[-\frac{\ln x}{x} - \frac{1}{x}\right]_1^b$

$\lim_{b\to\infty} \left[-\frac{\ln b}{b} - \frac{1}{b}\right] - \left[-\frac{\ln(1)}{1} - \frac{1}{1}\right]$

$\left(\frac{\infty}{\infty}\right)$ L'Hop: $\lim_{b\to\infty} -\frac{\frac{1}{b}}{1}$

$[0 - 0] + \cancel{\ln(1)}^{0} + 1$

Integral converges

$\boxed{\therefore \sum_{n=1}^{\infty} \frac{\ln(n)}{n^2} \text{ converges by Integral Test}}$

#16) $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ (Limit Comparison)

w/ $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n}$ p-series w/ $p=1$ diverges

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{b\to\infty} \frac{\left(\frac{n}{n^2+1}\right)}{\left(\frac{n}{n^2}\right)} = \lim_{b\to\infty} \frac{n^2}{n^2+1} = 1$

finite, positive (series "linked")

$\therefore$ Since $\sum_{n=1}^{\infty} \frac{n}{n^2}$ diverges

$\boxed{\sum_{n=1}^{\infty} \frac{n}{n^2+1} \text{ also diverges by Limit Comparison Test}}$

Determine whether the series diverges or converges (you must show work to justify using a valid and identify the test).

#17) $\sum_{n=1}^{\infty}(-2)^n$ (nth term)

$\lim_{n\to\infty} (-2)^n \neq 0$

$(-2)+(4)+(-8)+(16)\cdots$

$\boxed{\therefore \sum_{n=1}^{\infty} (-2)^n \text{ diverges by the nth term test}}$

#18) $\sum_{n=1}^{\infty}\frac{5n^4}{n^4+n^2+7}$ (nth term test)

$\lim_{n\to\infty} \frac{5n^4}{n^4+n^2+7} = 5 \neq 0$

$\boxed{\therefore \sum_{n=1}^{\infty} \frac{5n^4}{n^4+n^2+7} \text{ diverges by the nth term test}}$

#19) $\sum_{n=1}^{\infty}\frac{9^n}{n^5}$ (ratio test)

$\lim_{n\to\infty} \left| \frac{9^{n+1}}{(n+1)^5} \frac{n^5}{9^n} \right|$

$\lim_{n\to\infty} \left| \frac{9 \cdot 9^n \, n^5}{(n+1)^5 \, 9^n} \right|$

$\lim_{n\to\infty} \left| \frac{9n^5}{(n+1)^5} \right|$

$\lim_{n\to\infty} \left| \frac{9n^5}{n^5+\cdots} \right| = 9 > 1$

$\boxed{\therefore \sum_{n=1}^{\infty} \frac{9^n}{5^n} \text{ diverges by the Ratio Test}}$

#20) $\sum_{n=1}^{\infty}\frac{5}{n^{0.4}} = 5 \sum_{n=1}^{\infty} \frac{1}{n^{0.4}}$

p-series w/ $p = 0.4$

$\boxed{\therefore \sum_{n=1}^{\infty} \frac{5}{n^{0.4}} \text{ diverges by p-series Test}}$

Determine whether the series diverges or converges (you must show work to justify using a valid and identify the test).

#21) $\sum_{n=1}^{\infty}\left(\frac{n}{2n+1}\right)^n$ (root test)

$\lim_{n\to\infty} \sqrt[n]{\left(\frac{n}{2n+1}\right)^n}$

$\lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2} < 1$

$\boxed{\therefore \sum_{n=1}^{\infty}\left(\frac{n}{2n+1}\right)^n \text{ converges by the Root Test}}$

#22) $\sum_{n=1}^{\infty}(-1)^n\frac{1}{3^n}$ (alternating series test)

- $\lim_{n\to\infty} \frac{1}{3^n} = 0$ ✓
- $a_{n+1} \le a_n$? ✓

$\frac{1}{3^{n+1}} < \frac{1}{3^n}$

Since $\lim_{n\to\infty} a_n = 0$ and $a_{n+1} \le a_n$ the alternating series

$\boxed{\sum_{n=1}^{\infty}(-1)^n\frac{1}{3^n} \text{ converges by the Alternating Series Test}}$

Determine whether the series converges absolutely, conditionally, or diverges:

#23) $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n!}$

check $\sum|a_n| = \sum_{n=1}^{\infty}\frac{1}{n!}$ (ratio test)

$\lim_{n\to\infty}\left|\frac{\left(\frac{1}{(n+1)!}\right)}{\left(\frac{1}{n!}\right)}\right|$

$\lim_{n\to\infty}\left|\frac{n!}{(n+1)!}\right| = \lim_{n\to\infty}\left|\frac{n!}{(n+1)n!}\right|$

$\lim_{n\to\infty}\left|\frac{1}{n+1}\right| = 0 < 1$

$\therefore \sum_{n=1}^{\infty}\frac{1}{n!}$ converges (by Ratio Test)

Since $\sum_{n=1}^{\infty}\frac{1}{n!}$ converges,

$\boxed{\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n!} \text{ converges absolutely}}$

Determine whether the series converges absolutely, conditionally, or diverges:

#24) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{\sqrt{n}}$

check $\sum |a_n| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

$= \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$

p-series w/ $p = 1/2$

diverges

now, alternating Series test:

- $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$ ✓
- $a_{n+1} \overset{?}{\leq} a_n$

$\frac{1}{\sqrt{n+1}} \leq \frac{1}{\sqrt{n}}$ ✓

$\therefore \sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{\sqrt{n}}$ converges

Since $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges, but $\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{\sqrt{n}}$ converges,

$\boxed{\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{\sqrt{n}} \text{ converges conditionally}}$

#25) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2}{(n+1)^2}$

check $\sum |a_n| = \sum_{n=1}^{\infty} \frac{n^2}{(n+1)^2}$

(nth term test)

$\lim_{n\to\infty} \frac{n^2}{(n+1)^2}$

$\lim_{n\to\infty} \frac{n^2}{n^2+2n+1} = 1 \neq 0$

$\therefore \sum_{n=1}^{\infty} \frac{n^2}{(n+1)^2}$ diverges by nth term test

now, alternating series test:

- $\lim_{n\to\infty} \frac{n^2}{(n+1)^2} = 1 \neq 0$

(Alternating Series Test does not apply)

$\boxed{\therefore \sum_{n=1}^{\infty} (-1)^{n+1}\frac{n^2}{(n+1)^2} \text{ diverges by nth term test}}$

#26) Determine the minimum number of terms required to approximate the sum of the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{2n^3-1}$ with an error of less than 0.0005.

$|\text{error}| < |\text{1st neglected term}|$

$|(N+1)\text{ term}| < 0{,}0005$

$$\frac{1}{2(N+1)^3-1} < 0{,}0005$$

$$2(N+1)^3-1 > 2000$$

$$2(N+1)^3 > 2001$$

$$(N+1)^3 > \frac{2001}{2}$$

$$N+1 > \sqrt[3]{\frac{2001}{2}}$$

$$N > \sqrt[3]{\frac{2001}{2}} - 1 = 9.001666389 \nearrow$$

$\boxed{\text{include } N = 10 \text{ terms}}$