AP Calculus BC – Unit 7 Part 2 Extra Practice

7.9 – Extra Practice

State where the power series is centered.

#9b. $\sum_{n=1}^{\infty}(-1)^n\frac{2n-1}{2^n n!}(x)^n$ $(x-0)$

$\boxed{c=0}$

#10b. $\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n)!}(x-\pi)^{2n}$

$\boxed{c=\pi}$

Find the radius of convergence of the power series.

#11b. $\sum_{n=0}^{\infty}(3x)^n$

ratio test

$\lim_{n\to\infty}\left|\frac{(3x)^{n+1}}{(3x)^n}\right|$

$\lim_{n\to\infty}\left|\frac{(3x)(3x)^n}{(3x)^n}\right|$

$\lim_{n\to\infty}|3x|$

$|3x|<1$

$-1<3x<1$

$-\frac{1}{3}<x<\frac{1}{3}$ $\boxed{R=\frac{1}{3}}$

#12b. $\sum_{n=0}^{\infty}(-1)^n\frac{x^n}{5^n}$

ratio test

$\lim_{n\to\infty}\left|\frac{x^{n+1}}{5^{n+1}}\frac{5^n}{x^n}\right|$

$\lim_{n\to\infty}\left|\frac{x\,x^n\,5^n}{5\cdot5^n x^n}\right|$

$\lim_{n\to\infty}\left|\frac{x}{5}\right|$

$\left|\frac{x}{5}\right|<1$

$-1<\frac{x}{5}<1$

$-5<x<5$ $\boxed{R=5}$

#13b. $\sum_{n=0}^{\infty}\frac{(2n)!x^{2n}}{n!}$

ratio test

$\lim_{n\to\infty}\left|\frac{(2(n+1))!\,x^{2(n+1)}}{(n+1)!}\frac{n!}{(2n)!\,x^{2n}}\right|$

$\lim_{n\to\infty}\left|\frac{(2n+2)!\,x^{2n+2}}{(n+1)!}\frac{n!}{(2n)!\,x^{2n}}\right|$

$\lim_{n\to\infty}\left|\frac{(2n+2)(2n+1)(2n)!\,x^{2n}x^2\,n!}{(n+1)n!\,(2n)!\,x^{2n}}\right|$

$\left(\lim_{n\to\infty}\frac{(2n+2)(2n+1)}{n+1}\right)|x^2|$

$\left(\lim_{n\to\infty}\frac{4n^2+7n+2}{n+1}\right)|x^2|$

$\infty|x^2|<1$

only converges if $x=0$

$\therefore$ $\boxed{R=0}$

Find the interval of convergence of the power series (you must check each endpoint).

#14b. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{(n+1)4^{n+1}}(x-3)^{n+1}$

ratio test

$$\lim_{n\to\infty} \left| \frac{(x-3)^{(n+1)+1}}{((n+1)+1)\,4^{(n+1)+1}} \cdot \frac{(n+1)4^{n+1}}{(x-3)^{n+1}} \right|$$

can start here ↗

$$\lim_{n\to\infty} \left| \frac{(x-3)^{n+2}}{(n+2)4^{n+2}} \cdot \frac{(n+1)4^{n+1}}{(x-3)^{n+1}} \right|$$

:)

$$\lim_{n\to\infty} \left| \frac{(x-3)(x-3)^{n+1}(n+1)4^{n+1}}{(n+2)\,4 \cdot 4^{n+1}(x-3)^{n+1}} \right|$$

$$\lim_{n\to\infty} \left| \frac{(x-3)(n+1)}{(n+2)4} \right|$$

$$\left(\lim_{n\to\infty} \frac{n+1}{n+2}\right) \left| \frac{x-3}{4} \right|$$

$$(1) \left| \frac{x-3}{4} \right| < 1$$

$$-1 < \frac{x-3}{4} < 1$$

$$-4 < x-3 < 4$$

$$\underline{-1} < x < \underline{7}$$

(test endpoints)

$\underline{x=-1}$

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)4^{n+1}}(-1-3)^{n+1}$$

$$\sum_{n=0}^{\infty} \frac{(-4)^{n+1}}{(n+1)4^{n+1}}$$

$$\sum_{n=0}^{\infty} \frac{1}{n+1}\left(\frac{-4}{4}\right)^{n+1}$$

$$\sum_{n=0}^{\infty} (-1)^{n+1}\frac{1}{n+1}$$

alternating series test:

- $\lim_{n\to\infty} \frac{1}{n+1} = 0$ ✓
- $a_{n+1} < a_n$?

$$\frac{1}{n+2} < \frac{1}{n+1} \checkmark$$

$\underline{\text{converges}}$

$\underline{x=7}$

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)4^{n+1}}(7-3)^{n+1}$$

$$\sum_{n=0}^{\infty} \frac{1}{n+1}\left(\frac{4^{n+1}}{4^{n+1}}\right)$$

$$\sum_{n=0}^{\infty} \frac{1}{n+1}(1)$$

$$\sum_{n=0}^{\infty} \frac{1}{n+1}$$

Limit Compare w/

$$\sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} \frac{1}{n}$$

p-series w/ p=1 (diverges)

$$\lim_{n\to\infty} \frac{a_n}{b_n}$$

$$\lim_{n\to\infty} \frac{\left(\frac{1}{n+1}\right)}{\left(\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{n}{n+1} = 1$$

finite, positive "linked"

also $\underline{\text{diverges}}$

∴ interval of convergence is $\boxed{-1 \le x < 7 \text{ or } [-1,7)}$

Find the interval of convergence of the power series (you must check each endpoint).

#15b. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n2^n}(x-2)^n$

ratio test

$\lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{(n+1)2^{n+1}}\frac{n2^n}{(x-2)^n}\right|$

$\lim_{n\to\infty}\left|\frac{(x-2)(x-2)^n\, n2^n}{(n+1)2^1 2^n\,(x-2)^n}\right|$

$\lim_{n\to\infty}\left|\frac{(x-2)n}{(n+1)2}\right|$

$\left(\lim_{n\to\infty}\frac{n}{n+1}\right)\left|\frac{x-2}{2}\right|$

$(1)\left|\frac{x-2}{2}\right|<1$

$-1<\frac{x-2}{2}<1$

$-2<x-2<2$

$\underline{0}<x<\underline{4}$

(test ends)

$\underline{x=0}$

$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n2^n}(0-2)^n$

$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n2^n}(-2)^n$

$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}\left(\frac{-2}{2}\right)^n$

$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}(-1)^n$

$\sum_{n=1}^{\infty}(-1)^n(-1)\frac{1}{n}(-1)^n$

$\sum_{n=1}^{\infty}((-1)(-1))^n(-1)\frac{1}{n}$

$(-1)\sum_{n=1}^{\infty}(1)^n\frac{1}{n}$

$(-1)\sum_{n=1}^{\infty}\frac{1}{n}$ p-series w/ $p=1$ diverges

$\underline{x=4}$

$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n2^n}(4-2)^n$

$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n2^n}2^n$

$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}\left(\frac{2}{2}\right)^n$

$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}(1)^n$ just 1 can be removed

$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$

alternating series test

- $\lim_{n\to\infty}\frac{1}{n}=0$ ✓
- $a_{n+1}\le a_n$? $\frac{1}{n+1}\le\frac{1}{n}$ ✓

converges

$\therefore$ interval of convergence is $\boxed{0<x\le 4 \text{ or } (0,4]}$

#16b. (no additional problem – please see the lesson notes for examples of how to work #16).

7.10 – Extra Practice

Find the geometric power series for the function, centered at 0.

#5b. $f(x)=\dfrac{2}{5-x}$ goal: $\dfrac{a}{1-x}$

$\dfrac{2}{5\left(1-\left(\frac{x}{5}\right)\right)}$

$\dfrac{\left(\frac{2}{5}\right)}{1-\left(\frac{x}{5}\right)}$ $a=\frac{2}{5}$ $r=\frac{x}{5}$

$\boxed{\sum_{n=0}^{\infty}\left(\frac{2}{5}\right)\left(\frac{x}{5}\right)^n}$

#6b. $f(x)=\dfrac{1}{2+x}$ goal: $\dfrac{a}{1-x}$

$\dfrac{1}{2-(-x)}$

$\dfrac{1}{2\left(1-\left(-\frac{x}{2}\right)\right)}$

$\dfrac{\left(\frac{1}{2}\right)}{1-\left(-\frac{x}{2}\right)}$ $a=\frac{1}{2}$ $r=\frac{-x}{2}$

$\boxed{\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)\left(\frac{-x}{2}\right)^n}$

Find the geometric power series for the function, centered at c, and determine the interval of convergence.

#7b. $f(x)=\dfrac{2}{6-x}$ at $c=-2$ goal is $\dfrac{a}{1-(x+2)}$

$\dfrac{2}{6-x-2+2}$

$\dfrac{2}{6-(x+2)+2}$

$\dfrac{2}{8-(x+2)}$

$\dfrac{2}{8\left(1-\left(\frac{x+2}{8}\right)\right)}$

$\dfrac{\left(\frac{2}{8}\right)}{1-\left(\frac{x+2}{8}\right)}$ $a=\frac{2}{8}=\frac{1}{4}$ $r=\frac{x+2}{8}$

$\boxed{\sum_{n=0}^{\infty}\left(\frac{1}{4}\right)\left(\frac{x+2}{8}\right)^n}$

ratio test

$\lim_{n\to\infty}\left|\dfrac{\frac{1}{4}\left(\frac{x+2}{8}\right)^{n+1}}{\frac{1}{4}\left(\frac{x+2}{8}\right)^n}\right|$

$\lim_{n\to\infty}\left|\dfrac{\left(\frac{x+2}{8}\right)^n\left(\frac{x+2}{8}\right)}{\left(\frac{x+2}{8}\right)^n}\right|$

$\lim_{n\to\infty}\left|\dfrac{x+2}{8}\right|$

$\left|\dfrac{x+2}{8}\right|<1$

$-1<\dfrac{x+2}{8}<1$

$-8<x+2<8$

$\boxed{\text{for } -10<x<6}$

#8b. $f(x)=\dfrac{1}{1-5x}$ at $c=0$ goal is $\dfrac{a}{1-x}$ but can also be $\dfrac{a}{1-(kx)}$

$\dfrac{1}{1-(5x)}$ $a=1$ $r=5x$

$\boxed{\sum_{n=0}^{\infty}(1)(5x)^n}$

ratio test $\lim_{n\to\infty}\left|\dfrac{(5x)^{n+1}}{(5x)^n}\right|$

$\lim_{n\to\infty}\left|\dfrac{(5x)(5x)^n}{(5x)^n}\right|$

$\lim_{n\to\infty}|5x|$

$|5x|<1$

$-1<5x<1$

$\boxed{\text{for } -\frac{1}{5}<x<\frac{1}{5}}$

Find the geometric power series for the function, centered at c, and determine the interval of convergence.

#9b. $f(x)=\dfrac{6}{4-x^2}$ at $c=0$

goal is $\dfrac{a}{1-(\text{any function where } f(0)=0)}$

$\dfrac{6}{4-(x^2)}$

$\dfrac{6}{4(1-(\frac{x^2}{4}))}$

$\dfrac{(\frac{6}{4})}{1-(\frac{x^2}{4})}$ $\quad a=\frac{6}{4}=\frac{3}{2}$ $\quad r=\frac{x^2}{4}$

$$\boxed{\sum_{n=0}^{\infty}\left(\frac{3}{2}\right)\left(\frac{x^2}{4}\right)^n}$$

ratio test

$\lim_{n\to\infty}\left|\dfrac{\frac{3}{2}(\frac{x^2}{4})^{n+1}}{\frac{3}{2}(\frac{x^2}{4})^n}\right|$

$\lim_{n\to\infty}\left|\dfrac{(\frac{x^2}{4})(\frac{x^2}{4})^n}{(\frac{x^2}{4})^n}\right|$

$\lim_{n\to\infty}\left|\frac{x^2}{4}\right|$

$\left|\frac{x^2}{4}\right|<1$

$\frac{x^2}{4}<1$

$x^2<4$

$\boxed{\text{for } -2<x<2}$

Re-work the problem in #9b, using the fact that the denominator is factorable to re-write using Partial Fraction Expansion, then find the geometric power series (combining terms using properties and determine the interval of convergence.

#10b. $f(x)=\dfrac{6}{4-x^2}$ at $c=0$

$\dfrac{6}{4-x^2}=\dfrac{-6}{x^2-4}=\dfrac{-6}{(x-2)(x+2)}=\dfrac{A}{x-2}+\dfrac{B}{x+2}=\dfrac{(-\frac{3}{2})}{x-2}+\dfrac{(\frac{3}{2})}{x+2}$

$A(x+2)+B(x-2)=-6$

$Ax+2A+Bx-2B=-6$

$(A+B)x+(2A-2B)=(0)x+(-6)$

$\begin{cases}A+B=0\\2A-2B=-6\end{cases}$

$\begin{cases}A+B=0\\A-B=-3\end{cases}$

$2A=-3$

$A=-3/2 \quad B=3/2$

$\dfrac{(-\frac{3}{2})}{x-2}$

$\dfrac{(\frac{3}{2})}{2-x}$

$\dfrac{(\frac{3}{2})}{2(1-(\frac{x}{2}))}$

$\dfrac{(\frac{3}{4})}{1-(\frac{x}{2})}$ $\quad a=\frac{3}{4}$ $\quad r=\frac{x}{2}$

$\boxed{\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)\left(\frac{x}{2}\right)^n}$

$\lim_{n\to\infty}\left|\dfrac{\frac{3}{4}(\frac{x}{2})^{n+1}}{\frac{3}{4}(\frac{x}{2})^n}\right|$

$-1<\frac{x}{2}<1$

$(-2<x<2)$

$\dfrac{(\frac{3}{2})}{x+2}$

$\dfrac{(\frac{3}{2})}{2-(-x)}$

$\dfrac{(\frac{3}{2})}{2(1-(-\frac{x}{2}))}$

$\dfrac{(\frac{3}{4})}{1-(-\frac{x}{2})}$ $\quad a=\frac{3}{4}$ $\quad r=-\frac{x}{2}$

$\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)\left(-\frac{x}{2}\right)^n$

also $(-2<x<2)$

so $$\boxed{\sum_{n=0}^{\infty}\left(\frac{3}{4}\left(\frac{x}{2}\right)^n+\frac{3}{4}\left(-\frac{x}{2}\right)^n\right) \quad \text{for } -2<x<2}$$

Use the fact that the given function is a derivative of another function, find the power series of the other function, then integrate the result to find the power series of the original function.

#11b. $f(x) = \frac{-1}{(x+1)^2} = \frac{d}{dx}\left[\frac{1}{x+1}\right]$ at $c = 0$

verify: $\frac{d}{dx}\left[\frac{1}{x+1}\right] = \frac{d}{dx}\left[(x+1)^{-1}\right] = -(x+1)^{-2}(1)$

$= \frac{-1}{(x+1)^2}$

$\frac{1}{x+1}$

$\frac{1}{1-(-x)}$ $a=1$ $r=-x$

$\frac{1}{x+1} = \sum_{n=0}^{\infty} (1)(-x)^n$

ratio test

$\lim_{n\to\infty} \left| \frac{(-x)^{n+1}}{(-x)^n} \right|$

$\lim_{n\to\infty} \left| \frac{(-x)(-x)^n}{(-x)^n} \right|$

$\lim_{n\to\infty} |-x|$

$|-x| < 1$

$(-1 < x < 1)$

now, $f(x) = \frac{d}{dx}\left[\frac{1}{x+1}\right]$

$f(x) = \frac{d}{dx}\left[\sum_{n=0}^{\infty} (1)(-x)^n\right]$

$f(x) = \sum_{n=0}^{\infty} (1)\,n(-x)^{n-1}(-1)$

$$\boxed{f(x) = \sum_{n=0}^{\infty} (-1)\,n(-x)^{n-1}}$$

(recheck end pts)

$x = -1$

$\sum_{n=0}^{\infty} (-1)n(-(-1))^{n-1}$

$\sum_{n=0}^{\infty} (-1)n(1)^{n-1}$

$\sum_{n=0}^{\infty} (-1)n$

nth term test

$\lim_{n\to\infty} (-n) = -\infty \neq 0$

diverges

$x = 1$

$\sum_{n=0}^{\infty} (-1)n(-1)^{n-1}$

$= -\sum_{n=0}^{\infty} (-1)^{n-1} n$

alternating series test

$\lim_{n\to\infty} n = \infty \neq 0$

diverges by nth term test

so $\boxed{\text{interval of convergence is still } -1 < x < 1}$

7.11 – Extra Practice

Find the nth Maclaurin polynomial for the given function.

#4b. $f(x)=e^{-\frac{1}{2}x}$, $n=4$

$f(x)=e^{-1/2x}$ $\quad f(0)=1$

$f'(x)=-\frac{1}{2}e^{-1/2x}$ $\quad f'(0)=-\frac{1}{2}$

$f''(x)=\frac{1}{4}e^{-1/2x}$ $\quad f''(0)=\frac{1}{4}$

$f'''(x)=-\frac{1}{8}e^{-1/2x}$ $\quad f'''(0)=-\frac{1}{8}$

$f^{(4)}(x)=\frac{1}{16}e^{-1/2x}$ $\quad f^{(4)}(0)=\frac{1}{16}$

$P_4(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\frac{f^{(4)}(0)}{4!}x^4$

$P_4(x)=1-\frac{1}{2}x+\frac{(1/4)}{2!}x^2+\frac{(-1/8)}{3!}x^3+\frac{(1/16)}{4!}x^4$

$\boxed{P_4(x)=1-\frac{1}{2}x+\frac{1}{8}x^2-\frac{1}{48}x^3+\frac{1}{384}x^4}$

#5b. $f(x)=\cos(\pi x)$, $n=4$

$f(x)=\cos(\pi x)$ $\quad f(0)=1$

$f'(x)=-\pi\sin(\pi x)$ $\quad f'(0)=0$

$f''(x)=-\pi^2\cos(\pi x)$ $\quad f''(0)=-\pi^2$

$f'''(x)=\pi^3\sin(\pi x)$ $\quad f'''(0)=0$

$f^{(4)}(x)=\pi^4\cos(\pi x)$ $\quad f^{(4)}(0)=\pi^4$

$P_4(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\frac{f^{(4)}(0)}{4!}x^4$

$P_4(x)=1+0x+\frac{-\pi^2}{2!}x^2+0+\frac{\pi^4}{4!}x^4$

$\boxed{P_4(x)=1-\frac{\pi^2}{2!}x^2+\frac{\pi^4}{4!}x^4}$

$f(0)=0$

$f'(0)=0$

$f''(0)=2$

$f'''(0)=-6$

$f^{(4)}(0)=12$

$P_4(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\frac{f^{(4)}(0)}{4!}x^4$

$P_4(x)=0+0x+\frac{2}{2!}x^2+\frac{-6}{3!}x^3+\frac{12}{4!}x^4$

$\boxed{P_4(x)=x^2-x^3+\frac{1}{2}x^4}$

#6b. $f(x)=x^2e^{-x}$, $n=4$

$f(x)=x^2e^{-x}$

$f'(x)=x^2(-e^{-x})+e^{-x}(2x)$

$f''(x)=x^2(e^{-x})+(-e^{-x})(2x)+e^{-x}(2)+2x(-e^{-x})$

$f''=2e^{-x}-4xe^{-x}+x^2e^{-x}$

$f'''(x)=-2e^{-x}+(-4x)(-e^{-x})+e^{-x}(-4)+x^2(-e^{-x})+e^{-x}(2x)$

$=-6e^{-x}+6xe^{-x}-x^2e^{-x}$

$f^{(4)}(x)=6e^{-x}+6x(-e^{-x})+e^{-x}(6)+(-x^2)(-e^{-x})+e^{-x}(-2x)$

Find the nth-degree Maclaurin polynomial for the given function.

#7b. $f(x)=\frac{x}{x+1}$, $n=4$

$P_4(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\frac{f^{(4)}(0)}{4!}x^4$

$P_4(x)=0+1x+\frac{-2}{2!}x^2+\frac{6}{3!}x^3+\frac{-24}{4!}x^4$

$\boxed{P_4(x)=x-x^2+x^3-x^4}$

$f(x)=\frac{x}{x+1}$ $\quad f(0)=0$

$f'(x)=\frac{(x+1)(1)-x(1)}{(x+1)^2}=\frac{1}{(x+1)^2}=(x+1)^{-2}$ $\quad f'(0)=1$

$f''(x)=-2(x+1)^{-3}$ $\quad f''(0)=-2$

$f'''(x)=6(x+1)^{-4}$ $\quad f'''(0)=6$

$f^{(4)}(x)=-24(x+1)^{-5}$ $\quad f^{(4)}(0)=-24$

Find the nth-degree Taylor polynomial centered at c for the given function.

#8b. $f(x)=\sqrt[3]{x}$, $n=3$, $c=8$

$P_3(x)=f(8)+f'(8)(x-8)+\frac{f''(8)}{2!}(x-8)^2+\frac{f'''(8)}{3!}(x-8)^3$

$P_3(x)=2+\frac{1}{12}(x-8)+\frac{(-1/144)}{2!}(x-8)^2+\frac{(5/3456)}{3!}(x-8)^3$

$\boxed{P_3(x)=2+\frac{1}{12}(x-8)-\frac{1}{288}(x-8)^2+\frac{5}{20736}(x-8)^3}$

$f(x)=\sqrt[3]{x}=x^{1/3}$ $\quad f(8)=2$

$f'(x)=\frac{1}{3}x^{-2/3}$ $\quad f'(8)=\frac{1}{12}$

$f''(x)=-\frac{2}{9}x^{-5/3}$ $\quad f''(8)=-\frac{1}{144}$

$f'''(x)=\frac{10}{27}x^{-8/3}$ $\quad f'''(8)=\frac{5}{3456}$

#9b. $f(x)=x^2\cos(x)$, $n=2$, $c=\pi$

$P_2(x)=f(\pi)+f'(\pi)(x-\pi)+\frac{f''(\pi)}{2!}(x-\pi)^2$

$P_2(x)=(-\pi^2)+(-2\pi)(x-\pi)+\frac{(\pi^2-2)}{2!}(x-\pi)^2$

$\boxed{P_2(x)=-\pi^2-2\pi(x-\pi)+\frac{\pi^2-2}{2}(x-\pi)^2}$

$f(x)=x^2\cos x$

$f'(x)=x^2(-\sin x)+\cos x(2x)$

$f''(x)=x^2(-\cos x)+(-\sin x)(2x)+\cos x(2)+2x(-\sin x)$

$f(\pi)=\pi^2(-1)=-\pi^2$

$f'(\pi)=\pi^2(-1)(0)+(-1)2\pi=-2\pi$

$f''(\pi)=\pi^2(-1)(-1)+0+(-1)(2)+0=\pi^2-2$

Find the nth-degree Taylor <u>Series</u>, centered at c for the given function.

#10b. $f(x)=\frac{1}{1-x}$, $c=2$

start by writing some terms of a Taylor polynomial...

$f(x)=\frac{1}{1-x}=(1-x)^{-1}$

$f'(x)=-(1-x)^{-2}(-1)=(1-x)^{-2}$

$f''(x)=-2(1-x)^{-3}(-1)=2(1-x)^{-3}$

$f'''(x)=2(-3)(1-x)^{-4}(-1)=2(3)(1-x)^{-4}$

$f(2)=-1$

$f'(2)=1$

$f''(2)=-2$

$f'''(2)=6$

$P(x)=f(2)+f'(2)(x-2)+\frac{f''(2)}{2!}(x-2)^2+\frac{f'''(2)}{3!}(x-2)^3+\ldots$

$P(x)=-1+1(x-2)+\frac{-2}{2!}(x-2)^2+\frac{6}{3!}(x-2)^3+\ldots$

$P(x)=-1+(x-2)-(x-2)^2+(x-2)^3+\ldots$

look for a pattern to write a general expression

$n=\quad 0 \quad 1 \quad 2 \quad 3$

$$\boxed{\frac{1}{1-x}=\sum_{n=0}^{\infty}(-1)^{n+1}(x-2)^n}$$

Write out the terms for a Maclaurin polynomial for the given series, and find an expression for the nth-term. Then use this to write a Maclaurin <u>Series</u> for the function.

#11b. $f(x)=\cos(x)$

$P(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\frac{f^{(4)}(0)}{4!}x^4+\ldots$

$P(x)=1+0x+\frac{-1}{2!}x^2+\frac{0}{3!}x^3+\frac{1}{4!}x^4+\ldots$

$P(x)=1-\frac{1}{2!}x^2+\frac{1}{4!}x^4+\ldots$

now, renumber terms and look for a pattern

$(n=\quad 0 \quad 1 \quad 2)$

$f(x)=\cos x$

$f'(x)=-\sin x$

$f''(x)=-\cos x$

$f'''(x)=\sin x$

$f^{(4)}(x)=\cos x$

$f(0)=1$

$f'(0)=0$

$f''(0)=-1$

$f'''(0)=0$

$f^{(4)}(0)=1$

$$\boxed{\cos x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}}$$

7.12 – Extra Practice

Use the binomial series to find the Maclaurin Series for the function.

#4b. $f(x)=\sqrt{1+x^7} = (1+x^7)^{1/2}$

$$(1+x)^k = 1 + kx + \frac{k(k-1)x^2}{2!} + \frac{k(k-1)(k-2)x^3}{3!} + \frac{k(k-1)(k-2)(k-3)x^4}{4!} + \cdots$$

$$(1+x^7)^{1/2} = 1 + \tfrac{1}{2}(x^7) + \frac{\frac{1}{2}(-\frac{1}{2})(x^7)^2}{2!} + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})(x^7)^3}{3!} + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})(x^7)^4}{4!} + \cdots$$

$$\boxed{(1+x^7)^{1/2} = 1 + \tfrac{1}{2}x^7 - \tfrac{1}{8}x^{14} + \tfrac{1}{16}x^{21} - \tfrac{5}{128}x^{28} + \cdots}$$

↑ although not required to simplify the coefficients to be fully correct, a Maclaurin Series is a power series, which must be written with each term as $(\#)X^{\text{an integer power}}$

Use the list of basic Power Series to find the Maclaurin Series for the function.

#5b. $f(x)=e^{(-3x)}$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^{(-3x)} = \sum_{n=0}^{\infty} \frac{(-3x)^n}{n!} = \boxed{\sum_{n=0}^{\infty} \frac{(-3)^n}{n!} x^n} = \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{3^n}{n!} x^n}$$

#6b. $f(x)=\ln(1+x^2)$

$$\ln(x) = \sum_{n=0}^{\infty} (-1)^{n-1} \frac{(x-1)^n}{n}$$

$$\ln(1+x^2) = \sum_{n=0}^{\infty} (-1)^{n-1} \frac{(1+x^2-1)^n}{n} = \boxed{\sum_{n=0}^{\infty} (-1)^{n-1} \frac{x^{2n}}{n}}$$

could also be $(-1)^{n+1}$

Use the list of basic Power Series to find the Maclaurin Series for the function.

#7b. $f(x) = \cos(\pi x)$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$\cos(\pi x) = \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{(\pi x)^{2n}}{(2n)!}}$$

#8b. $f(x) = \cos^2(x)$ – two ways to solve…

trig identity; $\cos^2(x) = \frac{1+\cos(2x)}{2} = \frac{1}{2} + \frac{1}{2}\cos(2x)$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \quad \cos(2x) = \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n}}{(2n)!}$$

so $\boxed{f(x) = \frac{1}{2} + \frac{1}{2}\sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n}}{(2n)!}}$

–or– $f(x) = (\cos x)(\cos x)$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$f(x) = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)$$

$$f(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots - \frac{x^2}{2!} + \frac{x^4}{2!2!} - \frac{x^6}{2!4!} + \frac{x^8}{2!6!} + \cdots + \frac{x^4}{4!} - \frac{x^6}{4!2!} + \frac{x^8}{4!4!} - \frac{x^{10}}{4!6!} + \cdots$$

$$\boxed{f(x) = 1 - x^2 + \left(\frac{1}{4!} + \frac{1}{2!2!} + \frac{1}{4!}\right)x^4 - \left(\frac{1}{6!} + \frac{1}{2!4!} + \frac{1}{4!2!}\right)x^6 + \cdots}$$

$$\boxed{f(x) = 1 - x^2 + \frac{1}{3}x^4 - \frac{47}{1440}x^6 + \cdots}$$

7.13 – Extra Practice

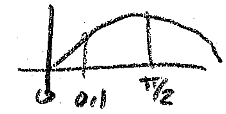
#7. (no matching problem) – hint on part c. Figure out what term this derivative would be a part of and match the coefficient.

Determine the degree of the Maclaurin polynomial required for the error in the approximation of the function at the indicated value to be less than 0.001.

#8b. $f(x)=\cos(x)$, *approximate* $f(0.1)$

$f(x)=\cos x$
$f'(x)=-\sin x$
$f''(x)=-\cos x$
$f'''(x)=\sin x$

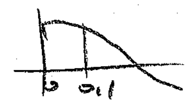
if sin:



$z=0.1$

$|\text{error}| \le \left|\frac{f^{(N+1)}(z)}{(N+1)!}(x-c)^{N+1}\right|$ $\quad x=0.1$, $c=0$

if cos



$z=0$
$\cos(0)=1$
so $f^{(N+1)}(z)$ max $=1$

try some N values and find error:

$\le \left|\frac{1}{(N+1)!}(0.1-0)^{N+1}\right|$

$N=1$ $\quad \left|\frac{(0.1)^2}{2!}\right| = 0.005 \nless 0.001$

$N=2$ $\quad \left|\frac{(0.1)^3}{3!}\right| = 0.0001667 < 0.001$ $\quad (1.667 \times 10^{-4})$

So need a $\boxed{\text{2nd degree polynomial}}$

Determine the degree of the Taylor polynomial centered at x = 1 required for the error in the approximation of the function at the indicated value to be less than 0.001.

#9b. $f(x)=\ln(x)$, *approximate* $f(1.25)$

$|\text{error}| \le \left|\frac{f^{(N+1)}(z)}{(N+1)!}(x-c)^{N+1}\right|$

$f(x)=\ln x$
$f'(x)=\frac{1}{x}$
form of derivative changes

change tactics: instead of using Lagrange error, let's just write out terms of the Taylor Series, truncate, and find actual error!

actual $\ln(1.25)=0.22314355 13$

$P(x)=(x-1)-\frac{(x-1)^2}{2}+\frac{(x-1)^3}{3}-\frac{(x-1)^4}{4}+\frac{(x-1)^5}{5}+\dots$

$P(1.25)=(1.25-1)-\frac{(1.25-1)^2}{2}+\frac{(1.25-1)^3}{3}-\frac{(1.25-1)^4}{4}+\frac{(1.25-1)^5}{5}+\dots$

stopping here $P(1.25)=.22317708 33$ (error $=3.3\times10^{-5}$ ✓)

stop here, $P(1.25)=.22279 17708$ (error $=1.617\times10^{-4}$ ✓)

Stop here, $P(1.25)=.2239583333$ (error $=8.147\times10^{-4}=.0008$ ✓)

Stop here, $P(1.25)=.21875$ (error $=.004$ ✗)

need $\boxed{\text{a 3rd degree polynomial}}$

#10b. If $\left|f^{(7)}(x)\right| \le 2$, find the Lagrange error bound if a sixth degree Taylor polynomial centered at x = 4 is used to approximate $f(4.7)$. (Assume the series converges for x = 4.)

$$|error| \le \left| \frac{f^{(N+1)}(z)}{(N+1)!}(x-c)^{N+1} \right|$$

$x = 4.7$
$c = 4$
$N = 6$

$$|error| \le \left| \frac{(2)}{7!}(4.7-4)^7 \right| = \boxed{3.268 \times 10^{-5}}$$

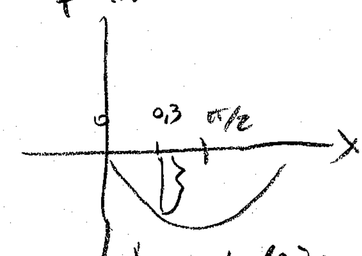
#11b. Find an upper limit for the error when the Taylor polynomial $T(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$ is used to approximate $f(x) = \cos(x)$ at $x = 0.3$.

$$|error| \le \left| \frac{f^{(N+1)}(z)}{(N+1)!}(x-c)^{N+1} \right|$$

$x = 0.3$
$c = 0$
$N = 4$

$f(x) = \cos x$
$f'(x) = -\sin x$
$f''(x) = -\cos x$
$f'''(x) = \sin x$
$f^{(4)}(x) = \cos x$
$f^{(5)}(x) = -\sin x$

$f^{(5)}(x) = -\sin x$



largest deriv. when $z = 0.3$

$f^{(5)}(z) = -\sin(0.3)$

$$|error| \le \left| \frac{-\sin(0.3)}{5!}(0.3-0)^5 \right|$$

$$\boxed{|error| \le 5.984 \times 10^{-6}}$$