AP Calculus BC – Unit 7 Part 2 Required Practice Name: answers only

7.9 – Required Practice

Find the radius of convergence for each series.

#1. $\sum_{n=0}^{\infty} 3(x-2)^n$

$\boxed{R=1}$

#2. $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

$\boxed{R=\infty}$

#3. $\sum_{n=0}^{\infty} n!x^n$

$\boxed{R=0}$

Find the interval of convergence for each series.

#4. $\sum_{n=1}^{\infty} \frac{x^n}{n}$

$$\boxed{\begin{array}{c} -1 \leq x < 1 \\ \text{or} \\ [-1, 1) \end{array}}$$

Find the interval of convergence for each series.

#5. $\sum_{n=0}^{\infty} \frac{(-1)^n (x+1)^n}{2^n}$

$$\boxed{\begin{array}{c} -3 < x < 1 \\ \text{or} \\ (-3, 1) \end{array}}$$

Find the interval of convergence for each series.

#6. $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$

$$\boxed{\begin{array}{c} -1 \le x \le 1 \\ \text{or} \\ [-1, 1] \end{array}}$$

Example: Find the intervals of convergence of $f(x)$, $f'(x)$, *and* $\int f(x)\,dx$ for $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(x-1)^{n+1}}{n+1}$

First, let's just find the series for the derivative and integral:

#7. $f'(x)$

$$= \boxed{\sum_{n=0}^{\infty} (-1)^{n+1} (x-1)^n}$$

#8. $\int f(x)\,dx$

$$\boxed{= \sum_{n=0}^{\infty} (-1)^{n+1} \frac{(x-1)^{n+2}}{(n+1)(n+2)} + C}$$

State where the power series is centered.

#9. $\sum_{n=0}^{\infty}\left(\frac{x}{4}\right)^n$

$\boxed{c=0}$

#10. $\sum_{n=1}^{\infty}\frac{1}{n^3}(x-2)^n$

$\boxed{c=2}$

Find the radius of convergence of the power series.

#11. $\sum_{n=0}^{\infty}(-1)^n\left(\frac{x^n}{n+1}\right)$

$\boxed{R=1}$

#12. $\sum_{n=1}^{\infty}\left(\frac{(4x)^n}{n^2}\right)$

$\boxed{R=\frac{1}{4}}$

#13. $\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}$

$\boxed{R=\infty}$

Find the interval of convergence of the power series (you must check each endpoint).

#14. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n9^n}(x-4)^n$

$\boxed{\begin{gathered} -5 < x \leq 13 \\ \text{or} \\ (-5, 13] \end{gathered}}$

Find the interval of convergence of the power series (you must check each endpoint).

#15. $\sum_{n=0}^{\infty}(-1)^{n+1}\frac{1}{n+1}(x-1)^{n+1}$

$\boxed{\begin{array}{c} 0<x\leq 2 \\ \text{or} \\ (0,2] \end{array}}$

#16. Given the series $f(x)=\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^{n}$

(a) Find the Power Series for $f'(x)$ *and* $\int f(x)\,dx$

(b) Find the interval of convergence for the series for $f(x)$

(c) Find the interval of convergence for the series for $f'(x)$ (just need to re-check the endpoints)

(a) $\boxed{f'(x) = \sum_{n=0}^{\infty} n\left(\frac{x}{3}\right)^{n-1}\left(\frac{1}{3}\right)}$ ← chain rule

$\boxed{\int f(x)\,dx = \sum_{n=0}^{\infty} \frac{1}{n+1}\left(\frac{x}{3}\right)^{n+1}(3) + C}$ ← from u-sub

(b) $\boxed{-3 < x < 3 \text{ or } (-3, 3)}$

(c) $\boxed{-3 < x < 3 \text{ or } (-3, 3)}$

7.10 – Required Practice

#1. Find a power series centered at x = -1 to represent $\frac{1}{1-x}$

$$\boxed{\begin{gathered}\sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{x+1}{2}\right)^n \\ (\text{for } -3<x<1)\end{gathered}}$$

#2. $f(x)=\frac{4}{x+2}$ centered at 0

$$\boxed{\begin{gathered}\sum_{n=0}^{\infty} (2)\left(-\frac{x}{2}\right)^n = 2\sum_{n=0}^{\infty} (-1)^n\left(\frac{x}{2}\right)^n \\ (\text{for } -2<x<2)\end{gathered}}$$

#3. $f(x)=\frac{1}{x}$ centered at 1

$$\boxed{\sum_{n=0}^{\infty} (1)(-x+1)^n \quad (\text{for } 0<x<2)}$$

#4. $f(x)=\frac{3x-1}{x^2-1}$
centered at 0

$$\boxed{\sum_{n=0}^{\infty} \left(-(x)^n+2(-x)^n\right) \quad (\text{for } -1<x<1)}$$

Find the geometric power series for the function, centered at 0.

#5. $f(x) = \dfrac{1}{4-x}$

$$\boxed{\sum_{n=0}^{\infty} \left(\tfrac{1}{4}\right)\left(\tfrac{x}{4}\right)^n}$$

#6. $f(x) = \dfrac{4}{3+x}$

$$\boxed{\sum_{n=0}^{\infty} \left(\tfrac{4}{3}\right)\left(\tfrac{-x}{3}\right)^n}$$

Find the geometric power series for the function, centered at c, and determine the interval of convergence.

#7. $f(x) = \dfrac{1}{3-x}$ $at\ \ c=1$

$$\boxed{\sum_{n=0}^{\infty} \left(\tfrac{1}{2}\right)\left(\tfrac{x-1}{2}\right)^n \quad \text{for } -1 < x < 3}$$

#8. $f(x) = \dfrac{1}{1-3x}$ $at\ \ c=0$

$$\boxed{\sum_{n=0}^{\infty} (1)(3x)^n \quad \text{for } -\tfrac{1}{3} < x < \tfrac{1}{3}}$$

Find the geometric power series for the function, centered at c, and determine the interval of convergence.

#9. $f(x)=\dfrac{2}{1-x^2}$ at $c=0$

$$\boxed{\sum_{n=0}^{\infty}(2)(x^2)^n=\sum_{n=0}^{\infty}(2)x^{2n} \quad \text{for } -1<x<1}$$

Re-work the problem in #9, using the fact that the denominator is factorable to re-write using Partial Fraction Expansion, then find the geometric power series (combining terms using properties and determine the interval of convergence.

#10. $f(x)=\dfrac{2}{1-x^2}$ at $c=0$

$$\boxed{\sum_{n=0}^{\infty}\left(x^n+(-x)^n\right) \quad \text{for } -1<x<1}$$

Use the fact that the given function is a derivative of another function, find the power series of the other function, then integrate the result to find the power series of the original function.

#11. $f(x) = \frac{1}{(1-x)^2} = \frac{d}{dx}\left[\frac{1}{1-x}\right]$ at $c = 0$

$$\boxed{f(x) = \sum_{n=0}^{\infty} n x^{n-1} \quad (\text{for } -1 < x < 1)}$$

7.11 – Required Practice

#1. Find the n=3 Taylor polynomial for $f(x)=\sqrt{x}$ at c = 4

$f(x)=\sqrt{x}=x^{1/2}$, $f(4)=2$

$f'(x)=\frac{1}{2}x^{-1/2}$, $f'(4)=\frac{1}{4}$

$f''(x)=-\frac{1}{4}x^{-3/2}$, $f''(4)=-\frac{1}{32}$

$f'''(x)=\frac{3}{8}x^{-5/2}$, $f'''(4)=\frac{3}{256}$

$P_3(x)=f(4)+f'(4)(x-4)+\frac{f''(4)}{2!}(x-4)^2+\frac{f'''(4)}{3!}(x-4)^3$

$P_3(x)=2+\frac{1}{4}(x-4)+\frac{(-1/32)}{2!}(x-4)^2+\frac{(3/256)}{3!}(x-4)^3$

$\boxed{P_3(x)=2+\frac{1}{4}(x-4)-\frac{1}{64}(x-4)^2+\frac{1}{512}(x-4)^3}$

#2. Find the nth Maclaurin polynomial for $f(x)=e^x$

$f(x)=e^x$, $f(0)=e^0=1$

$f'(x)=e^x$, $f'(0)=1$

$\vdots \qquad \vdots$

$f^{(n)}(x)=e^x \quad f^{(n)}(0)=1$

$P_n(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\dots+\frac{f^{(n)}(0)}{n!}x^n$

$$\boxed{\begin{gathered}P_n(x)=1+1x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\dots+\frac{1}{n!}x^n \\ P_n(x)=\sum_{n=0}^{\infty}\frac{x^n}{n!}\end{gathered}}$$

#3. Find the n=4 Taylor polynomial for $f(x)=\ln x$ at c = 1

$f(x)=\ln x$ $\qquad f(1)=0$

$f'(x)=\frac{1}{x}=x^{-1}$ $\qquad f'(1)=1$

$f''(x)=-x^{-2}=\frac{-1}{x^2}$ $\qquad f''(1)=-1$

$f'''(x)=2x^{-3}=\frac{2}{x^3}$ $\qquad f'''(1)=2$

$f^{(4)}(x)=-6x^{-4}=\frac{-6}{x^4}$ $\qquad f^{(4)}(1)=-6$

$P_4(x)=f(1)+f'(1)(x-1)+\frac{f''(1)}{2!}(x-1)^2+\frac{f'''(1)}{3!}(x-1)^3+\frac{f^{(4)}(1)}{4!}(x-1)^4$

$P_4(x)=0+1(x-1)+\frac{(-1)}{2!}(x-1)^2+\frac{2}{3!}(x-1)^3+\frac{(-6)}{4!}(x-1)^4$

$\boxed{P_4(x)=(x-1)-\frac{1}{2}(x-1)^2+\frac{1}{3}(x-1)^3-\frac{1}{4}(x-1)^4}$

Find the nth-degree Maclaurin polynomial for the given function.

#4. $f(x) = e^{4x}, \quad n = 4$

$$\boxed{P_4(x) = 1 + 4x + 8x^2 + \frac{32}{3}x^3 + \frac{32}{3}x^4}$$

#5. $f(x) = \sin(x), \quad n = 5$

$$\boxed{\begin{array}{l} P_5(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 \\ P_5(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 \end{array}}$$

#6. $f(x) = xe^x, \quad n = 4$

$$\boxed{P_4(x) = x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4}$$

Find the nth-degree Maclaurin polynomial for the given function.

#7. $f(x) = \dfrac{1}{x+1}, \quad n = 5$

$$\boxed{P_5(x) = 1 - x + x^2 - x^3 + x^4 - x^5}$$

Find the nth-degree Taylor polynomial centered at c for the given function.

#8. $f(x) = \sqrt{x}, \quad n = 3, \quad c = 9$

$$\boxed{P_3(x) = 3 + \frac{1}{6}(x-9) - \frac{1}{216}(x-9)^2 + \frac{1}{3888}(x-9)^3}$$

#9. $f(x) = \ln(x), \quad n = 4, \quad c = 2$

$$\boxed{P_4(x) = \ln(2) + \frac{1}{2}(x-2) - \frac{1}{8}(x-2)^2 + \frac{1}{24}(x-2)^3 - \frac{1}{64}(x-2)^4}$$

Find the nth-degree Taylor Series, centered at c for the given function.

#10. $f(x) = \frac{1}{x}, \quad c = 1$

$$\boxed{\frac{1}{x} = \sum_{n=0}^{\infty} (-1)^n (x-1)^n}$$

Write out the terms for a Maclaurin polynomial for the given series, and find an expression for the nth-term. Then use this to write a Maclaurin Series for the function.

#11. $f(x) = \sin(x)$ $\boxed{= \sum_{n=0}^{\infty} (-1)^{n+1} \frac{x^{2n+1}}{(2n+1)!}}$

7.12 – Required Practice

#1. Find the power series for $f(x)=\cos\left(\sqrt{x}\right)$ $= \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{(2n)!}}$

#2. Find the power series for $f(x)=e^x \arctan x$

$$\boxed{f(x) = x + x^2 + \left(-\frac{1}{3} + \frac{1}{2!}\right)x^3 + \frac{1}{3!}x^4 + \left(\frac{1}{5} - \frac{1}{3 \cdot 2!}\right)x^5 + \cdots}$$

#3. Use a power series to approximate $\int_0^1 e^{-x^2}\,dx$

$$\boxed{\int_0^1 e^{-x^2}dx \approx 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} + \cdots \approx 0.747}$$

Use the binomial series to find the Maclaurin Series for the function.

#4. $f(x)=\sqrt[4]{1+x}$

$$\boxed{\sqrt[4]{1+x} = 1 + \frac{1}{4}x + \frac{\frac{1}{4}\left(-\frac{3}{4}\right)x^2}{2!} + \frac{\frac{1}{4}\left(-\frac{3}{4}\right)\left(-\frac{7}{4}\right)x^3}{3!} + \frac{\frac{1}{4}\left(-\frac{3}{4}\right)\left(-\frac{7}{4}\right)\left(-\frac{11}{4}\right)x^4}{4!} + \ldots}$$

Use the list of basic Power Series to find the Maclaurin Series for the function.

#5. $f(x)=e^{\left(\frac{1}{2}x^2\right)}$ $= \boxed{\sum_{n=0}^{\infty} \frac{1}{2^n}\frac{1}{n!}x^{2n}}$

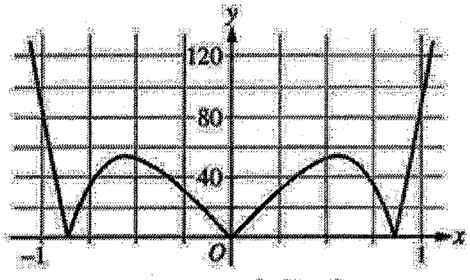
#6. $f(x)=\ln(1+x)$ $= \sum_{n=0}^{\infty} (-1)^{n-1} \frac{x^n}{n}$

or $(-1)^{n+1}$

Use the list of basic Power Series to find the Maclaurin Series for the function.

#7. $f(x)=\cos(4x)$ $\Rightarrow \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{(4x)^{2n}}{(2n)!}}$

#8. $f(x)=3+4e^{(x^3)}$ $\Rightarrow \boxed{3+4\sum_{n=0}^{\infty} \frac{x^{3n}}{n!}}$



Graph of $y = \left|f^{(5)}(x)\right|$

#7. Let $f(x) = \sin\left(x^2\right) + \cos x$. The graph of $y = \left|f^{(5)}(x)\right|$ is shown above.

(a) Write the first four nonzero terms of the Taylor series for $\sin x$ about $x = 0$, and write the first four nonzero terms of the Taylor series for $\sin\left(x^2\right)$ about $x = 0$.

(b) Write the first four nonzero terms of the Taylor series for $\cos x$ about $x = 0$. Use this series and the series for $\sin\left(x^2\right)$, found in part (a), to write the first four nonzero terms of the Taylor series for f about $x = 0$.

(c) Find the value of $f^{(6)}(0)$.

(d) Let $P_4(x)$ be the fourth-degree Taylor polynomial for f about $x = 0$. Using information from the graph of $y = \left|f^{(5)}(x)\right|$ shown above, show that $\left|P_4\left(\frac{1}{4}\right) - f\left(\frac{1}{4}\right)\right| < \frac{1}{3000}$.

(a) $\boxed{\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots}$

$\boxed{\sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} + \frac{x^{14}}{7!} + \cdots}$

(b) $\boxed{\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots}$

$\boxed{f(x) = 1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{121}{720}x^6 + \cdots}$

(c) $\boxed{f^{(6)}(0) = -121}$

(d) estimate $\left|f^{(5)}\left(\frac{1}{4}\right)\right| = 30$ from graph,

then $\boxed{|\text{error}| \le \frac{1}{4096}} < \frac{1}{3000}$

Determine the degree of the Maclaurin polynomial required for the error in the approximation of the function at the indicated value to be less than 0.001.

#8. $f(x)=\sin(x)$, *approximate* $f(0.3)$

$\boxed{N=3 \quad (3^{rd}\text{ degree polynomial})}$

#9. $f(x)=e^{x}$, *approximate* $f(0.6)$

$\boxed{\text{need a } 5^{th}\text{ degree polynomial}}$

#10. If $\left|f^{(4)}(x)\right| \leq 4$, find the Lagrange error bound if a third degree Taylor polynomial centered at x = 1 is used to approximate $f(2)$. (Assume the series converges for x = 2.)

$\boxed{|\text{error}| < 0.167}$

#11. Find an upper limit for the error when the Taylor polynomial $T(x) = x - \dfrac{x^3}{3!}$ is used to approximate $f(x) = \sin(x)$ at $x = 0.5$.

$\boxed{|\text{error}| \leq 0.00125}$

7.13 – Required Practice

Examples to help us see how this works in different cases...

#1. For the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n!}$

(a) Approximate the sum of the series by using the first 6 terms.

(b) Find the upper bound for the remainder for the approximation in part a.

(c) Find upper and lower bounds for the actual sum of the series.

(a) $\boxed{\frac{91}{144}}$

(b) $\boxed{\frac{1}{7!}}$

(c) $\boxed{\frac{91}{144}-\frac{1}{7!} < S < \frac{91}{144}+\frac{1}{7!}}$

#2. For the series $\sum_{n=0}^{\infty}(-1)^{n}\frac{1}{(2n)!}$

(a) Approximate the sum of the series with an error of less than 0.001.

(b) Which memorized Power Series matches this series form?
Use the function for the matching series to find the actual value of the given series.

(a) $\boxed{S \approx \frac{389}{720}}$

(b) $\boxed{\sum_{n=0}^{\infty}(-1)^{n}\frac{1}{(2n)!} = \cos(1) = 0.54030}$

#3. (a) Estimate e^2 using a Maclaurin polynomial of degree 10 for e^x

(b) Use the Lagrange form of the remainder (error) to find the upper bound of the error using this partial sum.

(c) What is the actual error $|f(x)-P(x)|$?

(a) $\boxed{e^2 \approx 7.38899470 9}$

(b) $\boxed{|\text{error}| \leq 3.791 \cdot 10^{-4} = 0.0003791}$

(c) $\boxed{|f(x)-P(x)| = 6.139 \cdot 10^{-5} = 0.00006139}$

#4. If $f^{(5)}(x) = 700\sin(x)$ and if $x = 0.7$ is in the convergence interval for the power series of f centered at $x = 0$, find an upper limit for the error when the fourth-degree Taylor polynomial is used to approximate $f(0.7)$

$\boxed{|\text{error}| \leq 0.6316}$

#5. If $f^{(6)}(x)$ is a positive, decreasing function, find the error bound when a 5th degree Taylor polynomial centered at $x=4$ is used to approximate $f(4.1)$

(Assume the series converges for $x=4.1$)

$$\boxed{|\text{error}| \le \left| \frac{f^{(6)}(4)}{6!} (4.1-4)^6 \right|}$$

#6. The Taylor series for $\ln(x)$, centered at $x=1$ is $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-1)^n}{n}$

Let f be the function given by the sum of the first three nonzero terms of this series.

The maximum value of $|\ln(x)-f(x)|$ for $0.3 \le x \le 1.7$ is:

(A) 0.030

(B) 0.039

$\boxed{\text{(C) } 0.145}$

(D) 0.153

(E) 0.529

- First, try Lagrange error.
- Next, try Alternating Series Error
- Finally, can we just compute the actual error over this x interval?

Unit 7 Part 2 Test Review

Find the Maclaurin series for the given function (use the basic Power Series list):

1) $g(x)=\frac{1}{x^5}$

$$\frac{1}{x}=\sum_{n=0}^{\infty}(-1)^n(x-1)^n$$

$$\boxed{g(x)=\sum_{n=0}^{\infty}(-1)^n(x^5-1)^n}$$

2) $g(x)=\sin\left(2-x^3\right)$

$$\sin x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$$

$$\boxed{g(x)=\sum_{n=0}^{\infty}(-1)^n\frac{(2-x^3)^{2n+1}}{(2n+1)!}}$$

3) $g(x)=\frac{1}{3+x^2}$

$$\frac{1}{x}=\sum_{n=0}^{\infty}(-1)^n(x-1)^n$$

$$g(x)=\boxed{\sum_{n=0}^{\infty}(-1)^n([3+x^2]-1)^n=\sum_{n=0}^{\infty}(-1)^n(2+x^2)^n}$$

Write the first 3 non-zero terms of a power expansion for the given function (use the basic Power Series list):

4) $f(x)=\cos\left(3x^5\right)$

$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$

$f(x) = 1 - \frac{(3x^5)^2}{2!} + \frac{(3x^5)^4}{4!} = \boxed{1 - \frac{9}{2!}x^{10} - \frac{81}{4!}x^{20}}$

a power series must be written as coefficients multiplied by $x^{(\text{integer power})}$ or $(x-c)^{(\text{integer power})}$

5) $f(x)=\sin(4x-3)$

$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$

$\boxed{f(x) = (4x-3) - \frac{(4x-3)^3}{3!} + \frac{(4x-3)^5}{5!}}$

(we believe College Board would be okay with this)

Find the specified Maclaurin or Taylor polynomial (for these you must use the Taylor procedure):

6) Find the 5th degree Maclaurin $(c=0)$ polynomial for the function $f(x)=\sin(3x)$

$f(x) = \sin(3x)$

$f'(x) = 3\cos(3x)$

$f''(x) = -9\sin(3x)$

$f'''(x) = -27\cos(3x)$

$f^{(4)}(x) = 81\sin(3x)$

$f^{(5)}(x) = 243\cos(3x)$

$P_5(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5$

$P_5(x) = 0 + 3x + \frac{0}{2!}x^2 + \frac{-27}{3!}x^3 + \frac{0}{4!}x^4 + \frac{243}{5!}x^5$

$\boxed{P_5(x) = 3x - \frac{27}{3!}x^3 + \frac{243}{5!}x^5}$

$f(0) = 0$

$f'(0) = 3$

$f''(0) = 0$

$f'''(0) = -27$

$f^{(4)}(0) = 0$

$f^{(5)}(0) = 243$

Find the specified Maclaurin or Taylor polynomial (for these you must use the Taylor procedure):

7) Find the 5^{th} degree Taylor polynomial centered at $c = 2$ for the function $f(x) = \ln(x)$

$f(x) = \ln x$

$f'(x) = \frac{1}{x} = x^{-1}$

$f''(x) = -x^{-2}$

$f'''(x) = 2x^{-3}$

$f^{(4)}(x) = -6x^{-4}$

$f^{(5)}(x) = 24x^{-5}$

$P_5(x) = f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3 + \frac{f^{(4)}(2)}{4!}(x-2)^4 + \frac{f^{(5)}(2)}{5!}(x-2)^5$

$P_5(x) = \ln(2) + \frac{1}{2}(x-2) + \frac{(-1/4)}{2!}(x-2)^2 + \frac{(1/4)}{3!}(x-2)^3 + \frac{(-3/8)}{4!}(x-2)^4 + \frac{(3/4)}{5!}(x-2)^5$

$P_5(x) = \ln(2) + \frac{1}{2}(x-2) - \frac{1}{8}(x-2)^2 + \frac{1}{24}(x-2)^3 - \frac{1}{64}(x-2)^4 + \frac{1}{160}(x-2)^5$

$f(2) = \ln(2)$

$f'(2) = \frac{1}{2}$

$f''(2) = -\frac{1}{4}$

$f'''(2) = \frac{1}{4}$

$f^{(4)}(2) = -\frac{3}{8}$

$f^{(5)}(2) = \frac{3}{4}$

8) Find the 4^{th} degree Maclaurin polynomial for the function $f(x) = xe^x$

$f(x) = xe^x$

$f'(x) = xe^x + e^x(1)$

$f''(x) = xe^x + e^x(1) + e^x = xe^x + 2e^x$

$f'''(x) = xe^x + e^x(1) + 2e^x = xe^x + 3e^x$

$f^{(4)}(x) = xe^x + e^x(1) + 3e^x = xe^x + 4e^x$

$P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$

$P_4(x) = 0 + 1x + \frac{2}{2!}x^2 + \frac{3}{3!}x^3 + \frac{4}{4!}x^4$

$P_4(x) = x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4$

$f(0) = 0$

$f'(0) = 1$

$f''(0) = 2$

$f'''(0) = 3$

$f^{(4)}(0) = 4$

Find the specified Maclaurin or Taylor polynomial (for these you must use the Taylor procedure):

9) Find the 4th degree Taylor polynomial centered at c = 9 for the function $f(x)=\sqrt{x}$

$f(x)=\sqrt{x}=x^{1/2}$

$f'(x)=\frac{1}{2}x^{-1/2}$

$f''(x)=-\frac{1}{4}x^{-3/2}$

$f'''(x)=\frac{3}{8}x^{-5/2}$

$f^{(4)}(x)=-\frac{15}{16}x^{-7/2}$

$P_4(x)=f(9)+f'(9)(x-9)+\frac{f''(9)}{2!}(x-9)^2+\frac{f'''(9)}{3!}(x-9)^3+\frac{f^{(4)}(9)}{4!}(x-9)^4$

$$\boxed{P_4(x)=3+\frac{1}{6}(x-9)+\frac{(-1/108)}{2!}(x-9)^2+\frac{(1/648)}{3!}(x-9)^3+\frac{(-15/34992)}{4!}(x-9)^4}$$

$f(9)=3$

$f'(9)=\frac{1}{6}$

$f''(9)=-\frac{1}{108}$

$f'''(9)=\frac{1}{648}$

$f^{(4)}(9)=\frac{-15}{16}(9)^{-7/2}=\frac{-15}{16}(9^{1/2})^{-7}=\frac{-15}{16}(3)^{-7}=\frac{-15}{34992}$

10) Find the 5th degree Maclaurin polynomial for the function $f(x)=e^{3x}$

$f(x)=e^{3x}$

$f'(x)=3e^{3x}$

$f''(x)=9e^{3x}$

$f'''(x)=27e^{3x}$

$f^{(4)}(x)=81e^{3x}$

$f^{(5)}(x)=243e^{3x}$

$P_5(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\frac{f^{(4)}(0)}{4!}x^4+\frac{f^{(5)}(0)}{5!}x^5$

$$\boxed{P_5(x)=1+3x+\frac{9}{2!}x^2+\frac{27}{3!}x^3+\frac{81}{4!}x^4+\frac{243}{5!}x^5}$$

$f(0)=1$

$f'(0)=3$

$f''(0)=9$

$f'''(0)=27$

$f^{(4)}(0)=81$

$f^{(5)}(0)=243$

For #11-13, use the basic Power Series list to write out terms:

11) Determine the degree of the Maclaurin polynomial centered at 0 required to approximate $f(0.4)$ for the function $f(x)=\sin(x)$ for the error to be less than 0.0002.

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$

$\sin(0.4) = (0.4) - \frac{(0.4)^3}{3!} \Big| + \frac{(0.4)^5}{5!}$

$(.0107) \quad (8.53 \times 10^{-5}$

$= 0.0000853$

< 0.002 ✓

truncate here

$\boxed{\text{3rd degree polynomial}}$

12) Determine the degree of the Maclaurin polynomial centered at 1 required to approximate $f(1.4)$ for the function $f(x)=\ln(x)$ for the error to be less than 0.0002.

$\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \dots$

$\ln(1.4) = (0.4) - \frac{(0.4)^2}{2} + \frac{(0.4)^3}{3} - \frac{(0.4)^4}{4} + \frac{(0.4)^5}{5} \Big| - \frac{(0.4)^6}{6}$

$(.08) \quad (.0213) \quad (.0064) \quad (.002048) \quad (6.83 \times 10^{-4}$

$= 0.000683$ ✓

< 0.0002

truncate here

$\boxed{\text{5th degree polynomial}}$

13) Determine the degree of the Maclaurin polynomial centered at 0 required to approximate $f(0.7)$ for the function $f(x)=e^x$ for the error to be less than 0.0004.

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$

$e^{0.7} = 1 + (0.7) + \frac{(0.7)^2}{2!} + \frac{(0.7)^3}{3!} + \frac{(0.7)^4}{4!} + \frac{(0.7)^5}{5!} \Big| + \frac{(0.7)^6}{6!}$

$(.245) \quad (.057) \quad (.010004) \quad (.0014) \quad (1.63 \times 10^{-4}$

$.000163 < .0004$ ✓

truncate here

$\boxed{\text{5th degree polynomial}}$

14) Find the interval of convergence of the series $\sum_{n=1}^{\infty} \frac{x^{2n}}{(2n)!}$

ratio test

$$\lim_{n\to\infty} \left| \frac{x^{2(n+1)}}{(2(n+1))!} \frac{(2n)!}{x^{2n}} \right|$$

$$\lim_{n\to\infty} \left| \frac{x^{2n+2}}{(2n+2)!} \frac{(2n)!}{x^{2n}} \right|$$

$$\lim_{n\to\infty} \left| \frac{x^{2n} x^{2} (2n)!}{(2n+2)(2n+1)(2n)!\, x^{2n}} \right|$$

$$\lim_{n\to\infty} \left| \frac{x^2}{(2n+2)(2n+1)} \right|$$

$$\left(\lim_{n\to\infty} \frac{1}{(2n+2)(2n+1)}\right) |x^2|$$

$$0 \cdot |x^2| < 1$$

converges for all x

$\therefore$ interval of convergence is $\boxed{-\infty < x < \infty \text{ or } (-\infty, \infty)}$

15) Find the interval of convergence of the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-4)^n}{n9^n}$

ratio test

$$\lim_{n\to\infty}\left|\frac{(x-4)^{n+1}}{(n+1)9^{n+1}}\,\frac{n9^n}{(x-4)^n}\right|$$

$$\lim_{n\to\infty}\left|\frac{(x-4)^n(x-4)\,n\,9^n}{(n+1)9^n 9\,(x-4)^n}\right|$$

$$\lim_{n\to\infty}\left|\frac{(x-4)\,n}{(n+1)9}\right|$$

$$\left(\lim_{n\to\infty}\frac{n}{n+1}\right)\left|\frac{x-4}{9}\right|$$

$$(1)\left|\frac{x-4}{9}\right|<1$$

$$-1<\frac{x-4}{9}<1$$

$$-9<x-4<9$$

$$-5<x<13$$

test ends

$x=-5$

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(-5-4)^n}{n9^n}$$

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(-9)^n}{n9^n}$$

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}\left(\frac{-9}{9}\right)^n$$

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}(-1)^n$$

$$\sum_{n=1}^{\infty}(-1)^n(-1)(-1)^n\frac{1}{n}$$

$$\sum_{n=1}^{\infty}((-1)(-1))^n(-1)\frac{1}{n}$$

$$\sum_{n=1}^{\infty}(1)^n(-1)\frac{1}{n}$$

$$\sum_{n=1}^{\infty}(1)(-1)\frac{1}{n}$$

$$-\sum_{n=1}^{\infty}\frac{1}{n}$$ p-series w/ $p=1$ diverges

$x=13$

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(13-4)^n}{n9^n}$$

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}\left(\frac{9}{9}\right)^n$$

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}\cancel{(1)^n}\ 1$$

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$$

alternating series test

$\lim_{n\to\infty}\frac{1}{n}=0$ ✓

$a_{n+1}\le a_n$?

$\frac{1}{n+1}<\frac{1}{n}$ ✓

converges

∴ interval of convergence is $\boxed{-5<x\le 13 \text{ or } (-5,13]}$

16) Find the interval of convergence of the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-1)^{n+1}}{n+1}$

ratio test

$$\lim_{n\to\infty}\left|\frac{(x-1)^{n+2}}{(n+2)}\frac{(n+1)}{(x-1)^{n+1}}\right|$$

$$\lim_{n\to\infty}\left|\frac{(x-1)(x-1)^{n+1}(n+1)}{(n+2)(x-1)^{n+1}}\right|$$

$$\lim_{n\to\infty}\left|\frac{(x-1)(n+1)}{n+2}\right|$$

$$\left(\lim_{n\to\infty}\frac{n+1}{n+2}\right)|x-1|$$

$$(1)\ |x-1|<$$

$$-1<x-1<1$$

$$0<x<2$$

test ends ?!

$x=0$

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(0-1)^{n+1}}{n+1}$$

$$\sum_{n=1}^{\infty}(-1)^{n+1}(-1)^{n+1}\frac{1}{n+1}$$

$$\sum_{n=1}^{\infty}[(-1)(-1)]^{n+1}\frac{1}{n+1}$$

$$\sum_{n=1}^{\infty}(1)^{n+1}\frac{1}{n+1}$$

$$\sum_{n=1}^{\infty}(1)\frac{1}{n+1}$$

$$\sum_{n=1}^{\infty}\frac{1}{n+1}$$

Limit compare w/ $\sum_{n=1}^{\infty} b_n=\sum_{n=1}^{\infty}\frac{1}{n}$

p-series w/ $p=1$ (diverges)

$$\lim_{n\to\infty}\frac{a_n}{b_n}$$

$$\lim_{n\to\infty}\frac{\left(\frac{1}{n+1}\right)}{\left(\frac{1}{n}\right)}=\lim_{n\to\infty}\frac{n}{n+1}=1$$

finite, positive "linked"

$\therefore \sum_{n=1}^{\infty}\frac{1}{n+1}$ also diverges

$x=2$

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(2-1)^{n+1}}{n+1}$$

$$\sum_{n=1}^{\infty}(-1)^{n+1}(1)^{n+1}\frac{1}{n+1}$$

$$\sum_{n=1}^{\infty}(-1)^{n+1}(1)\frac{1}{n+1}$$

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n+1}$$

Alternating Series Test

- $\lim_{n\to\infty}\frac{1}{n+1}=0$ ✓
- $a_{n+1}\le a_n$?

$\frac{1}{n+2}<\frac{1}{n+1}$ ✓

converges

$\therefore$ interval of convergence is $\boxed{0<x\le 2 \text{ or } (0,2]}$

17) If $f(x)=\sum_{n=1}^{\infty}\dfrac{x^{2n}}{2n+1}$ find the series expression for (a) $f'(x)$ (b) $\int f(x)\,dx$

(a) $\boxed{f'(x)=\sum_{n=1}^{\infty}\frac{1}{2n+1}(2n)x^{2n-1}}$

(b) $\boxed{\int f(x)dx=\sum_{n=1}^{\infty}\frac{1}{2n+1}\frac{x^{2n+1}}{(2n+1)}+C}$

18) If $\left|f^{(4)}(x)\right|\le 4$, find the Lagrange error upper bound if a third-degree Taylor polynomial centered at x = 7 is used to approximate $f(7.3)$. . (Assume the series converges at x = 7.3.)

$|\text{error}|\le\left|\frac{f^{(N+1)}(z)}{(N+1)!}(x-c)^{N+1}\right|$ $\quad x=7.3$, $c=7$, $N=3$

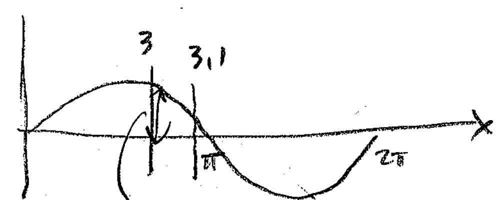
$\le\left|\frac{f^{(4)}(z)}{4!}(7.3-7)^4\right|$

$\le\left|\frac{4}{4!}(7.3-7)^4\right|=\boxed{0.00135}$

19) If $f^{(6)}(x)=40\sin(x)$, find the Lagrange error upper bound if a fifth-degree Taylor polynomial centered at x = 3 is used to approximate $f(3.1)$. . (Assume the series converges at x = 3.1.)

$|\text{error}|\le\left|\frac{f^{(N+1)}(z)}{(N+1)!}(x-c)^{N+1}\right|$ $\quad x=3.1$, $c=3$, $N=5$

$f^{(6)}(x)$



largest at $z=3$

$\le\left|\frac{f^{(6)}(z)}{6!}(3.1-3)^6\right|$

$\le\left|\frac{40\sin(3)}{6!}(3.1-3)^6\right|$

$=\boxed{7.841\cdot10^{-9}}$

20) Find the Lagrange error upper bound if a fourth-degree Taylor polynomial approximating $f(x) = e^x$ centered at x = 3 is used to approximate $f(3.1)$. . (Assume the series converges at x = 3.1.)

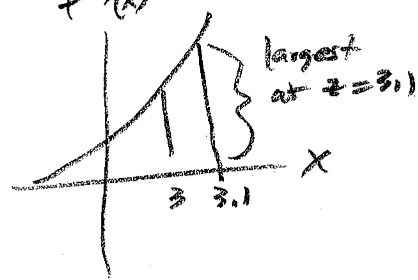
$$|\text{error}| \le \left| \frac{f^{(N+1)}(z)}{(N+1)!} (x-c)^{N+1} \right|$$

$x = 3.1$
$c = 3$
$N = 4$

$$\le \left| \frac{f^{(5)}(z)}{5!} (3.1-3)^5 \right|$$

$f(x) = e^x$
$f'(x) = e^x$
$\vdots$
$f^{(5)}(x) = e^x$

$f^{(5)}(x)$



$$\le \left| \frac{e^{(3.1)}}{5!} (3.1-3)^5 \right|$$

$$= \boxed{1.85 \times 10^{-6}}$$