AP Calculus BC – Unit 7 Extra Practice

8.1 – Extra Practice

Identify the type of each conic section.

#7b. $3x^2 - 5y^2 - 10y - 80 = 0$

$\boxed{\text{hyperbola}}$

#8b. $3x^2 + 4y^2 - 8y - 116 = 0$

$\boxed{\text{ellipse}}$

#9b. $-2x^2 + 20x + y - 47 = 0$

$\boxed{\text{parabola}}$

#10b. $9x^2 + 49y^2 + 291y = 0$

$\boxed{\text{ellipse}}$

#11b. $x^2 + y^2 - 4x + 8y + 19 = 0$

$\boxed{\text{circle}}$

#12b. $-x^2 + y^2 - 2x + 2y - 4 = 0$

$\boxed{\text{hyperbola}}$

Identify the center and radius of each circle.

#13b. $(x-5)^2+(y-2)^2=25$

center $=(5,2)$

$r=5$

#14b. $(x+13)^2+(y-5)^2=13$

center $=(-13,5)$

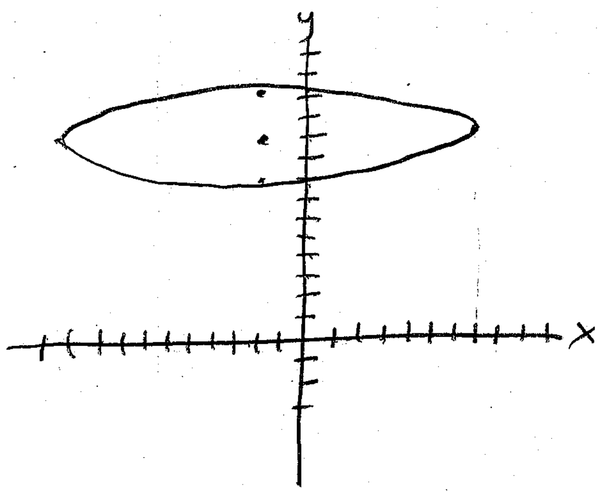
$r=\sqrt{13}$

Sketch the conic section already in standard form.

#15b. $\dfrac{(x+2)^2}{81}+\dfrac{(y-10)^2}{4}=1$ (ellipse)

$(9)^2$ $\quad$ $(2)^2$

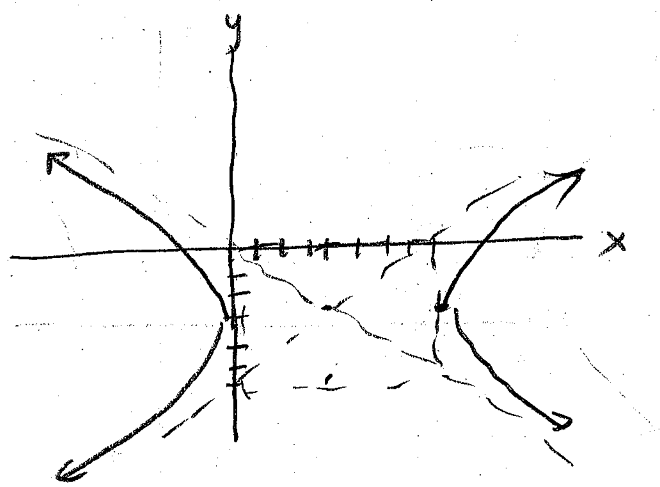
center $=(-2,10)$



#16b. $\dfrac{(x-4)^2}{16}-\dfrac{(y+3)^2}{9}=1$ (hyperbola)

$(4)^2$ $\quad$ $(3)^2$

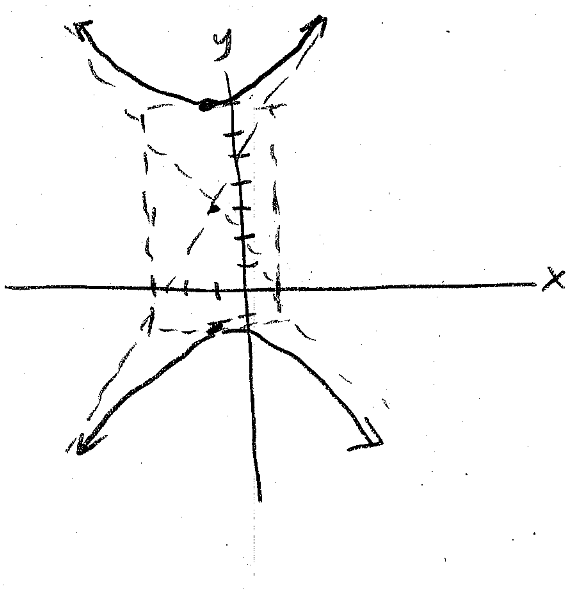
center $=(4,-3)$



#17b. $\dfrac{(y-3)^2}{16}-\dfrac{(x+1)^2}{4}=1$ (hyperbola)

$(4)^2$ $\quad$ $(2)^2$

center $=(-1,3)$

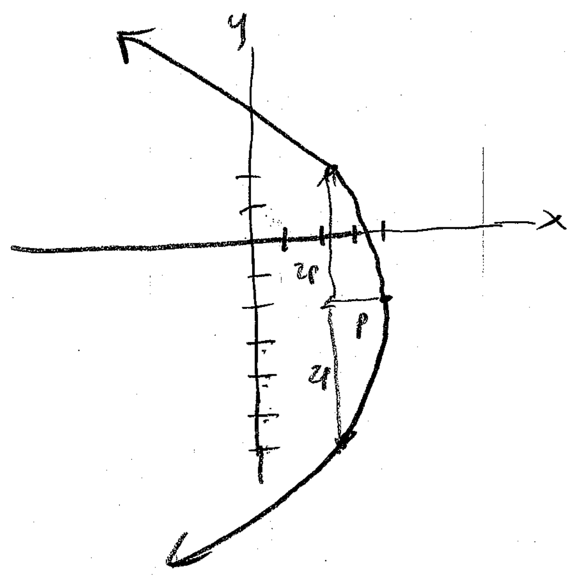


#18b. $(y+2)^2=-8(x-4)$ (parabola)

vertex $=(4,-2)$

$4p=-8$ $\quad$ $x=-y^2$

$p=-2$



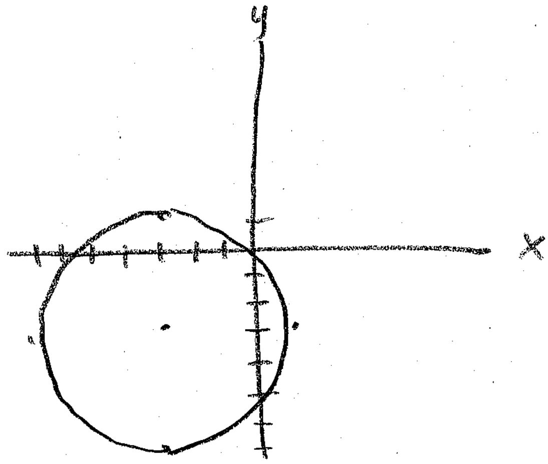
Convert the equation to standard form and sketch.

#19b. $x^2+y^2+6x+6y+2=0$

$(x^2+6x+9)+(y^2+6y+9)=-2+9+9$

$(x+3)^2+(y+3)^2=16$

circle, center $=(-3,-3)$, $r=4$



#20b. $-x^2+25y^2-150y+200=0$

$-(x-0)^2+(25y^2-150y)=-200$

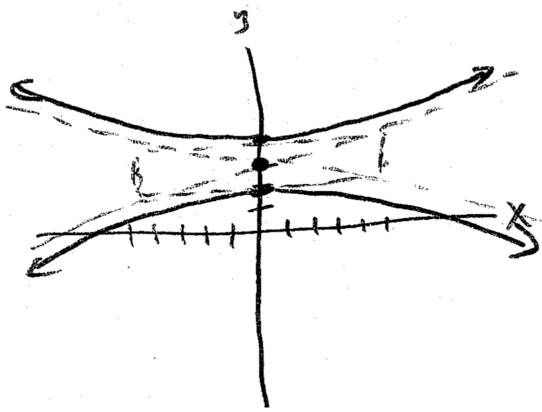
$-(x-0)^2+25(y^2-6y+9)=-200+225$

$-(x-0)^2+25(y-3)^2=25$

$25(y-3)^2-(x-0)^2=25$

$\frac{(y-3)^2}{1}-\frac{(x-0)^2}{25}=1$

hyperbola
center $=(0,3)$



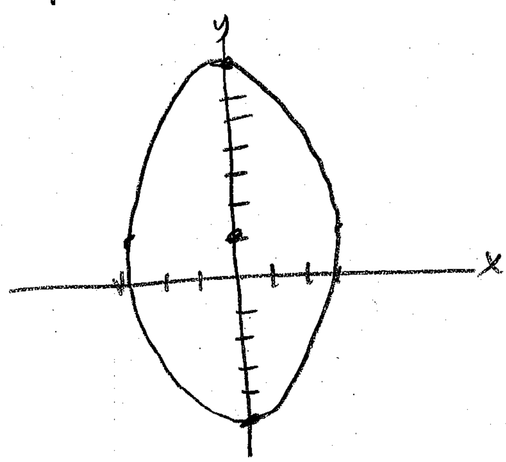
#21b. $4x^2+y^2-2y-35=0$

$(4x^2)+(y^2-2y+1)=35+1$

$4(x-0)^2+(y-1)^2=36$

$\frac{(x-0)^2}{9}+\frac{(y-1)^2}{36}=1$

ellipse
center $=(0,1)$

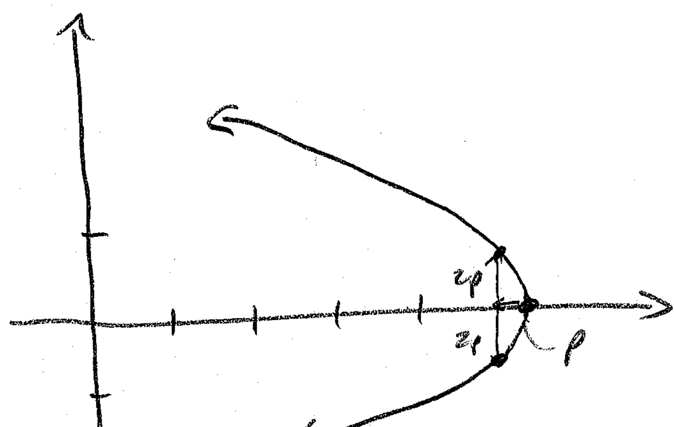


#22b. $y^2+x-5=0$

$y^2=-x+5$

$(y-0)^2=-(x-5)$

parabola
vertex $(5,0)$
$x=-y^2$
$4p=-1$, $p=1/4$



8.2 – Extra Practice

Sketch the curve of the parametric equation by either converting the equation to rectangular form, or using a table, then use your calculator to verify your sketch.

#9b. $x = 5 - 4t, \quad y = 2 + 5t$

$5t = y - 2$

$t = \frac{1}{5}y - \frac{2}{5}$

$x = 5 - 4\left(\frac{1}{5}y - \frac{2}{5}\right)$

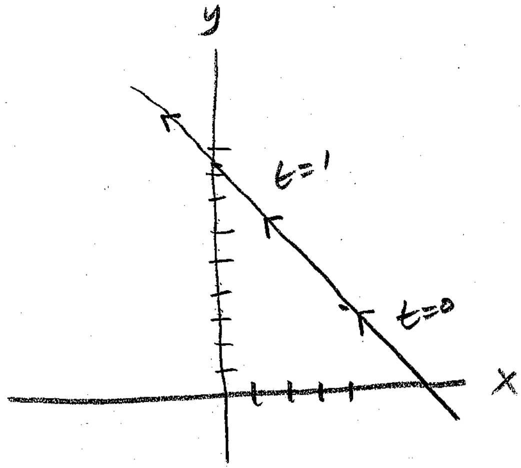
$x = 5 - \frac{4}{5}y + \frac{8}{5}$

$\frac{4}{5}y = -x + \frac{33}{5}$

$4y = -5x + 33$

$y = \frac{-5}{4}x + \frac{33}{4}$

t	x	y
0	5	2
1	1	7



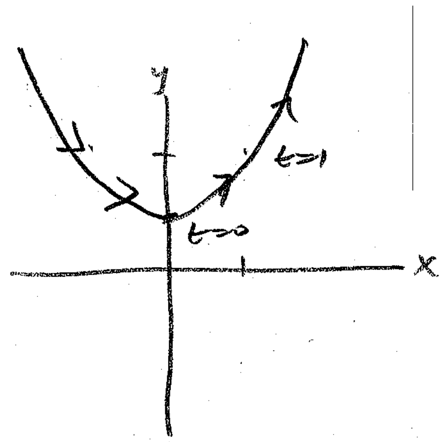
#10b. $x = t^2, \quad y = t^4 + 1$

$t^2 = x$

$y = (t^2)^2 + 1$

$y = x^2 + 1$

t	x	y
0	0	1
1	1	2



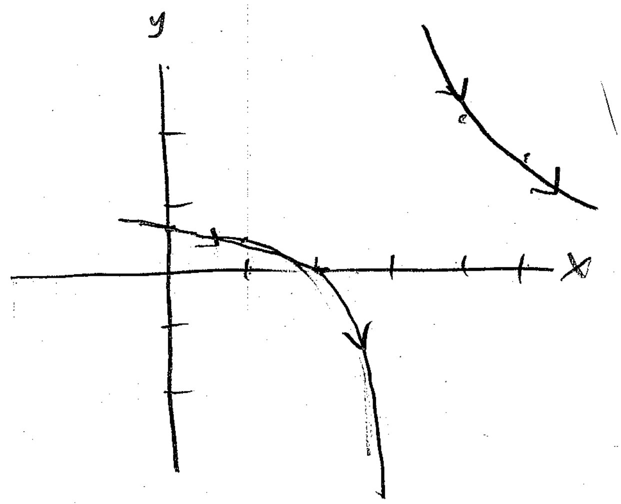
#11b. $x = t + 2, \quad y = \frac{t}{t-1}$

$t = x - 2$

$y = \frac{x-2}{x-2-1} = \frac{x-2}{x-3}$

switch to table

t	x	y
-2	0	.667
-1	1	.5
0	2	0
1	3	undef
2	4	2
3	5	1.5

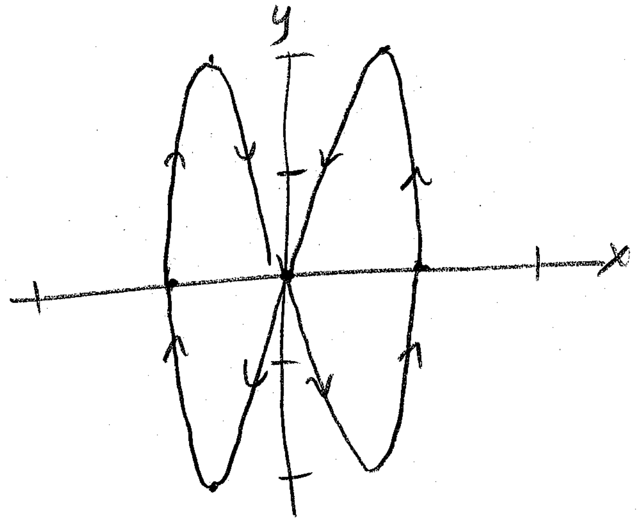


Sketch the curve of the parametric equation by either converting the equation to rectangular form, or using a table, then use your calculator to verify your sketch.

#12b. $x = \cos(\theta)$, $y = 2\sin(2\theta)$

use table

θ	x	y
-2π	1	0
$-3\pi/2$	0	0
$-\pi$	-1	0
$-\pi/2$	0	0
0	1	0
$\pi/2$	0	0
π	-1	0
$3\pi/2$	0	0
2π	1	0
$\pi/4$	.707	2
$\frac{3\pi}{4}$	$-.707$	-2
$\frac{5\pi}{4}$	$-.707$	-2



#13b. $x = t^3$, $y = 3\ln(t)$

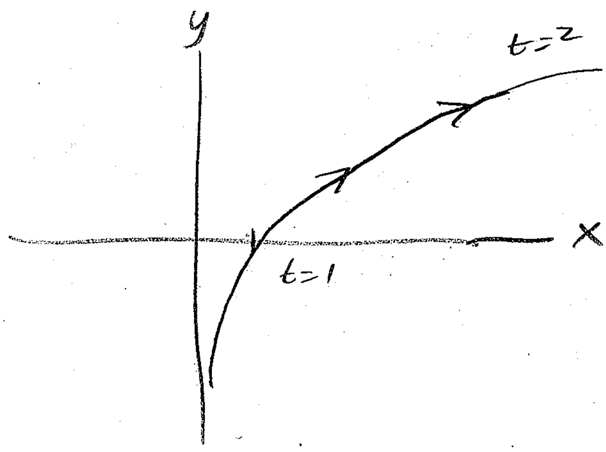
$t = x^{1/3}$

$y = 3\ln(x^{1/3})$

$y = 3\left(\frac{1}{3}\ln(x)\right)$

$y = \ln(x)$

t	x	y
1	1	0
2	8	$3\ln(2)$



8.3 day 1 – Extra Practice

Find $\frac{dy}{dx}$ for the given parametric equations.

#2b. $x = \sqrt[3]{t}, \quad y = 4 - t$

$x = t^{1/3}$

$\frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \boxed{\frac{-1}{\frac{1}{3}t^{-2/3}} = -3t^{2/3}}$

#3b. $x = 2e^{\theta}, \quad y = e^{\left(-\frac{1}{2}\theta\right)}$

$\frac{dy}{dx} = \frac{(dy/d\theta)}{(dx/d\theta)} = \boxed{\frac{-\frac{1}{2}e^{-1/2\theta}}{2e^{\theta}} = -\frac{1}{4}e^{-\frac{3}{2}\theta}}$

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ and find the slope and concavity (if possible) at the given value of the parameter.

#4b. $x = \sqrt{t}, \quad y = 3t - 1 \quad (at \;\; t = 1)$

$x = t^{1/2}$

$\frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \frac{3}{\frac{1}{2}t^{-1/2}} = 6\sqrt{t}$

$\left.\frac{dy}{dx}\right|_{t=1} = 6\sqrt{1} = \boxed{6}$

$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{(dx/dt)} = \frac{6\left(\frac{1}{2}t^{-1/2}\right)}{\frac{1}{2}t^{-1/2}} = \boxed{6}$

$\left.\frac{d^2y}{dx^2}\right|_{t=1} = \boxed{6}$

#5b. $x = \cos(\theta), \quad y = 3\sin(\theta) \quad (at \;\; t = 0)$

$\frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{3\cos\theta}{-\sin\theta} = -3\cot\theta$

$\left.\frac{dy}{dx}\right|_{t=0} = \frac{3\cos(0)}{-\sin(0)} \rightarrow \frac{3}{-0}$ $\boxed{\text{undefined}}$

(vertical tangent)

$\frac{d^2y}{dx^2} = \frac{\frac{d}{d\theta}\left[\frac{dy}{dx}\right]}{\left(\frac{dx}{d\theta}\right)} = \frac{-3(-\csc^2\theta)}{-\sin\theta} = \frac{-3\csc^2\theta}{\sin\theta}$

$= \frac{-3}{\sin^3\theta}$

$\left.\frac{d^2y}{dx^2}\right|_{t=0} = \frac{-3}{(\sin(0))^3} \leftarrow \div 0$ $\boxed{\text{undefined}}$

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ and find the slope and concavity (if possible) at the given value of the parameter.

#6b. $x=\sqrt{t+1},\quad y=\sqrt{t-1}\quad (at\ t=2)$

$$\frac{dx}{dt}=\frac{1}{2}(t+1)^{-1/2}(1)=\frac{1}{2\sqrt{t+1}}\qquad \frac{dy}{dt}=\frac{1}{2}(t-1)^{-1/2}(1)=\frac{1}{2\sqrt{t-1}}$$

$$\frac{dy}{dx}=\frac{(dy/dt)}{(dx/dt)}=\frac{\left(\frac{1}{2\sqrt{t-1}}\right)}{\left(\frac{1}{2\sqrt{t+1}}\right)}=\frac{\sqrt{t+1}}{\sqrt{t-1}}=\sqrt{\frac{t+1}{t-1}}$$

$$\left.\frac{dy}{dx}\right|_{t=2}=\sqrt{\frac{(2)+1}{(2)-1}}=\boxed{\sqrt{3}}$$

$$\frac{d^2y}{dx^2}=\frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{(dx/dt)}=\frac{\left(\frac{1}{2}\left(\frac{t+1}{t-1}\right)^{-1/2}\left[\frac{(t-1)(1)-(t+1)(1)}{(t-1)^2}\right]\right)}{\left(\frac{1}{2\sqrt{t+1}}\right)}$$

$$=\frac{1}{2}\left(\frac{t-1}{t+1}\right)^{1/2}\left[\frac{t-1-t-1}{(t-1)^2}\right]\cdot\left(\frac{2\sqrt{t+1}}{1}\right)$$

$$=\frac{1}{2}\sqrt{\frac{t-1}{t+1}}\ \frac{(-2)}{(t-1)^2}\ \frac{2\sqrt{t+1}}{1}$$

$$=\sqrt{\frac{(t-1)}{(t+1)}\frac{(t+1)}{1}}\ \frac{(-2)}{(t-1)^2}=\frac{-2\sqrt{t-1}}{(t-1)^2}$$

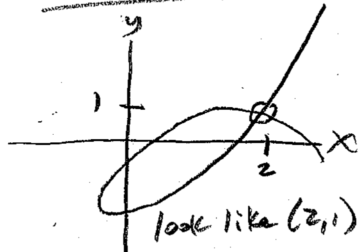
$$\left.\frac{d^2y}{dx^2}\right|_{t=2}=\frac{-2\sqrt{(2)-1}}{(2-1)^2}=\frac{-2\sqrt{1}}{1^2}=\boxed{-2}$$

Find the equations of the tangent lines at the point where the curve crosses itself.

#7b. $x=t^2-t,\quad y=t^3-3t-1$

$m=\dfrac{dy}{dx}=\dfrac{(dy/dt)}{(dx/dt)}=\dfrac{3t^2-3}{2t-1}$

calculator graph:



system $\begin{cases} x=t^2-t=2 \\ y=t^3-3t-1=1 \end{cases}$

$t^2-t=2$

$t^2-t-2=0$

$(t+1)(t-2)=0$

$t=-1,\ t=2$

$t^3-3t-1=1$

$t^3-3t-2=0$

by calc graph:

$t=-1,\ t=2$

at $t=-1$ and $t=2$

at $t=-1$

$x=(-1)^2-(-1)=2$

$y=(-1)^3-3(-1)-1=1$

$m=\dfrac{3(-1)^2-3}{2(-1)-1}=\dfrac{0}{-3}=0$

$\boxed{\begin{gathered}(y-1)=0(x-2)\\ y-1=0\\ y=1\end{gathered}}$

(horizontal tangent)

at $t=2$

$x=(2)^2-(2)=2$

$y=(2)^3-3(2)-1=1$

$m=\dfrac{3(2)^2-3}{2(2)-1}=\dfrac{9}{3}=3$

$\boxed{(y-1)=3(x-2)}$

Find all points (if any) of horizontal or vertical tangency to the curve.

#8b. $x=t+1,\quad y=t^2+3t$

$m=\dfrac{(dy/dt)}{(dx/dt)}$

horiz. tangent

when $\dfrac{dy}{dt}=0$

$2t+3=0$

$2t=-3$

$t=-3/2$

$x=\left(-\frac{3}{2}\right)+1=-\frac{1}{2}$

$y=\left(-\frac{3}{2}\right)^2+3\left(-\frac{3}{2}\right)$

$=\frac{9}{4}-\frac{9}{2}=-\frac{9}{4}$

$\boxed{\text{horiz. tangent at } \left(-\tfrac{1}{2},-\tfrac{9}{4}\right)}$

vertical tangent

when $\dfrac{dx}{dt}=0$

$1=0$

$\boxed{\text{nowhere}}$

Find all points (if any) of horizontal or vertical tangency to the curve.

#9b. $x = \cos(\theta)$, $y = 2\sin(2\theta)$

$m = \dfrac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$

horizontal tangent

when $\frac{dy}{dt} = 0$

$4\cos(2\theta) = 0$

$\cos(2\theta) = 0$

$2\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \ldots$

$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, \ldots$

$\theta = \frac{\pi}{4}$

$x = \cos(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$

$y = 2\sin(\frac{2\pi}{4}) = 2$

$(\frac{\sqrt{2}}{2}, 2)$

$\theta = \frac{3\pi}{4}$

$x = \cos(\frac{3\pi}{4}) = -\frac{\sqrt{2}}{2}$

$y = 2\sin(\frac{3\pi}{2}) = -2$

$(-\frac{\sqrt{2}}{2}, -2)$

$\theta = \frac{5\pi}{4}$

$x = \cos(\frac{5\pi}{4}) = -\frac{\sqrt{2}}{2}$

$y = 2\sin(\frac{5\pi}{2}) = 2$

$(-\frac{\sqrt{2}}{2}, 2)$

$\theta = \frac{7\pi}{4}$

$x = \cos(\frac{7\pi}{4}) = \frac{\sqrt{2}}{2}$

$y = 2\sin(\frac{7\pi}{2}) = -2$

$(\frac{\sqrt{2}}{2}, -2)$

$\boxed{\text{horizontal tangents at } (\frac{\sqrt{2}}{2}, 2), (-\frac{\sqrt{2}}{2}, -2), (-\frac{\sqrt{2}}{2}, 2), (\frac{\sqrt{2}}{2}, -2)}$

vertical tangent

when $\frac{dx}{dt} = 0$

$-\sin\theta = 0$

$\sin\theta = 0$

$\theta = 0, \pi, 2\pi \ldots$

$\theta = 0$

$x = \cos(0) = 1$

$y = 2\sin(0) = 0$

$(1, 0)$

$\theta = \pi$

$x = \cos(\pi) = -1$

$y = 2\sin(2\pi) = 0$

$(-1, 0)$

$\boxed{\text{vertical tangents at } (1,0) \text{ and } (-1,0)}$

Determine the open t-intervals on which the curve is concave up or concave down.

#10b. $x = t^2, \quad y = \ln(t)$

$$\frac{dy}{dx} = \frac{\left(dy/dt\right)}{\left(dx/dt\right)} = \frac{\left(\frac{1}{t}\right)}{2t} = \frac{1}{2t^2} = \frac{1}{2}t^{-2}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\left(\frac{dx}{dt}\right)} = \frac{-t^{-3}}{2t} = \frac{-1}{2t^4}$$

inflection pts where $\frac{d^2y}{dx^2} = 0$ or DNE, when $t = 0$

$-\infty$ ———— 0 ———— ∞

test (-1) $\quad \left.\frac{d^2y}{dx^2}\right|_{t=-1} \frac{(-)}{(+)} < 0$

test (1) $\quad \frac{d^2y}{dx^2} \frac{(-)}{(+)} < 0$

answer would be concave up: nowhere

concave down: $(-\infty, 0) \cup (0, \infty)$

but since $y = \ln(t)$ and domain of $\ln(t)$ is $t > 0$,

can't include any $t \leq 0$.

So

$$\boxed{\begin{array}{l} \text{concave up: nowhere} \\ \text{concave down: } 0 < t < \infty \end{array}}$$

8.3 day 2 – Extra Practice

Find the arc length of the curve on the given interval.

#2b. $x = 6t^2, \quad y = 2t^3 \quad (1 \le t \le 4)$

$\frac{dx}{dt} = 12t \quad \frac{dy}{dt} = 6t^2$

$$\boxed{\text{arc length} = \int_1^4 \sqrt{(12t)^2 + (6t^2)^2}\, dt}$$

$$\int_1^4 \sqrt{144t^2 + 36t^4}\, dt$$

$$\int_1^4 \sqrt{36t^2(4+t^2)}\, dt$$

$$\int_1^4 6t\sqrt{4+t^2}\, dt \qquad u = 4+t^2,\ du = 2t\,dt,\ t\,dt = \tfrac{1}{2}du$$

$$\int_5^{20} u^{1/2}\, 6\left(\tfrac{1}{2}\right) du$$

$$3\left[\tfrac{2}{3}u^{3/2}\right]_5^{20} = \boxed{3(20)^{3/2} - 3(5)^{3/2}}$$

practice evaluating integrals by hand if possible

#3b. $x = \arcsin(t), \quad y = \ln\left(\sqrt{1-t^2}\right) \quad \left(0 \le t \le \frac{1}{2}\right)$

$\frac{dx}{dt} = \frac{1}{\sqrt{1-t^2}} \qquad \frac{dy}{dx} = \frac{1}{\sqrt{1-t^2}}\left(\tfrac{1}{2}(1-t^2)^{-1/2}(-2t)\right) = \frac{-t}{\sqrt{1-t^2}\sqrt{1-t^2}} = \frac{-t}{1-t^2}$

$$\boxed{\text{arc length} = \int_0^{1/2} \sqrt{\left(\frac{1}{\sqrt{1-t^2}}\right)^2 + \left(\frac{-t}{1-t^2}\right)^2}\, dt}$$

$$= \int_0^{1/2} \sqrt{\frac{1}{(1-t^2)} + \frac{t}{(1-t^2)^2}}\, dt = \int_0^{1/2} \sqrt{\frac{1}{(1-t^2)}\frac{(1-t^2)}{(1-t^2)} + \frac{t}{(1-t^2)^2}}\, dt$$

$$= \int_0^{1/2} \sqrt{\frac{1-t^2+t}{(1-t^2)^2}}\, dt = \int_0^{1/2} \sqrt{\frac{-(t^2-t-1)}{(1-t^2)^2}}\, dt$$

can't see how to simplify further so for this one, math 9

$$= \boxed{0.549}$$