AP Calculus BC – Unit 8 Part 2 Extra Practice

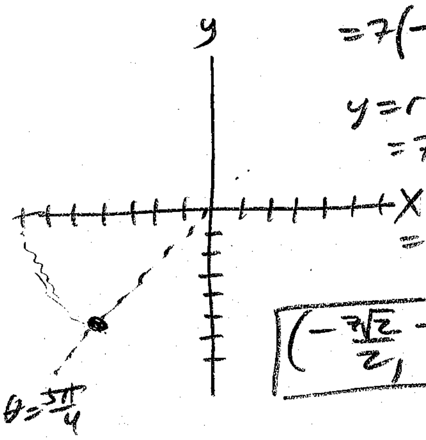
8.4 – Extra Practice

Plot the (r,θ) polar coordinate and find the corresponding rectangular (x,y) coordinate.

#13b. $\left(7, \dfrac{5\pi}{4}\right)$

$x = r\cos\theta$
$= 7\cos\left(\frac{5\pi}{4}\right)$
$= 7\left(-\frac{\sqrt{2}}{2}\right) = -\frac{7\sqrt{2}}{2}$

$y = r\sin\theta$
$= 7\sin\left(\frac{5\pi}{4}\right)$
$= 7\left(-\frac{\sqrt{2}}{2}\right) = -\frac{7\sqrt{2}}{2}$

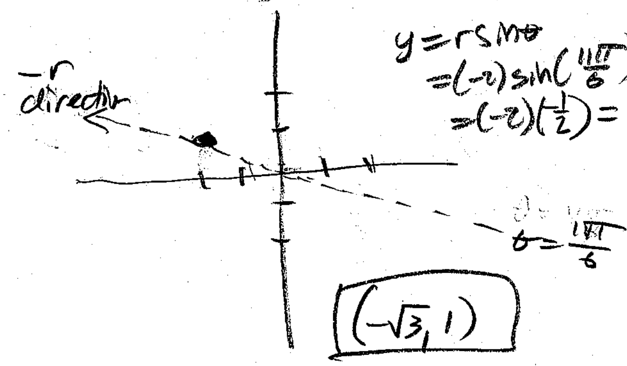


$\boxed{\left(-\frac{7\sqrt{2}}{2}, -\frac{7\sqrt{2}}{2}\right)}$

#14b. $\left(-2, \dfrac{11\pi}{6}\right)$

$x = r\cos\theta$
$= (-2)\cos\left(\frac{11\pi}{6}\right)$
$= (-2)\left(\frac{\sqrt{3}}{2}\right) = -\sqrt{3}$

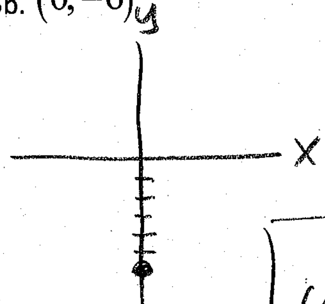
$y = r\sin\theta$
$= (-2)\sin\left(\frac{11\pi}{6}\right)$
$= (-2)\left(-\frac{1}{2}\right) = 1$



$\boxed{(-\sqrt{3}, 1)}$

The rectangular (x,y) coordinate is given. Plot the coordinate, the find two sets of polar coordinates for the same location with $0 \le \theta < 2\pi$

#15b. $(0,-6)$

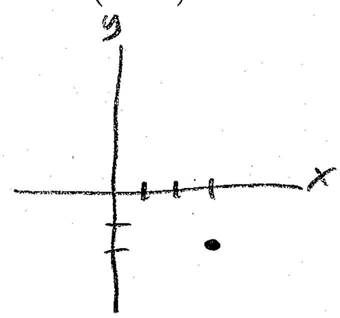


$r = \sqrt{0^2 + (-6)^2} = 6$

$\theta = \frac{3\pi}{2}$ or $-\frac{\pi}{2}$

r, θ
$(6, \frac{3\pi}{2})$
$(6, -\frac{\pi}{2})$

#16b. $(3,-2)$



$r = \sqrt{3^2 + (-2)^2} = \sqrt{13}$

$\tan\theta = \frac{y}{x} = \frac{-2}{3}$

$\theta = \arctan\left(\frac{-2}{3}\right)$

$\theta = -.588$

(correct quadrant – IV)
not add π

other angle is

$\theta = -.588 + 2\pi$

$\theta = 5.695$

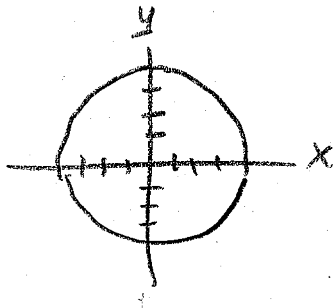
r, θ
$(\sqrt{13}, -0.588)$
$(\sqrt{13}, 5.695)$

Convert the rectangular equation to polar form and sketch its graph.

#17b. $x^2+y^2=16$

$r^2=16$

$\boxed{r=4}$



#18b. $x^2-y^2=4$

$(r\cos\theta)^2-(r\sin\theta)^2=4$

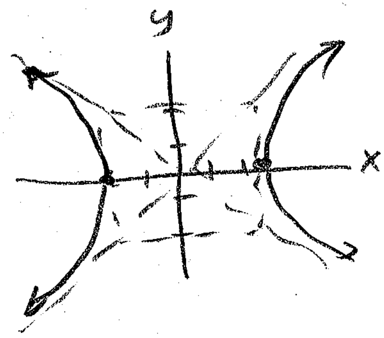
$r^2(\cos^2\theta-\sin^2\theta)=4$

$r^2=\dfrac{4}{\cos^2\theta-\sin^2\theta}$

$\boxed{r=\dfrac{2}{\sqrt{\cos^2\theta-\sin^2\theta}}}$

$\dfrac{x^2}{4}-\dfrac{y^2}{4}=1$

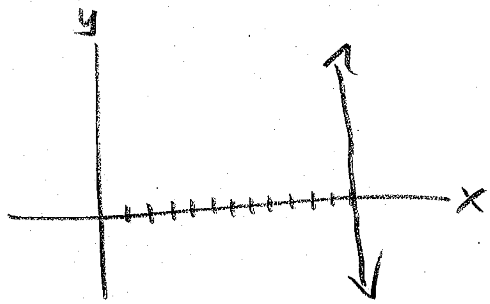
(hyperbola)



#19b. $x=12$

$r\cos\theta=12$

$\boxed{r=\dfrac{12}{\cos\theta}=12\sec\theta}$



#20b. $xy=4$

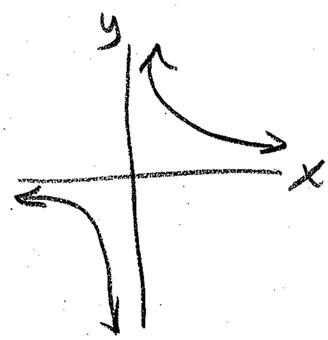
$(r\cos\theta)(r\sin\theta)=4$

$r^2\cos\theta\sin\theta=4$

$r^2=\dfrac{4}{\cos\theta\sin\theta}$

$\boxed{r=\dfrac{2}{\sqrt{\cos\theta\sin\theta}}}$

$y=\dfrac{4}{x}$



#21b. $(x^2+y^2)^2-9(x^2-y^2)=0$

$(r^2)^2-9((r\cos\theta)^2-(r\sin\theta)^2)=0$

$r^4-9r^2(\cos^2\theta-\sin^2\theta)=0$

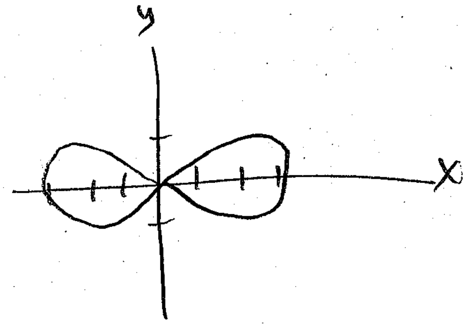
$r^2[r^2-9(\cos^2\theta-\sin^2\theta)]=0$

$\boxed{r=0}$ or $r^2=9(\cos^2\theta-\sin^2\theta)=0$

(origin)

$\boxed{r=\pm 3\sqrt{\cos^2\theta-\sin^2\theta}}$

used a calculator in polar mode for the graph:

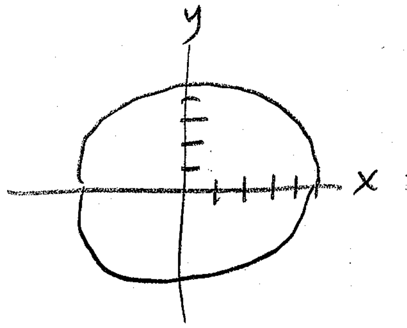


Convert the polar equation to rectangular form and sketch its graph.

#22b. $r = -5$

$r^2 = 25$

$\boxed{x^2 + y^2 = 25}$



#23b. $r = 5\cos(\theta)$

$r^2 = 5r\cos\theta$

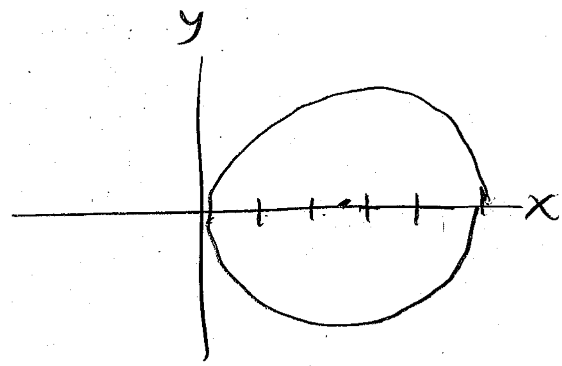
$\boxed{x^2 + y^2 = 5x}$ or $\left(x^2 - 5x + \frac{25}{4}\right) + y^2 = \frac{25}{4}$

$\boxed{\left(x - \frac{5}{2}\right)^2 + y^2 = \frac{25}{4}}$

Circle

center: $\left(\frac{5}{2}, 0\right)$

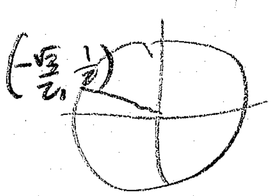
$r = \frac{5}{2}$



#24b. $\theta = \frac{5\pi}{6}$

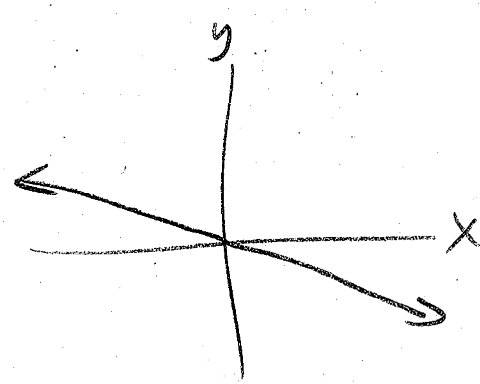
$\tan(\theta) = \tan\left(\frac{5\pi}{6}\right)$

$\frac{y}{x} = \frac{\sin\left(\frac{5\pi}{6}\right)}{\cos\left(\frac{5\pi}{6}\right)} = \frac{1/2}{-\sqrt{3}/2} = -\frac{1}{\sqrt{3}}$



$\frac{y}{x} = -\frac{1}{\sqrt{3}}$

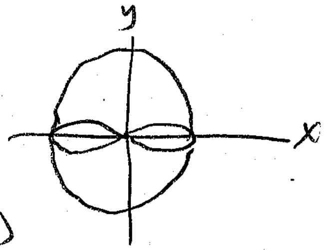
$\boxed{y = \left(-\frac{1}{\sqrt{3}}\right)x}$



Find the points of intersection of the graphs of the equations

#25b) $\begin{cases} r=3(1+\sin(\theta)) \\ r=3(1-\sin(\theta)) \end{cases}$

always graph in calculator first:



$3(1+\sin\theta)=3(1-\sin\theta)$

$1+\sin\theta=1-\sin\theta$

$2\sin\theta=0$

$\sin\theta=0$

$\theta=0, \pi \ldots$

$\underline{\theta=0}$ $r=3(1-\sin\theta)=3$

$x=r\cos\theta=3\cos(0)=3$

$y=r\sin\theta=3\sin(0)=0$

$\boxed{(3,0)}$

$\underline{\theta=\pi}$ $r=3(1-\sin(\pi))=3$

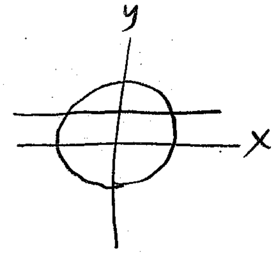
$x=r\cos\theta=3\cos(\pi)=-3$

$y=r\sin\theta=3\sin(\pi)=0$

$\boxed{(-3,0)}$

graph shows also $\boxed{(0,0)}$

#26b. $\begin{cases} r=3+\sin(\theta) \\ r=2\csc(\theta) \end{cases}$



$3+\sin\theta=2\csc\theta=\frac{2}{\sin\theta}$

$3\sin\theta+\sin^2\theta=2$

use calculator in x,y mode graph

$Y1=3\sin\theta+\sin^2\theta$

$Y2=2$ intersection

$\theta=0.59626125$

$\theta=2.5453314$

$\underline{\theta=0.59626125}$ $r=\frac{2}{\sin(.596\ldots)}=3.56155$

$x=r\cos\theta=3.56155\cos(.5962\ldots)=2.94$

$y=r\sin\theta=3.56155\sin(.5962\ldots)=2$

$\boxed{(2.947, 2)}$

$\underline{\theta=2.5453314}$

$r=\frac{2}{\sin(2.5453314)}=3.56155$

$x=r\cos\theta=3.56155\cos(2.545\ldots)=-2.947$

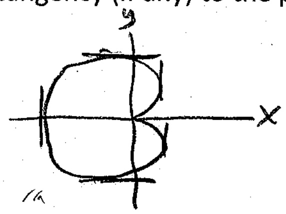
$y=r\sin\theta=3.56155\sin(2.545\ldots)=2$

$\boxed{(-2.947, 2)}$

8.5 – Extra Practice

Find the points of vertical and horizontal tangency (if any) to the polar curve.

#4b. $r=1-\cos(\theta)$



vertical tangents

when $\frac{dx}{d\theta}=0$

$x = r\cos\theta = (1-\cos\theta)\cos\theta = \cos\theta - \cos^2\theta$

$\frac{dx}{d\theta} = -\sin\theta + 2\cos\theta\sin\theta$

$\frac{dx}{d\theta} = -\sin\theta(1-2\cos\theta) = 0$

$\sin\theta = 0 \qquad 1-2\cos\theta = 0$

$\theta = 0, \pi \qquad \cos\theta = \frac{1}{2}$

$\theta = \frac{\pi}{3}, \frac{5\pi}{3}$

$\theta = \frac{\pi}{3}$ $\left(r = 1-\cos\left(\frac{\pi}{3}\right) = 1-\left(\frac{1}{2}\right) = \frac{1}{2}\right)$

$x = \frac{1}{2}\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}\left(\frac{1}{2}\right) = \frac{1}{4}$ $\quad \left(\frac{1}{4}, \frac{\sqrt{3}}{4}\right)$

$y = \frac{1}{2}\sin\left(\frac{\pi}{3}\right) = \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{4}$

$\theta = \frac{5\pi}{3}$ $\left(r = 1-\cos\left(\frac{5\pi}{3}\right) = 1-\left(\frac{1}{2}\right) = \frac{1}{2}\right)$

$x = \frac{1}{2}\cos\left(\frac{5\pi}{3}\right) = \frac{1}{2}\left(\frac{1}{2}\right) = \frac{1}{4}$ $\quad \left(\frac{1}{4}, -\frac{\sqrt{3}}{4}\right)$

$y = \frac{1}{2}\sin\left(\frac{5\pi}{3}\right) = \frac{1}{2}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\sqrt{3}}{4}$

$\theta = 0$ $\left(r = 1-\cos(0) = 1-1 = 0\right)$

origin? (no)

$\theta = \pi$ $\left(r = 1-\cos(\pi) = 1-(-1) = 2\right)$

$x = 2\cos(\pi) = 2(-1) = -2$ $\quad (-2, 0)$

$y = 2\sin(\pi) = 2(0) = 0$

vertical tangents at

$\left(\frac{1}{4}, \frac{\sqrt{3}}{4}\right)\left(\frac{1}{4}, -\frac{\sqrt{3}}{4}\right)(-2, 0)$

x y

horizontal tangents

when $\frac{dy}{d\theta}=0$

$y = r\sin\theta = (1-\cos\theta)\sin\theta = \sin\theta - \sin\theta\cos\theta$

$\frac{dy}{d\theta} = \cos\theta - (\sin\theta(-\sin\theta) + \cos\theta\cos\theta)$

$\frac{dy}{d\theta} = \cos\theta + \sin^2\theta - \cos^2\theta \quad (\sin^2\theta = 1-\cos^2\theta)$

$\frac{dy}{d\theta} = \cos\theta + (1-\cos^2\theta) - \cos^2\theta$

$\frac{dy}{d\theta} = -2\cos^2\theta + \cos\theta + 1 \quad (u = \cos\theta)$

$\frac{dy}{d\theta} = -(2\cos^2\theta - \cos\theta - 1)$

$-(2u^2 - u - 1)$

$-(2u+1)(u-1)$

$\frac{dy}{d\theta} = -(2\cos\theta+1)(\cos\theta-1)$

$2\cos\theta + 1 = 0 \qquad \cos\theta - 1 = 0$

$\cos\theta = -\frac{1}{2} \qquad \cos\theta = 1$

$\theta = \frac{2\pi}{3}, \frac{4\pi}{3} \qquad \theta = 0$

$\theta = \frac{2\pi}{3}$ $\left(r = 1-\cos\left(\frac{2\pi}{3}\right) = 1-\left(-\frac{1}{2}\right) = \frac{3}{2}\right)$

$x = \frac{3}{2}\cos\left(\frac{2\pi}{3}\right) = \frac{3}{2}\left(-\frac{1}{2}\right) = -\frac{3}{4}$ $\quad \left(-\frac{3}{4}, \frac{3\sqrt{3}}{4}\right)$

$y = \frac{3}{2}\sin\left(\frac{2\pi}{3}\right) = \frac{3}{2}\left(\frac{\sqrt{3}}{2}\right) = \frac{3\sqrt{3}}{4}$

$\theta = \frac{4\pi}{3}$ $\left(r = 1-\cos\left(\frac{4\pi}{3}\right) = 1-\left(-\frac{1}{2}\right) = \frac{3}{2}\right)$

$x = \frac{3}{2}\cos\left(\frac{4\pi}{3}\right) = \frac{3}{2}\left(-\frac{1}{2}\right) = -\frac{3}{4}$ $\quad \left(-\frac{3}{4}, -\frac{3\sqrt{3}}{4}\right)$

$y = \frac{3}{2}\sin\left(\frac{4\pi}{3}\right) = \frac{3}{2}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{3\sqrt{3}}{4}$

$\theta = 0$ $\left(r = 1-\cos(0) = 1-1 = 0\right)$

origin? (no)

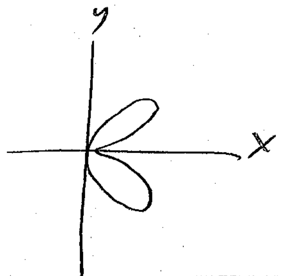
horizontal tangents at

$\left(-\frac{3}{4}, \frac{3\sqrt{3}}{4}\right)\left(-\frac{3}{4}, -\frac{3\sqrt{3}}{4}\right)$

x y

Find the arc length.

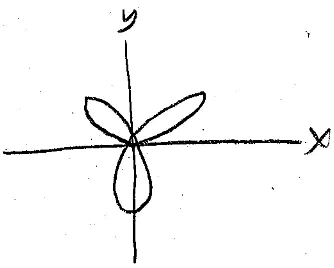
#5b. $r = \sin(3\cos(\theta)), \quad 0 \le \theta \le \pi$



$$\frac{dr}{d\theta} = \cos(3\cos\theta)(-3\sin\theta) = -3\sin(\theta)\cos(3\cos(\theta))$$

$$\boxed{\text{arclength} = \int_0^{\pi} \sqrt{(\sin(3\cos(\theta)))^2 + (-3\sin(\theta)\cos(3\cos(\theta)))^2}\, d\theta = 4.388}$$

#6b. One petal of $r = 5\sin(3\theta)$



$$\frac{dr}{d\theta} = 15\cos(3\theta)$$

$$\boxed{\text{arclength} = \int_0^{\pi/3} \sqrt{(5\sin(3\theta))^2 + (15\cos(3\theta))^2}\, d\theta = 11.137}$$

(limits of integration could also be: $\int_{\pi/3}^{2\pi/3}$ or $\int_{2\pi/3}^{\pi}$)

start/end at origin, $r = 0$

$5\sin(3\theta) = 0$

$\sin(3\theta) = 0$

$3\theta = 0, \pi, 2\pi, 3\pi \ldots$

$\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi \ldots$

suggest one petal $0 \le \theta \le \frac{\pi}{3}$

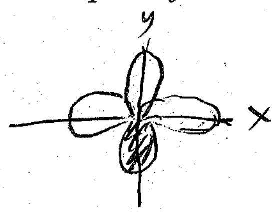
verify w/ calculator graph

✓

8.6 – Extra Practice

Write (and evaluate with calculator) an integral that represents the area of the entire figure.

#5b. *One petal of* $r=\cos(2\theta)$



start/end at $r=0$, $\cos(2\theta)=0$

$2\theta=\frac{\pi}{2},\frac{3\pi}{2},\frac{5\pi}{2},\frac{7\pi}{2}\ldots$

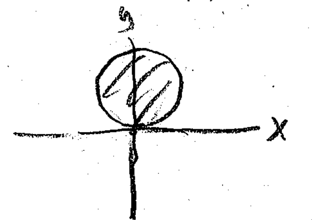
$\theta=\frac{\pi}{4},\frac{3\pi}{4},\frac{5\pi}{4},\frac{7\pi}{4}$

one petal?

verify w/ polar plot window

$$\boxed{A=\frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}(\cos(2\theta))^2\,d\theta=0.393}$$

#6b. *interior of* $r=6\sin(\theta)$



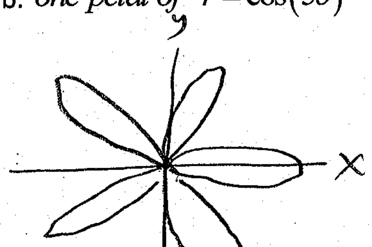
start/end at $r=0$, $6\sin\theta=0$

$\theta=0,\pi,2\pi$

(verify w/ calc. plot)

$$\boxed{A=\frac{1}{2}\int_0^{\pi}(6\sin\theta)^2\,d\theta=28.274}$$

#7b. *one petal of* $r=\cos(5\theta)$



start/end at $r=0$, $\cos(5\theta)=0$

$5\theta=\frac{\pi}{2},\frac{3\pi}{2},\frac{5\pi}{2},\frac{7\pi}{2},\frac{9\pi}{2}\ldots$

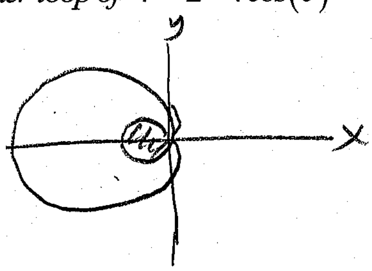
$\theta=\frac{\pi}{10},\frac{3\pi}{10},\frac{5\pi}{10},\frac{7\pi}{10}\ldots$

one petal? (verify) ✓

(may need to lower θstep)

$$\boxed{A=\frac{1}{2}\int_{\frac{\pi}{10}}^{\frac{3\pi}{10}}(\cos(5\theta))^2\,d\theta=0.157}$$

#8b. *inner loop of* $r=2-4\cos(\theta)$



start/end at $r=0$, $2-4\cos\theta=0$

$\cos\theta=\frac{1}{2}$

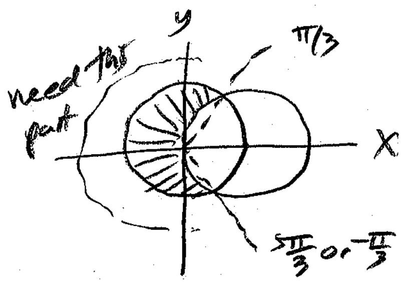
$\theta=\frac{\pi}{3},\frac{5\pi}{3}$ also $-\frac{\pi}{3},\frac{\pi}{3}$

outer loop   inner loop

$$\boxed{A=\frac{1}{2}\int_{-\pi/3}^{\pi/3}(2-4\cos\theta)^2\,d\theta=2.174}$$

Write (and evaluate with calculator) an integral that represents the indicated area.

#9b. *inside* $r=1$ *and outside* $r=2\cos(\theta)$



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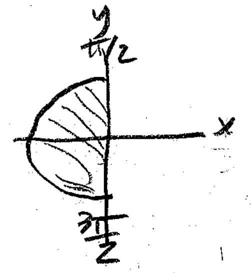

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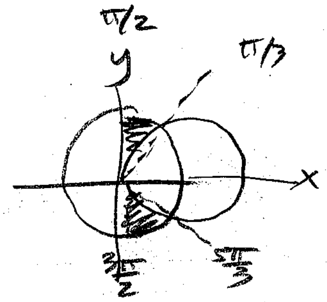


intersections: $2\cos\theta = 1$

$\cos\theta = \frac{1}{2}$

$\theta = -\frac{\pi}{3}, \frac{\pi}{3}, \underbrace{\frac{5\pi}{3}}$

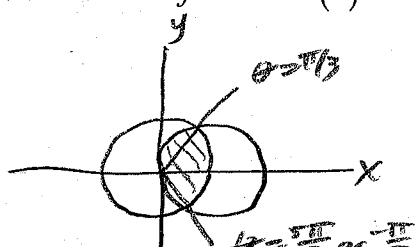
(verify w/calc. plot window)

$$A = \frac{1}{2}\int_{\pi/2}^{3\pi/2} (1)^2\,d\theta + 2\left[\frac{1}{2}\int_{\pi/3}^{\pi/2} (1)^2\,d\theta - \frac{1}{2}\int_{\pi/3}^{\pi/2} (2\cos\theta)^2\,d\theta\right]$$

$$= \frac{\pi}{2} + 2[\,0.2617993878 - 0.0905\,8607$$

$$= \boxed{1.913}$$

#10b. *common interior of* $r=4\cos(\theta)$ *and* $r=2$



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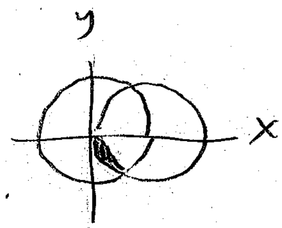

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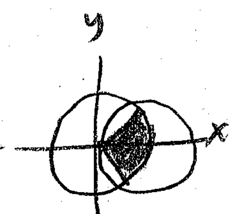

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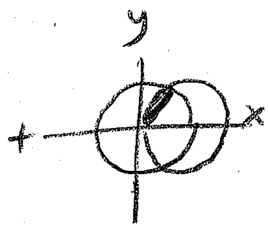

intersections: $4\cos\theta = 2$

$\cos\theta = \frac{1}{2}$

$\theta = -\frac{\pi}{3}, \frac{\pi}{3}, \frac{5\pi}{3}$

(outer radius changes here)

$$\boxed{A = \frac{1}{2}\int_{-\pi/2}^{-\pi/3} (4\cos\theta)^2\,d\theta + \frac{1}{2}\int_{-\pi/3}^{\pi/3} (2)^2\,d\theta + \frac{1}{2}\int_{\pi/3}^{\pi/2} (4\cos\theta)^2\,d\theta}$$

$$= \boxed{4.913}$$