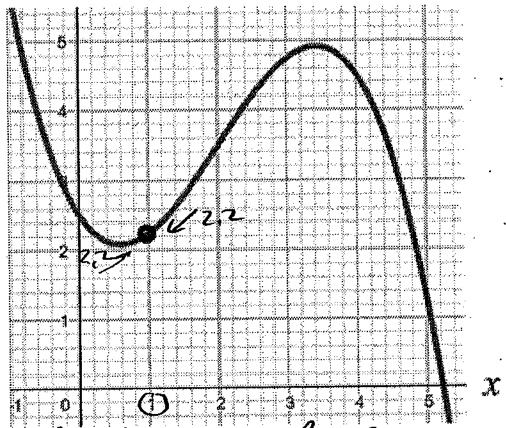
AP Calc BC – Lesson Notes – Unit 1 : Limits and Continuity

Unit 1.1: Finding Limits Graphically and Numerically

The idea of a limit

The idea of a limit: A limit exists if, as you get closer and closer to a specific x-value (but <u>not</u> reaching the x-value), the y-value of the function approaches the same number whether you approach from lower or higher x-values.

$f(x)$



$\lim_{x \to 1^-} f(x) = 2.2$ $\lim_{x \to 1^+} f(x) = 2.2$

Here is a more formal definition of a limit:

$$\lim_{x \to c} f(x) = N$$

Read:
"The limit of f of x as x approaches c equals the number N."

This means:
For all x approximately equal to c, but not equal to c, the value f(x) is approximately equal to N.

The 3 methods for evaluating limits

There are 3 ways to evaluate a limit. Let's evaluate this limit using all 3 methods: $\lim_{x \to 5} \frac{x^2-25}{x-5}$

1) **<u>Numerically</u>**: Plug numbers into the function getting closer and closer to the target x-value. Note: you must plug in numbers from both below and above the number, because they may not result in the same value.

From the left (from lower values):

x =	4	4.5	4.9	4.99	4.999	Seems to be approaching
$\frac{x^2-25}{x-5}$ =	9.0	9.5	9.9	9.99	9.999	$\to 10$

$$\lim_{x \to 5^-} \frac{x^2-25}{x-5} = 10$$

From the right (from higher values):

x =	6	5.5	5.1	5.01	5.001	
$\frac{x^2-25}{x-5}$ =	11.0	10.5	10.1	10.01	10.001	$\to 10$

$$\lim_{x \to 5^+} \frac{x^2-25}{x-5} = 10$$

$$\lim_{x \to 5} \frac{x^2-25}{x-5} = \boxed{10}$$

Work this one in your practice packet...

#1. $\lim_{x \to 4} \frac{x-4}{x^2-5x+4} = \boxed{\frac{1}{3}}$

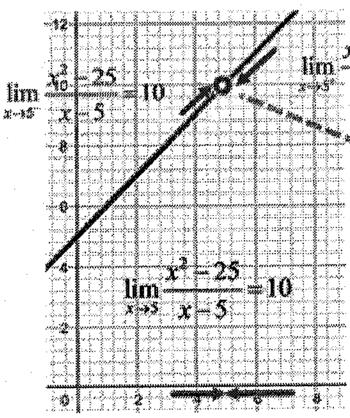
$$\lim_{x \to 4^-} \frac{x-4}{x^2-5x+4} = .33\overline{3} = \frac{1}{3} \qquad \lim_{x \to 4^+} \frac{x-4}{x^2-5x+4} = .33\overline{3} = \frac{1}{3}$$

x	3.9	3.99	3.999	4	4.001	4.01	4.1
f(x)	.34483	.33445	.33344	? (.33333)	.33322	.33223	.32258

The 3 methods for evaluating limits

$$\lim_{x\to 5}\frac{x^2-25}{x-5}$$

2) **Graphically**: If you graph the function, you can visually see what number is being approached from the left and the right:



Note: although graphing technology often won't show it, there is actually a hole in the domain right at x = 5.

But that doesn't affect the limit value. The limit is the value being approached, not the value right at x = 5.

Some limits do not exist

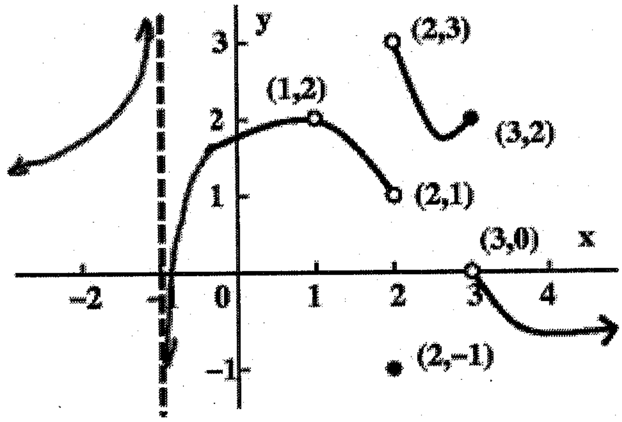
For a given function, it is possible that for every *x* there will be a limit for the function at that *x*. But it is very common for functions to have some *x* values for which there is no limit (the limit does not exist).

Things that cause a limit not to exist at a given x value:

1) If the value being approach from either left or right is not a number (is, for example, infinity because we are approaching a vertical asymptote).

2) If the values being approached from the left and right are both numbers, but they are not the same number.

#2.



$\lim_{x\to 2^-} f(x)=$ 1

$\lim_{x\to 2^+} f(x)=$ 3

$f(2)=$ −1

$\lim_{x\to 2} f(x)=$ Does not exist (DNE)

$\lim_{x\to -1^+} f(x)=$ $-\infty$

$\lim_{x\to -1^-} f(x)=$ ∞

$\lim_{x\to -1} f(x)=$ DNE

The 3 methods for evaluating limits

3) **Analytically**: If the function is well-behaved in the region of the target x, sometimes we can find the value being approached by simply plugging the target value itself into the function:

#3. $\lim_{x\to 3}\frac{x^2-25}{x-5}=\frac{(3)^2-25}{(3)-5}$

$=\frac{9-25}{-2}$

$=\frac{-16}{-2}$

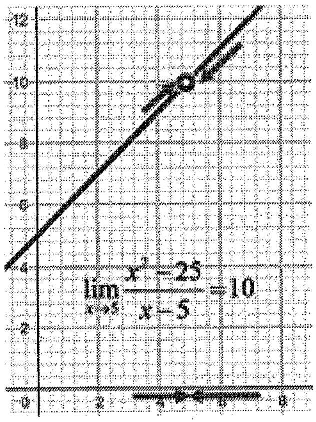
$=\boxed{8}$

The 3 methods for evaluating limits

3) **Analytically:** But sometimes plugging in the target x results in something undefined:

#4. $\lim\limits_{x\to 5}\dfrac{x^2-25}{x-5} \neq \dfrac{0}{0}$ (indeterminate form), but $= \lim\limits_{x\to 5}\dfrac{\cancel{(x-5)}(x+5)}{\cancel{(x-5)}\,1} = \lim\limits_{x\to 5}(x+5) = (5)+5 = \boxed{10}$

If we can't plug in, we could just go back to using numerical (table) or graphing to evaluate. Let's graph this in a calculator. What y-value appears to be approached when x is near 5?



If we just plug 5 in for x...

$$\frac{(5)^2-25}{(5)-5}=\frac{0}{0}$$

...the function is undefined (because x=5 is not in the domain of the function)

But the graph doesn't look like there is a 'problem' at x = 5 (there is no vertical asymptote here). So before we plug in x=5, we can try any algebraic simplification techniques we can think of to find an equivalent algebraic expression for the function. Here, factoring would work:

$$\frac{x^2-25}{x-5}=\frac{(x-5)(x+5)}{(x-5)}=x+5$$

$$\lim_{x\to 5}\frac{x^2-25}{x-5}=\lim_{x\to 5}x+5=(5)+5=10$$

This is the fastest and most convenient way to find a limit, so we usually try this first and only resort to using graphing or numerical methods when needed (or when directed to by the problem). In the next section, we will learn additional algebraic simplifying methods we can use besides factoring before plugging in.

#5. $\lim\limits_{x\to 0}\dfrac{2x}{x^2+4x} \neq \dfrac{0}{0}$

$= \lim\limits_{x\to 0}\dfrac{\cancel{x}(2)}{\cancel{x}(x+4)}$

$= \lim\limits_{x\to 0}\dfrac{2}{x+4}$

$= \dfrac{2}{0+4}$

$= \boxed{\dfrac{1}{2}}$

#6.

$$f(x)=\begin{cases}3x-1 & x<1\\ 2 & x=1\\ 3x & x>1\end{cases}$$

$\lim\limits_{x\to 1}f(x)=$

$\lim\limits_{x\to 1^-}(3x-1)$ $\qquad$ $\lim\limits_{x\to 1^+}3x$

$3(1)-1$ $\qquad$ $3(1)$

$2 \neq 3$

$\therefore \lim\limits_{x\to 1}f(x)\ \boxed{\text{DNE}}$

Sometimes, you can use synthetic division to aid in factoring...

#7. $\lim\limits_{x\to 1}\dfrac{x-1}{x^3-1} \neq \dfrac{0}{0}$

$x^3-1 \div x-1$

$1\,|\ 1\quad 0\quad 0\quad -1$

$\quad\ \ \ \ \ \ 1\quad 1\quad 1$

$\quad\ 1\quad 1\quad 1\ |\ 0$ ✓

x^2+x+1

$= \lim\limits_{x\to 1}\dfrac{(x-1)\,1}{(x-1)(x^2+x+1)}$

$= \lim\limits_{x\to 1}\dfrac{1}{x^2+x+1}$

$= \dfrac{1}{1^2+1+1} = \boxed{\dfrac{1}{3}}$

Unit 1.2: Finding Limits Analytically

Methods for Evaluating Limits Show/Hide Screen Shade

1) **Numerically** (table of values approaching from each side)

2) **Graphically**

3) **Analytically** (plug in target value)
   If result is undefined, can try...
   - Factoring / synthetic division

Properties of Limits

Our textbook defines a number of useful properties about limits. Here is a sample of the most important ones:

$\lim\limits_{x\to c} b = b$ If you are taking a limit of a constant, the limit is the constant for limits at any x.

$\lim\limits_{x\to c} x = c$ You can plug in the x-value to the function to determine the limit (if well-behaved).

$\lim\limits_{x\to c}\left[f(x)\pm g(x)\right]=\lim\limits_{x\to c} f(x)\pm\lim\limits_{x\to c} g(x)$

$\lim\limits_{x\to c}\left[f(x)\cdot g(x)\right]=\lim\limits_{x\to c} f(x)\cdot\lim\limits_{x\to c} g(x)$

When evaluating limits with multiple functions, you can combine the results of the separate limit calculations algebraically. This also applies to algebraic operations on a single limit...compute the limit, and apply the algebraic operations.

$\lim\limits_{x\to c} f(g(x))=f\left(\lim\limits_{x\to c} g(x)\right)$ If you are taking the limit of a composition of functions, you can take the limit of the inner function and then apply the outer function to the result.

example...

$\lim\limits_{x\to 0}\sqrt{\frac{1}{x^2}}=\sqrt{\lim\limits_{x\to 0}\left(\frac{1}{x^2}\right)}$ "The limit of the square root is the square root of its limit."

Here are some other techniques for evaluating analytically...

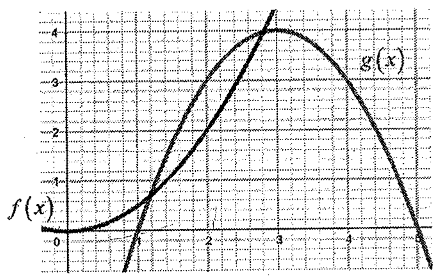
1) **Numerically** (table of values approaching from each side)

2) **Graphically**

3) **Analytically** (plug in target value)
   If result is undefined, can try...
   - Factoring / synthetic division
   - Rationalizing (when you have a radical)
   - Special trig limits

#1. Evaluate the limit

$$\lim_{x\to 2}\frac{2f(x)+1}{4-g(x)}$$



$$=\frac{2(3)+1}{4-(2)}=\boxed{\frac{7}{2}}$$

#2. $\lim\limits_{x\to 9}\frac{x^2-81}{\sqrt{x}-3}$ $\left(\frac{0}{0}\right)$

$$=\lim_{x\to 9}\frac{(x^2-81)(\sqrt{x}+3)}{(\sqrt{x}-3)(\sqrt{x}+3)}$$

$$=\lim_{x\to 9}\frac{(x-9)(x+9)(\sqrt{x}+3)}{x-9}$$

$$=\lim_{x\to 9}(x+9)(\sqrt{x}+3)$$

$$=((9)+9)(\sqrt{9}+3)=18(6)=\boxed{108}$$

#3. $\lim\limits_{x\to 0}\frac{\cos(x)\tan(x)}{x}=$

$$=\lim_{x\to 0}\frac{\cos x}{1}\,\frac{1}{x}\,\frac{\sin x}{\cos x}$$

$$=\lim_{x\to 0}\frac{\cos x}{\cos x}\,\frac{\sin x}{x}$$

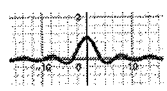
$$=(1)(1)$$

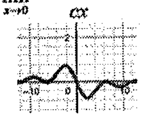
$$=\boxed{1}$$

Evaluating Limits by using the 'special limits'

In some cases, we can rearrange an expression to factor it into one or more 'special limit' form where we have 'pre-evaluated' the limit using a graph...these special limits are:

$$\lim_{x\to 0}\frac{\sin(cx)}{cx}=\lim_{x\to 0}\frac{cx}{\sin(cx)}=1 \qquad \lim_{x\to 0}\frac{\cos(cx)-1}{cx}=0$$





(You need to memorize these special limit values)

Example: Evaluate the limit $\lim\limits_{x\to 0}\frac{\tan(3x)}{\tan(5x)}=\lim\limits_{x\to 0}\frac{\sin(3x)}{1}\,\frac{1}{\cos(3x)}\,\frac{1}{\sin(5x)}\,\frac{\cos(5x)}{1}$

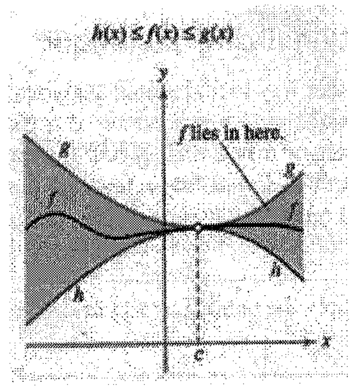
$$=\lim_{x\to 0}\frac{3x}{1}\,\frac{\sin(3x)}{3x}\,\frac{1}{\cos(3x)}\,\frac{5x}{\sin(5x)}\,\frac{1}{5x}\,\frac{\cos(5x)}{1}$$

$$=\lim_{x\to 0}\frac{3x}{5x}\,\frac{\sin(3x)}{3x}\,\frac{1}{\cos(3x)}\,\frac{5x}{\sin(5x)}\,\frac{\cos(5x)}{1}$$

$$=\frac{3}{5}\,(1)\,\frac{1}{(1)}\,(1)\,\frac{1}{1}$$

$$=\boxed{\frac{3}{5}}$$

The Squeeze Theorem



Sometimes it is difficult to evaluate a limit directly, but you can show that the function's value in an interval is always between two other functions whose limits are easier to evaluate. If these other two functions evaluate to the same value, then the given function in the middle is 'squeezed' in between, and must also have the same limit.

Here is the formal theorem:

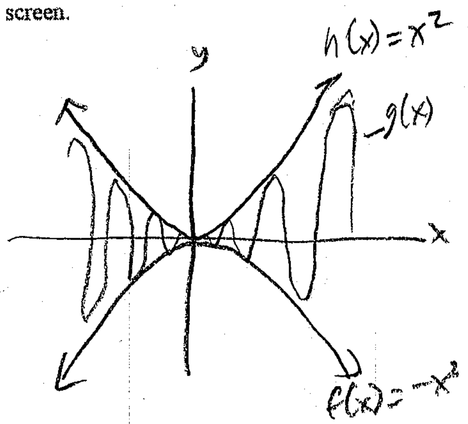
> ***If $h(x) \le f(x) \le g(x)$ for all x in an open interval containing c, except possibly at c itself, and if***
>
> $$\lim_{x\to c} h(x) = L = \lim_{x\to c} g(x)$$
>
> ***then $\lim_{x\to c} g(x)$ exists and is equal to L.***

#4. Use the Squeeze Theorem to show that $\lim_{x\to 0} x^2 \cos 20\pi x = 0$.

Illustrate by graphing the functions

$f(x) = -x^2$, $g(x) = x^2 \cos 20\pi x$, and $h(x) = x^2$

on the same screen.

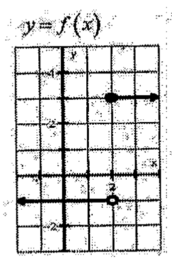


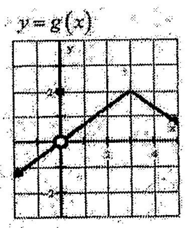
since $f(x) \le g(x) \le h(x)$

and $\lim_{x\to 0} f(x) = \lim_{x\to 0} -x^2 = 0$ and $\lim_{x\to 0} h(x) = \lim_{x\to 0} x^2 = 0$

$\lim_{x\to 0} g(x) = 0$ by the Squeeze Theorem

#5. Evaluate $\lim_{x\to 3} f(g(x))$





$\lim_{x\to 3} f(g(x))$ define: $y = \lim_{x\to 3} g(x) = 2$

hit from lower values

so

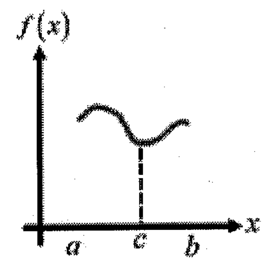
$= \lim_{y\to 2^-} f(y)$

$= \boxed{-1}$

Unit 1.3: Continuity and One-sided Limits

Definition of Continuity of a Function

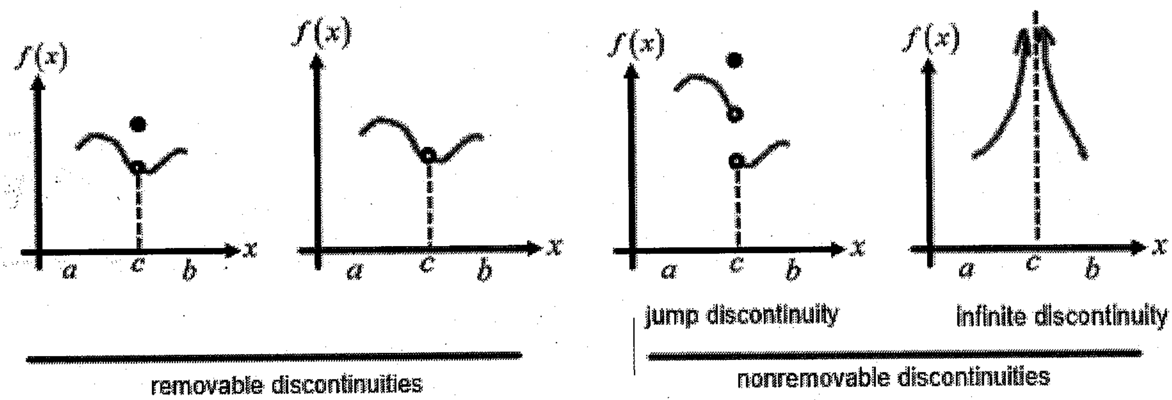
A function can be determined to be **continuous** or **discontinuous** at every x-value in its domain. The general idea of continuity is that a function is continuous at an x-value if, as you are drawing the function from left to right your pencil stays on the paper as you go through this x-value:



This function is said to be **continuous over the interval (a, b)** because, for every value c in the interval, there is no 'break' in the function.

Definition of Continuity of a Function

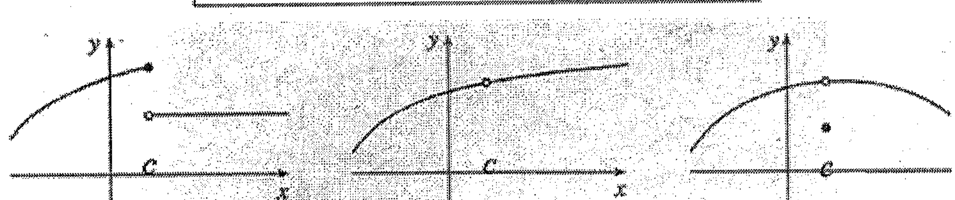
A function can be **discontinuous** at an input value in various ways which are labelled as follows:



A discontinuity is called '**removable**' if the function can be made continuous by appropriately defining or re-defining f at c. (If you could change things so that the 'hole is filled in'.)

Verifying/Proving the Continuity of a Function at $x = c$

Conditions for a Function to Be Continuous at c
To summarize, a function f is continuous at c provided that three conditions are met:
Condition 1 $f(c)$ is defined;
that is, c is in the domain of the function
Condition 2 $\lim_{x \to c} f(x)$ exists
Condition 3 $\lim_{x \to c} f(x) = f(c)$



f is discontinuous at c (violates condition 2)

f is discontinuous at c (violates condition 1)

f is discontinuous at c (violates condition 3)

Verifying/Proving the Continuity of a Function at $x = c$

Determine if the function is continuous at a:

$$f(x)=\begin{cases}\dfrac{x^2-2x-8}{x-4} & \text{if } x\neq 4\\ 3 & \text{if } x=4\end{cases}\qquad a=4$$

Conditions for a Function to Be Continuous at c
To summarize, a function f is continuous at c provided that three conditions are met:
Condition 1 $f(c)$ is defined; that is, c is in the domain of the function
Condition 2 $\lim_{x\to c} f(x)$ exists
Condition 3 $\lim_{x\to c} f(x) = f(c)$

✓1) $f(4)=3$

✓2) $\lim_{x\to 4} f(x)$

$\lim_{x\to 4^-} \frac{x^2-2x-8}{x-4}$

$=\lim_{x\to 4^-} \frac{(x-4)(x+2)}{x-4}$

$=\lim_{x\to 4^-} (x+2) = 4+2 = 6$

$\lim_{x\to 4^+} \frac{x^2-2x-8}{x-4}$

$=\lim_{x\to 4^+} \frac{(x-4)(x+2)}{x-4}$

$=\lim_{x\to 4^+} (x+2 = 4+2 = 6$

$=$

$\therefore \lim_{x\to 4} f(x) = 6$

X3) $f(4) \neq \lim_{x\to 4} f(x)$

$\therefore$ f(x) is discontinuous at $x=4$
(a removable discontinuity)

Determine if the function is continuous at a:

$$f(x)=\begin{cases}\dfrac{x^2-2x-8}{x-4} & \text{if } x\neq 4\\ 6 & \text{if } x=4\end{cases}\qquad a=4$$

✓1) $f(4)=6$

✓2) $\lim_{x\to 4} f(x)$

$\lim_{x\to 4^-} \frac{x^2-2x-8}{x-4}$

$=\lim_{x\to 4^-} \frac{(x-4)(x+2)}{x-4}$

$=\lim_{x\to 4^-} (x+2) = 4+2$
$=6$

$\lim_{x\to 4^+} \frac{x^2-2x-8}{x-4}$

$=\lim_{x\to 4^+} \frac{(x-4)(x+2)}{x-4}$

$=\lim_{x\to 4^+} (x+2) = 4+2$
$=6$

$=$

$\therefore \lim_{x\to 4} f(x) = 6$

✓ 3) $f(4) = \lim_{x\to 4} f(x)$

$\therefore$ f(x) is continuous at $x=4$

#1. Is g(x) continuous at x=2? Is g(x) continuous at x=3?

$$g(x)=\begin{cases}x^2 & x<2\\ -3 & x=2\\ 3x & x>2\end{cases}$$

$x=2$

✓ 1) $g(2)=-3$

X 2) $\lim_{x\to 2} g(x)$

$\lim_{x\to 2^-} x^2 = 2^2 = 4$

$\lim_{x\to 2^+} 3x = 3(2) = 6$

$4 \neq 6$

$\therefore \lim_{x\to 2} g(x)$ DNE

$\therefore$ g(x) is discontinuous at $x=2$
(a jump discontinuity)

$x=3$

✓1) $g(3) = 3(3) = 9$

✓2) $\lim_{x\to 3} g(x)$

$\lim_{x\to 3^-} 3(x) = 3(3) = 9$

$\lim_{x\to 3^+} 3(x) = 3(3) = 9$

$9 = 9$

$\therefore \lim_{x\to 3} g(x) = 9$

✓ 3) $g(3) = \lim_{x\to 3} g(x)$

$\therefore$ g(x) is continuous at $x=3$

Determine if the function is continuous at a:

$$f(x) = \ln|x-2| \qquad a = 2$$

$\times$ 1) $f(2) = \ln|2-2| = \ln(0)$ DNE

$\therefore f(x)$ is discontinuous at $x=2$
(an infinite discontinuity)

Adjusting a function to make it continuous

#2. Find the constant c that makes g continuous on $(-\infty, \infty)$

$$g(x) = \begin{cases} x^2 - c^2 & \text{if } x < 4 \\ cx + 20 & \text{if } x \geq 4 \end{cases}$$

possible discontinuity at $x=4$

make

$x^2 - c^2 = cx + 20$
at $x = 4$

$4^2 - c^2 = c(4) + 20$

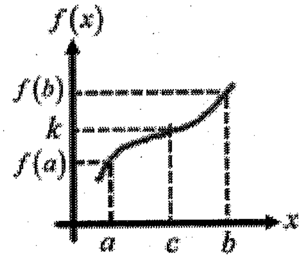
$c^2 + 4c + 4 = 0$

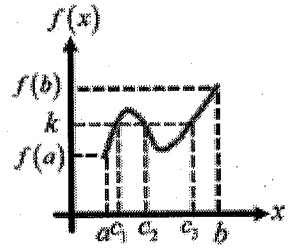
$(c+2)^2 = 0$

$c = -2$

The Intermediate Value Theorem

> *If f is continuous on the closed interval $[a,b]$, $f(a) \neq f(b)$, and k is any number between $f(a)$ and $f(b)$, then there is at least one number c in $[a,b]$ such that $f(c) = k$.*





Note: This theorem doesn't provide a method for finding the value(s) c, and doesn't indicate the number of c values which map to k, it only guarantees the existence of at least one number c such that $f(c) = k$.

#3. Use the Intermediate Value Theorem to show that the following polynomial has a zero in the interval [0,1].

$$f(x) = x^3 + 2x - 1$$

f is continuous over $[0,1]$

$f(0) = 0^3 + 2(0) - 1 = -1$

$f(1) = 1^3 + 2(1) - 1 = 2$

Since $-1 \leq 0 \leq 2$

the Intermediate Value Theorem guarantees c, $0 \leq c \leq 1$, such that $f(c) = 0$ (therefore there must be at least one zero for $f(x)$ in $[0,1]$)

#4. The height of an object changes with time as described by the function

$$h(t) = 2t - 3t^2 + 10$$

Can you guarantee that there is a time value, t in the interval [0,2] at which the height of the object is 4? Explain.

$h(t)$ is continuous over $[0,2]$

$h(0) = 2(0) - 3(0)^2 + 10 = 10$

$h(2) = 2(2) - 3(2)^2 + 10 = 2$

Since $2 \leq 4 \leq 10$

the Intermediate Value Theorem guarantees c, $0 \leq c \leq 2$, such that $h(c) = 4$,

The connection between limits and asymptotes

#5. Find the limit and any asymptotes: $\lim\limits_{x \to 5} \dfrac{x+2}{x-5}$

vertical asymptotes at uncanceled denominator zeros:

$x = 5$

horizontal asymptotes:

$\lim\limits_{x \to -\infty} \dfrac{x+2}{x-5} \left(\dfrac{-\infty}{-\infty}\right)$

$\approx \lim\limits_{x \to -\infty} \dfrac{x}{x}$

$= \lim\limits_{x \to -\infty} 1$

$= 1$

$\lim\limits_{x \to \infty} \dfrac{x+2}{x-5} \left(\dfrac{\infty}{\infty}\right)$

$\approx \lim\limits_{x \to \infty} \dfrac{x}{x}$

$= \lim\limits_{x \to \infty} 1$

$= 1$

$y = 1$ (both sides)

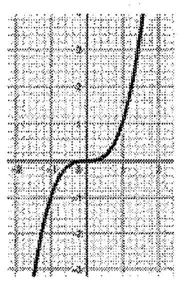
Unit 1.4: Infinite Limits, Limits at Infinity, and Asymptotes

Infinite Limits

An **infinite limit** is a limit that, when evaluated, is increasing without bound to $+\infty$ or decreasing without bound to $-\infty$. Technically, these limits Do Not Exist because they are not numbers, however we usually do indicate whether the value is approaching positive or negative infinity.

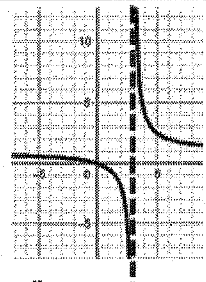
There are three general situations where we see infinite limits:

$x \to \infty$ *or* $x \to -\infty$
In Polynomials



$\lim\limits_{x\to-\infty} x^3 = -\infty \quad \lim\limits_{x\to\infty} x^3 = \infty$

$x \to c$
in some Rational Functions
where denominator goes to zero



$\lim\limits_{x\to 3^-} \dfrac{x}{x-3} = -\infty \quad \lim\limits_{x\to 3^+} \dfrac{x}{x-3} = \infty$

at a Vertical Asymptote
x = 3

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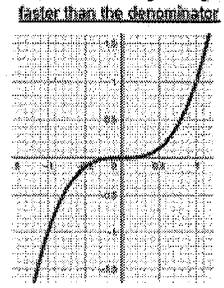
$x \to \infty$ *or* $x \to -\infty$
in Rational Functions
where numerator gets larger faster than the denominator



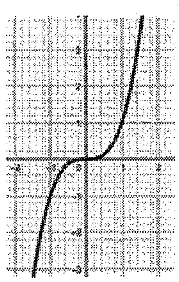
$\lim\limits_{x\to-\infty} \dfrac{x^5}{x^2} = -\infty \quad \lim\limits_{x\to\infty} \dfrac{x^5}{x^2} = \infty$

Limits at Infinity

A **limit at infinity** is a limit where x is approaching either $+\infty$ or $-\infty$.

When evaluate limits at infinity by imagining what will happen as x gets very large (either in the positive or negative direction). The resulting limit value may be zero, a constant, or an infinite limit, depending upon the function:

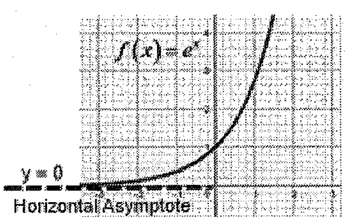
$x \to \infty$ *or* $x \to -\infty$
In Polynomials



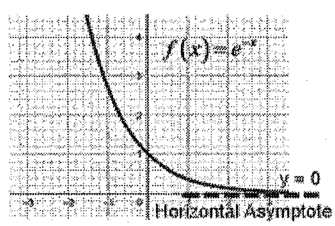
$\lim\limits_{x\to-\infty} x^3 = -\infty \quad \lim\limits_{x\to\infty} x^3 = \infty$

you get an infinite limit

$x \to \infty$ *or* $x \to -\infty$
In Exponentials



$\lim\limits_{x\to-\infty} e^x = 0 \quad \lim\limits_{x\to\infty} e^x = \infty$



$\lim\limits_{x\to-\infty} e^{-x} = \infty \quad \lim\limits_{x\to\infty} e^{-x} = 0$

If, as x approaches either positive or negative infinity the value of the limit approaches zero or any other constant, then there is a **horizontal asymptote** at this y value.

Evaluating limits without graphing

For many limits, you can evaluate without resorting to graphing the function...just consider what happens as x approaches the target value.

$$\lim_{x\to\infty} \frac{x^5}{e^x}$$

The numerator and denominator are both getting very large:

$\dfrac{\infty}{\infty}$ this is an **indeterminant form** (the numerator and denominator are 'fighting for control')

...in this case, the exponential increases faster than the power, so the denominator will eventually be much larger than the numerator:

$$\lim_{x\to\infty} \frac{x^5}{e^x} = 0$$

$$\lim_{x\to-\infty} x^2(x-1)$$

Here, two number which are multiplied are both getting very large, but because x is negative and one is squared but the other is not, we have...

$$\infty(-\infty)$$

...so:

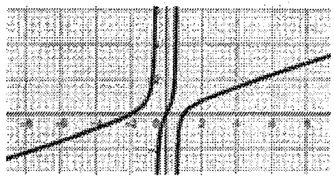
$$\lim_{x\to-\infty} x^2(x-1) = -\infty$$

Evaluating Rational Function limits

If you have a limit of a rational function with polynomials for numerator and denominator, three things can happen:

numerator degree is larger:

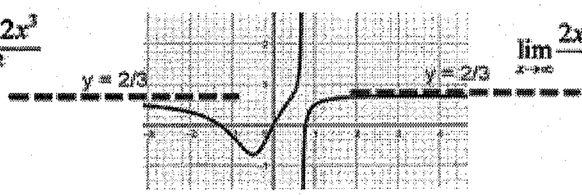
$$\lim_{x\to\infty}\frac{x^5+x^4-2x^3}{3x^4-x^2}$$



$$\lim_{x\to\infty}\frac{x^5+x^4-2x^3}{3x^4-x^2}=\boxed{\infty}$$

numerator and denominator degrees are the same:

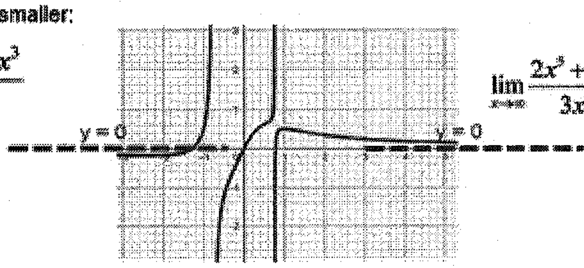
$$\lim_{x\to\infty}\frac{\boxed{2}x^5+x^4-2x^3}{\boxed{3}x^5-x^2}$$



$$\lim_{x\to\infty}\frac{2x^5+x^4-2x^3}{3x^5-x^2}=\boxed{\frac{2}{3}}$$

numerator degree is smaller:

$$\lim_{x\to\infty}\frac{2x^5+x^4-2x^3}{3x^6-x^2}$$



$$\lim_{x\to\infty}\frac{2x^5+x^4-2x^3}{3x^6-x^2}=\boxed{0}$$

Evaluating Rational Function limits without graphing

There is a procedure you can use to evaluate rational function limits without graphing.

- If numerator degree is higher than denominator, then the limit is an infinite limit (be careful of the sign).
- If denominator degree is higher than numerator, then the limit is 0.
- If the degrees of the numerator and denominator are the same, divide every term in both the numerator and denominator by $x^{\text{degree of denominator}}$. Then cancel within each term, and all terms with a constant over a power of x go to zero.

$$\lim_{x\to\infty}\frac{2x^5+x^4-2x^3}{x^5-x^2}$$ divide everything by x^5

$$\lim_{x\to\infty}\frac{\frac{2x^5}{x^5}+\frac{x^4}{x^5}-\frac{2x^3}{x^5}}{\frac{3x^5}{x^5}-\frac{x^2}{x^5}}$$

$$\lim_{x\to\infty}\frac{2+\frac{1}{x}-\frac{2}{x^2}}{3-\frac{1}{x^3}}$$

$$\lim_{x\to\infty}\frac{2+0-0}{3-0}=\boxed{\frac{2}{3}}$$

#1. $\lim_{r\to\infty} \frac{r^4 - r^2 + 1}{r^5 + r^3 - r}$

$= 0$ or

$= \lim_{r\to\infty} \frac{1 - \frac{r^2}{r^4} + \frac{1}{r^4}}{\frac{r^5}{r^4} + \frac{r^3}{r^4} - \frac{r}{r^4}}$

$= \lim_{r\to\infty} \frac{1 - \frac{1}{r^2} + \frac{1}{r^4}}{r + \frac{1}{r} - \frac{1}{r^3}}$

$= \lim_{r\to\infty} \frac{1 - 0 + 0}{r + 0 - 0}$

$= \lim_{r\to\infty} \frac{1}{r} = 0$

#2. $\lim_{x\to-\infty} \frac{3x^3 - x}{x^2 + 2x + 1}$

$\Rightarrow -\infty$ (or DNE)

$\lim_{x\to-\infty} \frac{\frac{3x^3}{x^2} - \frac{x}{x^2}}{1 - \frac{2x}{x^2} + \frac{1}{x^2}}$

$= \lim_{x\to-\infty} \frac{3x - \frac{1}{x}}{1 - \frac{2}{x} + \frac{1}{x^2}}$

$= \lim_{x\to-\infty} \frac{3x - 0}{1 - 0 + 0}$

$= \lim_{x\to-\infty} 3x \to -\infty$

#3. $\lim_{t\to-\infty} \frac{6t^2 + 5t}{(1-t)(2t-3)}$

$= \lim_{t\to-\infty} \frac{6t^2+5t}{-2t^2+5t-3}$

$= -3$ — or —

$= \lim_{t\to-\infty} \frac{\frac{6t^2}{t^2} + \frac{5t}{t^2}}{\frac{-2t^2}{t^2} + \frac{5t}{t^2} - \frac{3}{t^2}}$

$= \lim_{t\to-\infty} \frac{6 + \frac{5}{t}}{-2 + \frac{5}{t} - \frac{3}{t^2}}$

$= \lim_{t\to-\infty} \frac{6+0}{-2+0-3}$

$= \lim_{t\to-\infty} \frac{6}{-2} = -3$

#4. $\lim_{x\to\infty} e^{-x^2}$

$= \lim_{x\to\infty} \frac{1}{e^{x^2}}$

$\to \frac{1}{\infty}$

$= 0$

#5. Find the horizontal and vertical asymptotes of the curve. Check by graphing.

$y = \frac{x^2+4}{x^2-1}$

vertical asymptotes at uncanceled denominator zeros:

$x^2 - 1 = 0$ $\quad x = 1, x = -1$

$(x-1)(x+1) = 0$

horizontal asymptotes:

$\lim_{x\to-\infty} \frac{x^2+4}{x^2-1} = 1$ $\quad \lim_{x\to\infty} \frac{x^2+4}{x^2-1} = 1$

$y = 1$ $\quad y = 1$

(both sides)