AP Calculus BC – Study Guide: Unit 4 – Integral Evaluation

Antiderivative shortcuts…

$\int 0\ dx =$	$\int 0\ dx = C$
$\int c\ dx =$	$\int c\ dx = cx + C$
$\int x^n\ dx =$	$\int x^n\ dx =$
$\int e^x\ dx =$	$\int e^x\ dx = e^x + C$
$\int e^{ax}\ dx =$	$\int e^{ax}\ dx = \frac{e^{ax}}{a} + C$
$\int a^x\ dx =$	$\int a^x\ dx = \frac{a^x}{\ln(a)} + C$
$\int \frac{1}{x}\ dx =$	$\int \frac{1}{x}\ dx = \ln\lvert x\rvert + C$
$\int \sin(x)\ dx =$	$\int \sin(x)\ dx = -\cos(x) + C$
$\int \cos(x)\ dx =$	$\int \cos(x)\ dx = \sin(x) + C$
$\int \sec^2(x)\ dx =$	$\int \sec^2(x)\ dx = \tan(x) + C$
$\int \csc^2(x)\ dx =$	$\int \csc^2(x)\ dx = -\cot(x) + C$
$\int \tan(x)\ dx =$	$\int \tan(x)\ dx = \ln\lvert\sec(x)\rvert + C = -\ln\lvert\cos(x)\rvert + C$
$\int \cot(x)\ dx =$	$\int \cot(x)\ dx = -\ln\lvert\csc(x)\rvert + C = \ln\lvert\sin(x)\rvert + C$
$\int \sec(x)\tan(x)\ dx =$	$\int \sec(x)\tan(x)\ dx = \sec(x) + C$
$\int \csc(x)\cot(x)\ dx =$	$\int \csc(x)\cot(x)\ dx = -\csc(x) + C$
$\int \sec(x)\ dx =$	$\int \sec(x)\ dx = \ln\lvert\sec(x) + \tan(x)\rvert + C$
$\int \csc(x)\ dx =$	$\int \csc(x)\ dx = \ln\lvert\csc(x) - \cot(x)\rvert + C$
$\int \frac{1}{\sqrt{a^2 - u^2}}\ du =$	$\int \frac{1}{\sqrt{a^2 - u^2}}\ du = \sin^{-1}\left(\frac{u}{a}\right) + C$
$\int \frac{1}{a^2 + u^2}\ du =$	$\int \frac{1}{a^2 + u^2}\ du = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$
$\int \frac{1}{\lvert u\rvert\sqrt{u^2 - a^2}}\ du =$	$\int \frac{1}{\lvert u\rvert\sqrt{u^2 - a^2}}\ du = \frac{1}{a}\sec^{-1}\left(\frac{u}{a}\right) + C$

Integral properties/procedures…

$\int c\ f(x)\,dx =$

$\int c\ f(x)\,dx = c\int f(x)\,dx$ (constants can be moved out)

$\int [f(x) \pm g(x)]\,dx =$

$\int [f(x) \pm g(x)]\,dx = \int [f(x)]\,dx \pm \int [g(x)]\,dx$

(can split into separate integrals for each term)

$\int_b^a f(x)\,dx =$

$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

1) u-substitution (integral version of chain rule)

ex: $\int x\cos(x^2)\,dx$

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$\int x\cos(x^2)\,dx \quad u = x^2$

$$\frac{du}{dx} = 2x, \quad du = 2xdx, \quad xdx = \frac{1}{2}du$$

substitute into original integral:

$$\int \cos(u)\frac{1}{2}du = \frac{1}{2}\int \cos(u)\,du = \frac{1}{2}\sin(u) = \frac{1}{2}\sin(x^2) + C$$

2) by parts (integral version of product rule)

ex: $\int x\ln(x)\,dx$

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$\int x\ln(x)\,dx \quad u = \ln(x) \qquad dv = x\,dx$

$$\frac{du}{dx} = \frac{1}{x} \qquad \int dv = \int x\,dx$$

$$du = \frac{1}{x}dx \qquad v = \frac{1}{2}x^2$$

substitute into pattern:

$$uv - \int v\,du = (\ln(x))\left(\frac{1}{2}x^2\right) - \int \frac{1}{2}x^2\frac{1}{x}dx$$

$$= \frac{1}{2}x^2\ln(x) - \frac{1}{2}\int xdx$$

$$= \frac{1}{2}x^2\ln(x) - \frac{1}{4}x^2 + C$$

3) trigonometric integrals

ex: $\int \sin^3 x\cos^3 x\,dx$

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$\int \sin^3 x\cos^3\,dx$ (*split off something to form du*)

$\int \sin^3 x\cos^2 x\cos xdx$

$\int \sin^3 x(1-\sin^2 x)\cos xdx$

$\int (\sin^3 x - \sin^5 x)\cos xdx$

$\int \sin^3 x\cos xdx - \int \sin^5 x\cos xdx$

$u = \sin x, \dfrac{du}{dx} = \cos x, \quad \cos xdx = du$

$\int u^3du - \int u^5du$

$$\frac{1}{4}u^4 - \frac{1}{6}u^6 + C = \frac{1}{4}\sin^4 x - \frac{1}{6}\sin^6 x + C$$

4) <u>trigonometric substitution</u>

ex: $\int \frac{1}{x^3\sqrt{x^2-1}}\,dx$

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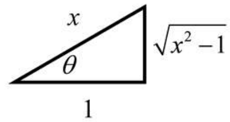

$$\cos\theta = \frac{1}{x} \qquad \tan\theta = \frac{\sqrt{x^2-1}}{1}$$

$$x = \frac{1}{\cos\theta} = \sec\theta \qquad \sqrt{x^2-1} = \tan\theta$$

$$\frac{dx}{d\theta} = \sec\theta\tan\theta \qquad dx = \sec\theta\tan\theta\, d\theta$$

5) <u>partial fraction expansion</u>

ex: $\int \frac{1}{x^2-5x+6}\,dx$

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$$\int \frac{1}{x^2-5x+6}\,dx \qquad \frac{1}{(x-3)(x-2)} = \frac{A}{x-3} + \frac{B}{x-2}$$

$$\int \frac{1}{(x-3)(x-2)}\,dx \qquad A(x-2) + B(x-3) = 1$$

$$Ax - 2A + Bx - 3B = 1$$

$$(A+B)x + (-2A-3B) = (0)x + (1)$$

$$system: \begin{cases} A+B=0 \\ -2A-3B=1 \end{cases} \qquad A=1,\ B=-1$$

$$1\int \frac{1}{x-3}\,dx - 1\int \frac{1}{x-2}\,dx$$

$$\ln|x-3| - \ln|x-2| + C = \ln\left|\frac{x-3}{x-2}\right| + C$$

6) <u>complete the square to arctan form</u>

ex: $\int \frac{1}{x^2-4x+13}\,dx$

6) <u>complete the square to arctan form</u>

$$\int \frac{1}{x^2-4x+13}\,dx \qquad x^2 - 4x + \underline{\underline{4}} + 13 - \underline{\underline{4}}$$

$$(x-2)^2 + 9$$

$$\int \frac{1}{(x-2)^2+9}\,dx \quad now\ u-sub:\ u = x-2,\ \frac{du}{dx} = 1,\ du = dx$$

$$\int \frac{1}{u^2+3^2}\,dx$$

$$\frac{1}{3}\tan^{-1}\left(\frac{u}{3}\right) + C = \frac{1}{3}\tan^{-1}\left(\frac{x-2}{3}\right) + C$$

<u>Improper Integrals</u>:

$$\int_1^\infty \frac{1}{x^2}\,dx =$$

$$\int_1^\infty \frac{1}{x}\,dx =$$

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^2}\,dx = \lim_{b\to\infty}\left[-\frac{1}{b} - \left(-\frac{1}{1}\right)\right] = -\frac{1}{\infty} + 1 = 0 + 1 = 1$$

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty}\Big[\ln|b| - (\ln|1|)\Big] = \infty - 0 = \infty$$

(integral may converge to a number, or diverge)

Integral as inverse operation of derivative:

$$\frac{d}{dx}\left(\int_{2}^{3x^2}\left(t^3-4t\right)dt\right)=$$

$$\frac{d}{dx}\left(\int_{x^5}^{3x^2}\left(t^3-4t\right)dt\right)=$$

Integral as inverse operation of derivative:

$$\frac{d}{dx}\left(\int_{a}^{b(x)}f(t)\,dt\right)=f\left(b(x)\right)\bullet b'(x)\ \left[chain\ rule\right]$$

$$\frac{d}{dx}\left(\int_{2}^{3x^2}\left(t^3-4t\right)dt\right)=\left(\left(3x^2\right)^3-4\left(3x^2\right)\right)\bullet\left(6x\right)$$

$$\frac{d}{dx}\left(\int_{a(x)}^{b(x)}f(t)\,dt\right)=f\left(b(x)\right)\bullet b'(x)-f\left(a(x)\right)\bullet a'(x)$$

$$\frac{d}{dx}\left(\int_{x^5}^{3x^2}\left(t^3-4t\right)dt\right)=\left(\left(3x^2\right)^3-4\left(3x^2\right)\right)\bullet\left(6x\right)-\left(\left(x^5\right)^3-4\left(x^5\right)\right)\bullet\left(5x^4\right)$$