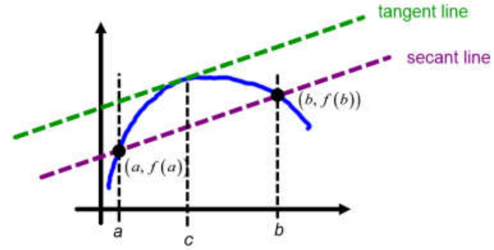
**AP Calculus BC – Study Guide: Unit 3 – Derivative Applications**

**Important Theorems…**

What is the Mean Value Theorem?

Mean Value Theorem

*Let $f$ be continuous on $[a,b]$, and differentiable on $(a,b)$, then there exists a number $c$ in $(a,b)$ such that* $f'(c)=\dfrac{f(b)-f(a)}{b-a}$.




In other words, you can find a mean (average) rate of change across and interval, and there is some input value where the instantaneous rate of change equals the mean rate of change.

(Special case when slope = 0 is called 'Rolle's Theorem')

What do each of these tell us about $f$ ?

$f(x)$ is

$f'(x)$ is

$f''(x)$ is

Critical points occur when…

Inflection points occur when…

Relative (local) max occurs when…

Relative (local) min occurs when…

What do each of these tell us about $f$ ?

$f(x)$ *is the y-value at x*

$f'(x)$ *is the instantaneous rate of change ('slope') at x*

$f'(x)>0$ $f$ *is increasing* 

$f'(x)<0$ $f$ *is decreasing* 

$f''(x)$ *is the concavity ('curvature') at x*

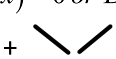
$f''(x)>0$ $f$ *is concave up* 

$f''(x)<0$ $f$ *is concave down* 

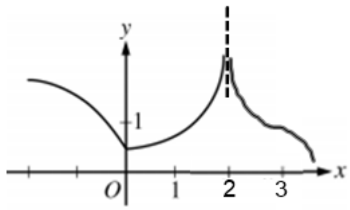
Critical points occur when $f'(x)=0$ *or DNE* and the sign of $f'(x)$ changes.

Inflection points occur when $f''(x)=0$ *or DNE* and the sign of $f''(x)$ changes.

Relative (local) max occurs when $f'(x)=0$ *or DNE* and the sign of $f'(x)$ goes from + to - 

Relative (local) min occurs when $f'(x)=0$ *or DNE* and the sign of $f'(x)$ goes from - to + 

Using a graph of the curve $f$




Graph of $f$

Where is $f$ increasing? decreasing?

Where is $f$ concave up? concave down?

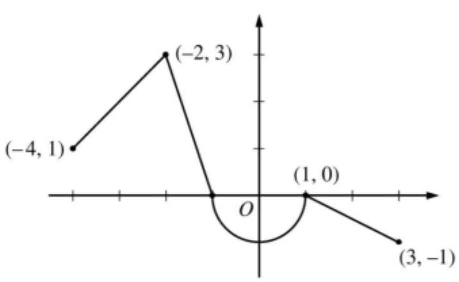
Where is $f$ continuous?

Where is $f$ differentiable?

Where are the following for $f$ ?

- critical points
- relative maxima
-relative minima
- inflection points

What are the absolute max/min over [-2,1]?

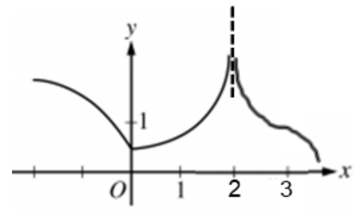



Graph of $f$

$$\int_{-4}^{3} f(x)\,dx =$$

$$\int_{3}^{-4} f(x)\,dx =$$

Using a graph of the curve $f$




Graph of $f$

$f$ *is increasing over* $(0,2)$ $[f \text{ going up}]$

*decreasing over* $(-2,0)\cup(2,4)$ $[f \text{ going down}]$

$f$ *is concave up over* $(0,2)\cup(2,3)$

*concave down over* $(-2,0)\cup(3,4)$

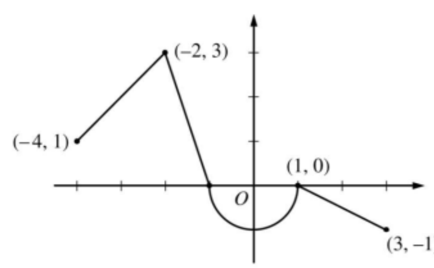
$f$ *is continuous over* $(-2,2)\cup(2,4)$

$f$ *is differentiable over* $(-2,0)\cup(0,2)\cup(2,4)$

Where are the following for $f$ ?

critical points at (-2,2), (0,0.5), (2.5,1)
no relative maxima
relative minima at (0,0.5)
inflection points at (0,0.5), (3,1)

What are the absolute max/min over [-2,1]?
Absolute min at (0,0.5), absolute max at (-2,2)

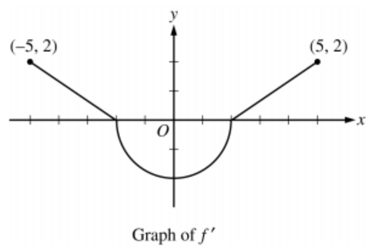



Graph of $f$

$$\int_{-4}^{3} f(x)\,dx = areas = 4+2-\frac{\pi}{2}-1 = 5-\frac{\pi}{2}$$

$$\int_{3}^{-4} f(x)\,dx = -\int_{-4}^{3} f(x)\,dx = -5+\frac{\pi}{2}$$

Using a graph of the derivative $f'$




Where is $f$ increasing? decreasing?

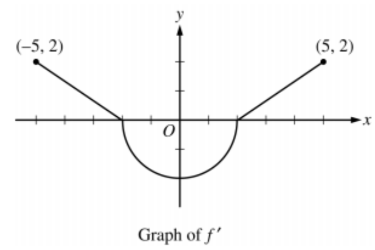
Where is $f$ concave up? concave down?

Where are the following for $f$ ?

- critical points
- relative maxima
-relative minima
- inflection points

$If\ f(2)=1,\ then\ f(-5)=$

Using a graph of the derivative $f'$




$f$ *is increasing over* $(-5,-2)\cup(2,5)$ $[f'>0]$
*decreasing over* $(-2,2)$ $[f'<0]$

$f$ *is concave up over* $(0,5)$ $[f'\ going\ up]$
*concave down over* $(-5,0)$ $[f'\ going\ down]$

Where are the following for $f$ ?

critical points at x=-2, x=2 [f' = 0]
relative minimum at x = 2 [f' from – to +]
relative maximum at x = -2 [f' from + to -]
inflection point at x =0 [f' graph changing direction]

*We can use the Net Change Theorem*
(*part of the Fundamental Theorem of Calculus*):

$$\int_a^b f(x)\,dx = F(b)-F(a)$$

evaluate definite integral by plugging limits into antiderivative
*This also means an integral of a derivative of something is equal to the accumulation* (*net change*) *in the value this is a derivative of* :

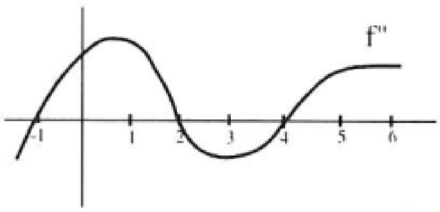
$$\int_a^b f'(x)\,dx = f(b)-f(a)$$

Pick one limit to be what you have and the other what you need :

$$\int_{-5}^{2} f'(x)\,dx = f(2)-f(-5) \text{ and evaluate integral using areas}$$

$$3-\frac{1}{2}\pi(2)^2 = 1-f(-5), \qquad f(-5)=1-3+2\pi=2\pi-2$$

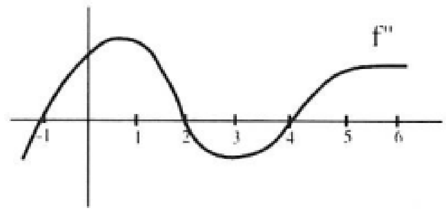
Using a graph of the concavity $f''$




Where is $f$ concave up? concave down?

Where inflection points for $f$ ?

Using a graph of the concavity $f''$




$f$ *is concave up over* $(-1,2)\cup(4,6)$ $[f''>0]$

*concave down over* $(-2,-1)\cup(2,4)$ $[f''<0]$

Where inflection points for $f$ ?

at x =-1, x=2, x=4 [f'' = 0 and sign is changing]

**Tangent lines…**

*For* $(x-2)^2+(y+3)^2=4$

$(a)$ *Write the equation of the tangent line at* $\left(1,-3+\sqrt{3}\right)$

$(b)$ *Where does this curve have horizontal tangents?*

$(c)$ *Where does this curve have vertical tangents?*

*For* $(x-2)^2+(y+3)^2=4$

$(a)$ *Write the equation of the tangent line at* $\left(1,-3+\sqrt{3}\right)$

$m=\frac{dy}{dx}$ [*use implicit differentiation if needed*]:

$$2(x-2)(1)+2(y+3)\left(\frac{dy}{dx}\right)=0,\quad \frac{dy}{dx}=\frac{-x+2}{y+3}=\frac{-(1)+2}{\left(-3+\sqrt{3}\right)+3}=\frac{1}{\sqrt{3}}$$

$$\left(y-\left(-3+\sqrt{3}\right)\right)=\frac{1}{\sqrt{3}}(x-1)$$

$(b)$ *Where does this curve have horizontal tangents?*

*where* $\frac{dy}{dx}=0$ (*numerator* $=0$), $-x+2=0$, *at* $x=2$ (2 *points*)

$(c)$ *Where does this curve have vertical tangents?*

*where* $\frac{dy}{dx}=DNE$ (*denominator* $=0$), $y+3=0$, *at* $y=-3$ (2 *points*)

**Position, Velocity (speed), Acceleration…**

<u>In 1D</u>:

An object moves in one direction with position x given by $x(t) = t^3 - 4t^2 + 3$.

(a) Find velocity as function of time.
(b) What acceleration as a function of time.
(c) What is the position of the particle at *t* = 2?
(d) What is the speed of the particle at *t* = 2?

<u>In 1D</u>:

$x(t) = t^3 - 4t^2 + 3$

$(a)\quad v(t) = x'(t) = 3t^2 - 8t$

$(b)\quad a(t) = v'(t) = 6t - 8$

$(c)\quad x(2) = (2)^3 - 4(2)^2 + 3 = -5 \quad (\textit{include units if given in problem})$

$(d)\quad speed = |v(2)| = |3(2)^2 - 8(2)| = |-4| = 4$

An object is launched upward with an initial velocity of $30\ m/s$ from an initial height of $10\ m$ in gravity field with $a(t) = -9.8\, m/s^2$.

(a) Find velocity as a function of time.
(b) Find height as a function of time.
(c) At what time does the object reach maximum height and what is the max height?
(d) At what time does the object hit the ground?

$a(t) = -9.8$

$(a)\quad v(t) = \int a(t)\,dt = \int(-9.8)\,dt = -9.8t + C_1$

$v(0) = 30,\ so\ 30 = -9.8(0) + C_1,\ C_1 = 30$

$v(t) = -9.8t + 30$

$(b)\quad x(t) = \int v(t)\,dt = \int(-9.8t + 30)\,dt = -4.9t^2 + 30t + C_2$

$x(0) = 10,\ so\ 10 = -4.9(0)^2 + 30(0) + C_2,\ C_2 = 10$

$x(t) = -4.9t^2 + 30t + 10$

$(c)\quad Max\ height\ when\ v = 0:\ -9.8t + 30 = 0,\ t = 3.06122\text{ sec}$

$x(3.06122) = 55.91837\ m$

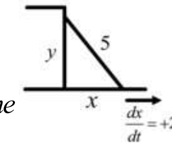
$(d)\quad On\ ground\ when\ x = 0:\ -4.9t^2 + 30t + 10 = 0$

$$at\ t = \frac{-30 \pm \sqrt{(30)^2 - 4(-4.9)(10)}}{2(-4.9)} = \cancel{-0.3169},\ 6.439\text{ sec}$$

**Related Rates Problems…**

A 5-foot long ladder is leaning against a building. If the foot of the ladder is sliding away from the building at a rate of 2 ft/sec, how fast is the top of the ladder moving and in what direction when the foot of the ladder is 4 feet from the building?

*Draw a picture and assign variables to things which vary then find equations which relate the variables:*




$$x^2 + y^2 = 5^2$$

*Anything changing is a derivative with respect to time $(+\text{ if value is increasing})$*

$$\frac{dx}{dt} = +2$$

*At this snapshot in time, variables have 'snaphot' values:*

$$(4)^2 + y^2 = 5^2, \quad y = 3$$

*Differentiate implicitly WRT time, plug in values, and solve:*

$$x^2 + y^2 = 25$$

$$\frac{d}{dt}\left[x^2\right] + \frac{d}{dt}\left[y^2\right] = \frac{d}{dt}[25]$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$2(4)(2) + 2(4)\frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = -\frac{16}{8} = -2 \ ft/\sec$$

*negative b/c top of ladder is moving so y is decreasing (downward)*

**Optimization Problems…**

A cylindrical can (with circular base) is made with a material for the lateral side which costs \$3/cm$^2$, and a material for the top and bottom circular sides which costs \$5/cm$^2$. If the can must enclose a volume of $20\pi\, cm^3$ what should the radius and height be to minimize the material cost?

*Need functions for the objective function (what is being optimized) and any constraints.*

| Objective Function | Constraint |
| --- | --- |
| $Cost, C = \left(A_{lateral}\right)\left(\frac{\$3}{cm^3}\right) + \left(A_{top/bottom}\right)\left(\frac{\$5}{cm^3}\right)$ | $V = 20\pi\ cm^3$ |
| $C = (2\pi rh)(3) + (2)(\pi r^2)(5)$ | $\pi r^2 h = 20\pi$ |

$C = 6\pi rh + 10\pi r^2$ *cost in terms of r and h*

*Now solve constraint for one variables, substitute into objective function:*

$$h = \frac{20\pi}{\pi r^2} = \frac{20}{r^2} \qquad so \qquad C = 6\pi r\left(\frac{20}{r^2}\right) + 10\pi r^2 = 120\pi r^{-1} + 10\pi r^2$$

*Now find* min *by taking derivative and finding where $C'(r) = 0$*

$$C'(r) = -120\pi r^{-2} + 20\pi r = 0$$

$$20\pi r = \frac{120\pi}{r^2}, \quad r^3 = \frac{120}{20} = 6, \quad r = \sqrt[3]{6} = (6)^{\frac{1}{3}} = 1.81712\ cm$$

*Use constraint equation to find other dimension:*

$$h = \frac{20}{r^2} = \frac{20}{(1.81712)^2} = 6.057\ cm$$

*Should use* 2nd*-derivative to verify this is a* min *not a* max:

$C''(r) = 240\pi r^{-3} + 20\pi$ *is + for + r, so concave up, so this is a* min.