AP Calculus BC – Study Guide: Unit 5 – Integral Applications

Use Fundamental Theorem of Calculus PT1 to evaluate the definite integral:

$$\int_1^2 x^2\,dx =$$

Use Fundamental Theorem of Calculus PT1 to evaluate the definite integral:

$$\int_1^2 x^2\,dx = \left[\frac{1}{3}x^3\right]_1^2 = \left(\frac{1}{3}(2)^3\right) - \left(\frac{1}{3}(1)^3\right) = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$$

Using the Fundamental Theorem of Calculus PT2 (net change theorem): The rate of change of the altitude of a hot-air balloon is given by $r(t) = t^3 - 4t^2 + 6 \quad (0 \le t \le 8)$. Find the change in altitude of the balloon during the time when the altitude is decreasing.

First graph $r(t)$ *in calculator and find that this rate is negative for* $1.572 < t < 3.514$

Then, since $r(t)$ *is the derivative of altitude:*

$$\int_{1.572}^{3.514} a'(t)\,dt = \int_{1.572}^{3.514} \left(t^3 - 4t^2 + 6\right)dt = a(3.514) - a(1.572)$$

is the change in altitude $= -4.431$ (*Math* 9)

Using the Fundamental Theorem of Calculus PT2 to find a y-value from another given derivative: If $f'(x) = x^2 - 5x,\ and\ f(1) = 2$ find $f(4)$.

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

$$\int_1^4 \left(3x^2 - \frac{5}{2}x\right)dx = f(4) - f(1)$$

$$\left[x^3 - 5x^2\right]_1^4 = f(4) - 2$$

$$\left((4)^3 - 5(4)^2\right) - \left((1)^3 - 5(1)^2\right) = f(4) - 2$$

$$-12 = f(4) - 2, \quad f(4) = -10$$

Integral as inverse operation of derivative:

$$\frac{d}{dx}\left(\int_2^{3x^2} \left(t^3 - 4t\right)dt\right) =$$

$$\frac{d}{dx}\left(\int_{x^5}^{3x^2} \left(t^3 - 4t\right)dt\right) =$$

Integral as inverse operation of derivative:

$$\frac{d}{dx}\left(\int_a^{b(x)} f(t)\,dt\right) = f(b(x)) \bullet b'(x) \ [\textit{chain rule}]$$

$$\frac{d}{dx}\left(\int_2^{3x^2} \left(t^3 - 4t\right)dt\right) = \left(\left(3x^2\right)^3 - 4\left(3x^2\right)\right) \bullet (6x)$$

$$\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(t)\,dt\right) = f(b(x)) \bullet b'(x) - f(a(x)) \bullet a'(x)$$

$$\frac{d}{dx}\left(\int_{x^5}^{3x^2} \left(t^3 - 4t\right)dt\right) = \left(\left(3x^2\right)^3 - 4\left(3x^2\right)\right) \bullet (6x) - \left(\left(x^5\right)^3 - 4\left(x^5\right)\right) \bullet \left(5x^4\right)$$

Average value of a function:

If $f(x) = x^2 - 5x$ find the average value of $f(x)\ over\ [2,6]$

Average value of a function:

$$\textit{average value of } f(x) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$\frac{1}{6-2}\int_2^6 \left(x^2 - 5x\right)dx - \frac{1}{4}\left[\frac{1}{3}x^3 - \frac{5}{2}x^2\right]_2^6 = \frac{8}{3}$$

NOTE: This is different than 'average rate of change of $f(x)$*'*

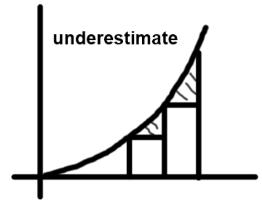
which would instead be: $\dfrac{f(6) - f(2)}{6-2}$

Riemann Sums (approximation of definite integral):

Use a left-endpoint Riemann Sum with two subintervals of equal length to approximate $\int_2^{2.4} x^2\,dx$

Does this estimate under- or over-estimate the value?

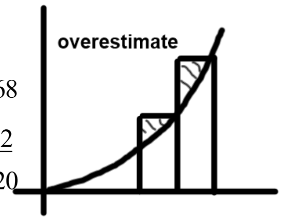
interval	x_i	$f(x_i)\cdot\Delta x = area$
$[2,2.2]$	2	$(2)^2\cdot 0.2 = 0.8$
$[2.2,2.4]$	2.2	$(2.2)^2\cdot 0.2 = \underline{0.968}$
		1.7744



Use a right-endpoint Riemann Sum with two subintervals of equal length to approximate $\int_2^{2.4} x^2\,dx$

Does this estimate under- or over-estimate the value?

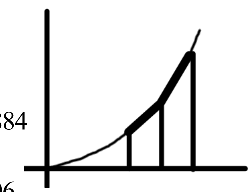
interval	x_i	$f(x_i)\cdot\Delta x = area$
$[2,2.2]$	2.2	$(2.2)^2\cdot 0.2 = 0.968$
$[2.2,2.4]$	2.4	$(2.4)^2\cdot 0.2 = \underline{1.152}$
		2.120



Use the trapezoidal rule with two subintervals of equal length to approximate $\int_2^{2.4} x^2\,dx$

$$A_{trapezoid} = \frac{1}{2}\left(f\left(x_{left}\right) + f\left(x_{right}\right)\right)\cdot\Delta x$$

interval	$A_{trapezoid}$
$[2,2.2]$	$\frac{1}{2}\left((2)^2 + (2.2)^2\right)\cdot 0.2 = 0.884$
$[2.2,2.4]$	$\frac{1}{2}\left((2.2)^2 + (2.4)^2\right)\cdot 0.2 = \underline{1.06}$
	1.944



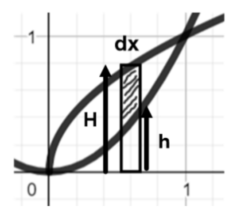
Area between curves (rectangular):

Find the area enclosed by $f(x) = x^2$ and $g(x) = \sqrt{x}$.

$$A = \int_a^b H\,dx - \int_a^b h\,dx$$

$$= \int_0^1 \sqrt{x}\,dx - \int_0^1 x^2\,dx$$

$$= \int_0^1 \left(\sqrt{x} - x^2\right)dx = 0.333$$

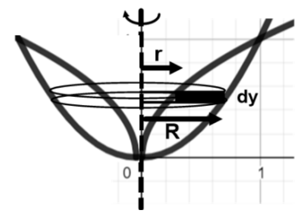


Volumes:

Find the volume formed by rotating the area enclosed by $f(x)=x^2$ and $g(x)=\sqrt{x}$ around y-axis (disc method).

disc method ('perpendiscular')

(rectangle ⊥ rotation axis)

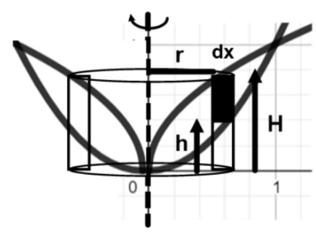


$$V=\int_a^b \pi R^2\,dh-\int_a^b \pi r^2\,dh$$
$$=\int_a^b \pi R^2\,dy-\int_a^b \pi r^2\,dy$$
$$=\int_0^1 \pi\left(\sqrt{y}\right)^2 dy-\int_0^1 \pi\left(y^2\right)^2 dy$$
$$=\pi\int_0^1\left(y-y^4\right)dy=0.942$$

Find the volume formed by rotating the area enclosed by $f(x)=x^2$ and $g(x)=\sqrt{x}$ around y-axis (shell method).

shell method ('parashell')

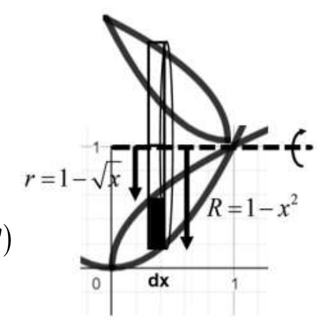
(rectangle || rotation axis)



$$y=x^2\rightarrow x=\sqrt{y}$$
$$y=\sqrt{x}\rightarrow x=y^2$$

$$V=\int_a^b 2\pi rHdr-\int_a^b 2\pi rhdr$$
$$=\int_a^b 2\pi rHdx-\int_a^b 2\pi rhdx$$
$$=\int_0^1 2\pi x\left(\sqrt{x}\right)dx-\int_0^1 2\pi x\left(x^2\right)dx$$
$$=2\pi\int_0^1\left(x\sqrt{x}-x^3\right)dx=0.942$$

Find the volume formed by rotating the area enclosed by $f(x)=x^2$ and $g(x)=\sqrt{x}$ around the line $y=1$.

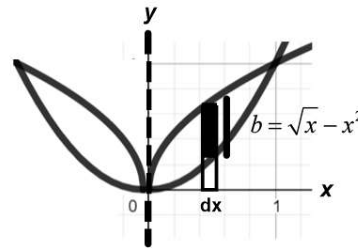


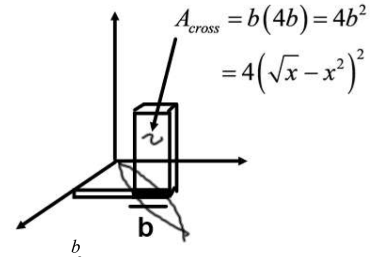
disc method ('perpendiscular')

(rectangle ⊥ rotation axis)

$$V=\int_a^b \pi R^2\,dh-\int_a^b \pi r^2\,dh$$
$$=\int_a^b \pi\left(1-x^2\right)^2 dx-\int_a^b \pi\left(1-\sqrt{x}\right)^2 dx$$
$$=\pi\int_0^1\left(\left(1-x^2\right)^2-\left(1-\sqrt{x}\right)^2\right)dx=1.152$$

The region R enclosed by $f(x) = x^2$ and $g(x) = \sqrt{x}$ forms the base of a solid. For this solid, each cross section Perpendicular to the *x*-axis is a rectangle whose Height is 4 times the length of its base in region *R*. Find the volume of this solid.





$$V = \int_a^b A_{cross}\, dx$$

$$= \int_0^1 4\left(\sqrt{x} - x^2\right)^2\, dx = 0.514$$

<u>Arclength (rectangular)</u>:

If $f(x) = \frac{x^3}{6} + \frac{1}{2x}$ find the length of this curve for $\frac{1}{2} \le x \le 1$.

$$arclength = \int_a^b \sqrt{1 + \left[f'(x)\right]^2}\, dx$$

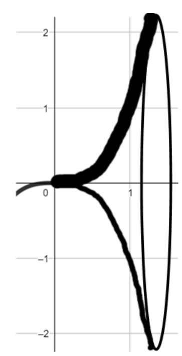
$$f(x) = \frac{1}{6}x^3 + \frac{1}{2}x^{-1}, \quad f'(x) = \frac{1}{2}x^2 - \frac{1}{2}x^{-2}$$

$$= \int_{1/2}^1 \sqrt{1 + \left[\frac{1}{2}x^2 - \frac{1}{2}x^{-2}\right]^2}\, dx = 0.646$$

<u>Surface area of surface of revolution</u>:

Find the area of the surface obtained by rotating The curve $y = x^3,\ 0 \le x \le 2$ about the *x*-axis.

<u>Surface area of surface of revolution</u>:



$$arclength = \int_a^b \sqrt{1 + \left[f'(x)\right]^2}\, dx$$

concept is : if you rotate each 'piece' of arc around the axis, this piece forms a 'strip' with surface area $= 2\pi(d\text{'arclength'})$, *so* :

$$surface\ area = \int_a^b 2\pi r \sqrt{1 + \left[f'(x)\right]^2}\, dx \ \ and\ r = f(x)$$

$$f(x) = x^3, \quad f'(x) = 3x^2$$

$$surface\ area = \int_0^2 2\pi x^3 \sqrt{1 + \left[3x^2\right]^2}\, dx = 203.044$$