AP Calculus BC – Study Guide: Unit 6 – Differential Equations

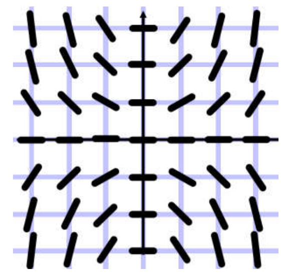
Slope fields:

Sketch a slope field for $\frac{dy}{dx}=\frac{1}{2}xy$

Slope fields:

plug various (x,y) into

$\frac{dy}{dx}$ to get slopes at pts



Which of the following could be a specific solution to the Differential equation with the given slope field:

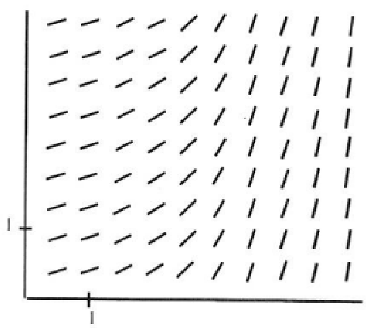
(A) $y=e^{-x}$

(B) $y=\sin x$

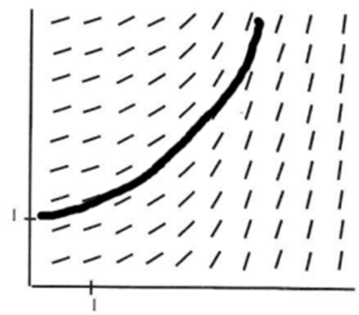
(C) $y=\sqrt{x}$

(D) $y=\ln x$

(E) $y=e^{0.5x}$



Solution curves follow the direction in slope field (E) [this is an exponential curve shape]



Solving separable differential equations:

Find the particular solution for $\frac{dy}{dx}=3x^2y,\quad y(2)=1$.

Solving separable differential equations:

$$\textit{Separate the variables: } \frac{1}{y}dy=3x^2dx$$

$$\textit{integrate both sides: } \int\frac{1}{y}dy=\int 3x^2dx$$

$$(\textit{general, implicit solution}): \ln|y|=x^3+C$$

$$\textit{plug in initial condition: } \ln(1)=(2)^3+C$$

$$\textit{solve for } C: \quad 0=8+C,\quad C=-8$$

$$\textit{write the particular, implicit solution}:$$

$$\ln|y|=x^3-8$$

$$\textit{if needed, solve for } y\ (\textit{explicit solution}):$$

$$e^{\ln|y|}=e^{(x^3-8)},\quad y=e^{(x^3-8)}=e^{-8}e^{x^3}$$

Euler's method:

$f(x)$ is the solution to the differential equation $\frac{dy}{dx}=x^2y,\quad y(1)=2$. Use Euler's method with a step size of 0.1 to approximate $f(1.3)$.

(h can be negative)

Euler's method:

(x,y)	$y_{n+1}=y_n+h\left(\frac{dy}{dx}\right)$
$(1,2)$	$y=2+(0.1)\left((1)^2(2)\right)=2.2$
$(1.1,2.2)$	$y=2.2+(0.1)\left((1.1)^2(2.2)\right)=2.4662$
$(1.2,2.4662)$	$y=2.4662+(0.1)\left((1.2)^2(2.4662)\right)=2.8213$
$(1.3,2.8213)$	$f(1.3)\approx 2.8213$

Differential Equation Models:

Write the differential equation and solution equations:

Unrestricted population growth:

Radioactive Decay:

Logistic Model Growth:

Unrestricted population growth example:
A rabbit population with an initial size of 500 grows at A rate proportional to its size. If there are1200 rabbits At t = 10 days, when was the rabbit population 900?

Differential Equation Models:

Unrestricted population growth:

$$DE:\ \frac{dP}{dt}=kP \qquad solution:\ P=P_0e^{kt}$$

Continuously compounded interest is same form:

$$\frac{dA}{dt}=kA \qquad A=Pe^{rt}$$

Radioactive Decay:

$$DE:\ \frac{dQ}{dt}=kQ \qquad solution:\ Q=Q_0e^{kt}$$

$$(halflife = time\ for\ quantity\ to\ reduce\ by\ half)$$

Logistic Model Growth:

growth limited by environment

maximum population = carrying capacity, L

$$DE:\ \frac{dP}{dt}=kP\left(1-\frac{P}{L}\right) \qquad solution:\ P=\frac{L}{1+Ce^{-kt}}$$

$$\left(population\ grows\ fastest\ when\ P=\frac{1}{2}L\right)$$

$$\frac{dP}{dt}=kP,\ solution:\ P=P_0e^{kt}$$

$$P=500e^{kt},\ use\ P(10)=1200:$$

$$1200=500e^{k(10)},\ e^{10k}=\frac{1200}{500},\ 10k=\ln\left(\frac{1200}{500}\right)$$

$$k=0.087547,\ so\ P=500e^{0.087547t}$$

Now, solve for t when P = 900:

$$900=500e^{0.087547t},\ e^{0.087547t}=\frac{900}{500},$$

$$0.087547t=\ln\left(\frac{900}{500}\right),\ t=6.714\ days$$

Logistic growth example:

The number of moose in a national park is modeled by the function $M(t)$ that satisfies the logistical differential equation $\frac{dM}{dt}=\frac{3}{5}M-\frac{3}{1000}M^2$ *and* $M(0)=50$.

(a) What is $\lim_{t\to\infty} M(t)$?

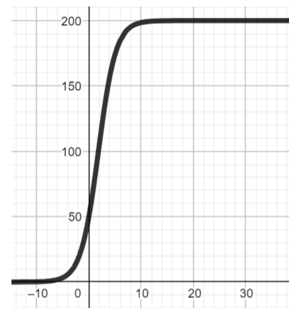
(b) What is the population of moose when the number of moose is growing most rapidly?

(c) At what time does max rate of growth occur?

(a) *logistic DE form:* $\frac{dP}{dt}=kP\left(1-\frac{P}{L}\right)$

factoring to get the 1: $\frac{dM}{dt}=\frac{3}{5}M\left(1-\frac{1}{200}M\right)$

$\frac{dM}{dt}=\frac{3}{5}M\left(1-\frac{M}{200}\right)$ *so carrying capacity* $=200$



From curve shape,

$\lim_{t\to\infty} M(t)=$ *carrying capacity* $=200$

(b) *Fastest growth for logistic occurs when population is half the carrying capacity:*

When there are 100 *moose.*

(c) *DE form:* $\frac{dM}{dt}=\frac{3}{5}M\left(1-\frac{M}{200}\right)=kP\left(1-\frac{P}{L}\right)$

solution form: $M=\frac{L}{1+Ce^{-kt}}=\frac{200}{1+Ce^{-\frac{3}{5}t}}$

using initial condition: $M(0)=50$:

$$50=\frac{200}{1+Ce^{-\frac{3}{5}(0)}}=\frac{200}{1+C},\quad 1+C=\frac{200}{50},\quad C=3$$

final solution equation: $M=\frac{200}{1+3e^{-\frac{3}{5}t}}$

Now, solve for time when $M=100$:

$$100=\frac{200}{1+3e^{-\frac{3}{5}t}},\quad 1+3e^{-\frac{3}{5}t}=\frac{200}{100},\quad 3e^{-\frac{3}{5}t}=1,$$

$$-\frac{3}{5}t=\ln\left(\frac{1}{3}\right),\qquad t=1.831\ \textit{years.}$$