AP Calculus BC – Study Guide: Unit 7 – Infinite Series

Series convergence tests: For each, state procedure (and conditions for use) result for convergence, result for divergence…	Series convergence tests:
nth-term test	nth term test $\lim_{n\to\infty} a_n \neq 0\ [diverges]$ $(cannot\ be\ used\ to\ show\ convergence)$
Geometric series	Geometric series $form: \sum_{n=0}^{\infty} ar^n$ $\lvert r\rvert < 1 \quad [converges]$ $\lvert r\rvert \geq 1 \quad [diverges]$
p-series	p-series $form: \sum_{n=1}^{\infty} \frac{1}{n^p}$ $p > 1 \quad [converges]$ $0 < p \leq 1 \quad [diverges]$
Alternating series test	Alternating series test $form: \sum_{n=1}^{\infty} (-1)^{n-1} a_n$ $1)\ \lim_{n\to\infty} a_n = 0 \quad and$ $2)\ a_{n+1} \leq a_n \quad [converges]$ $if\ either\ not\ met, inconclusive$
Alternating Series are absolutely convergent if…	$\sum_{n=1}^{\infty} \left\lvert (-1)^{n-1} a_n \right\rvert\ converges$ $\left(by\ theorem, \sum_{n=1}^{\infty} (-1)^{n-1} a_n\ also\ converges\right)$
Alternating Series are conditionally convergent if…	$\sum_{n=1}^{\infty} \left\lvert (-1)^{n-1} a_n \right\rvert\ diverges\ but\ \sum_{n=1}^{\infty} (-1)^{n-1} a_n\ converges$
Error of truncated Alternating Series…	$\lvert error\rvert \leq \lvert 1st\ neglected\ term\rvert$

Integral test

Integral test

$$form: \sum_{n=1}^{\infty} a_n \quad a_n = f(n) \quad f(n)\ terms\ positive\ and\ decreasing$$

$$evaluate \int_1^{\infty} f(x)\,dx$$

if integral converges, series converges

if integral diverges, series diverges

Root test

Root test

$$\sum_{n=1}^{\infty} \sqrt[n]{|a_n|} < 1 \qquad [converges]$$

$$\sum_{n=1}^{\infty} \sqrt[n]{|a_n|} > 1\ or\ \infty \qquad [diverges]$$

Ratio test

Ratio test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1 \qquad [converges]$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1\ or\ \infty \qquad [diverges]$$

Direct Comparison

Direct Comparison

$$if\ 0 < a_{orig} \le a_{new}\ and \sum_{n=1}^{\infty} a_{new}\ converges,\ then\ \sum_{n=1}^{\infty} a_{orig}\ converges$$

$$f\ 0 < a_{new} \le a_{orig}\ and \sum_{n=1}^{\infty} a_{new}\ diverges,\ then\ \sum_{n=1}^{\infty} a_{orig}\ diverges$$

Limit Comparison

Limit Comparison

$$If\ \lim_{n\to\infty}\frac{a_{orig}}{a_{new}} > 0\ (a\ finite,\ positive\ number)$$

then series are 'linked' so...

$$If \sum_{n=1}^{\infty} a_{new}\ converges,\ then\ \sum_{n=1}^{\infty} a_{orig}\ converges$$

$$If \sum_{n=1}^{\infty} a_{new}\ diverges,\ then\ \sum_{n=1}^{\infty} a_{orig}\ diverges$$

Taylor Polynomials/Power Series…

Taylor Polynomial form:

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$$P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

Maclaurin means centered at…

Maclaurin means centered at *x* = 0.

Max Error (Lagrange Error)…

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$$max\ error = \frac{f^{(n)}(z)}{(n+1)!}(x-c)^{n+1}$$

$$where\ f^{(n)}(z)\ is\ \max\ value\ of\ derivative$$

Memorized Power Series:

$e^x =$

$\sin x =$

$\cos x =$

Memorized Power Series:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \ldots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \ldots$$

Find the radius of convergence of

$$\sum_{n=1}^{\infty} \frac{n+1}{2n+1}\frac{(x-3)^n}{2^n}$$

Find the radius of convergence of

$$ratio\ test:\ \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{n+2}{2n+3}\frac{(x-3)^{n+1}}{2^{n+1}}\frac{2n+1}{n+1}\frac{2^n}{(x-3)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{n+2}{2n+3}\frac{(x-3)\cancel{(x-3)^n}}{2\,\cancel{2^n}}\frac{2n+1}{n+1}\frac{\cancel{2^n}}{\cancel{(x-3)^n}}\right|$$

$$= \lim_{n\to\infty}\frac{(n+2)(2n+1)}{2(2n+3)(n+1)}|x-3| = \lim_{n\to\infty}\frac{2n^2+\ldots\ldots}{4n^2+\ldots.}|x-3|$$

$$\frac{1}{2}|x-3| < 1,\quad |x-3| = 2,\quad radius\ of\ convergence = 2$$

Find the <u>interval</u> of convergence

First, find the radius of convergence, then individually

Test the endpoints by plugging in the x values into

the original series and determining convergence.

The Maclaurin series for $\frac{1}{1-x}$ is $\sum_{n=0}^{\infty} x^n$.

Find a power series expansion for $\frac{x^2}{1-x^2}$

You can do algebraic operations with power series :

let $u = x^2$

$$\frac{x^2}{1-x^2} = x^2 \frac{1}{1-u} = x^2 \sum_{n=0}^{\infty} u^n = x^2 \sum_{n=0}^{\infty} \left(x^2\right)^n = x^2 \sum_{n=0}^{\infty} x^{2n} = \sum_{n=0}^{\infty} x^2 x^{2n} = \sum_{n=0}^{\infty} x^{2n+2}$$

$$= x^2 + x^4 + x^6 + x^8 + \ldots$$

The function f is defined by the power series

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \ldots$$

Show that $1 - \frac{1}{3!}$ approximates $f(1)$ with an error less than $\frac{1}{100}$

$$\max \textit{error} \le \left| \textit{first neglected term} \right|$$

$$\max \textit{error} \le \left| \frac{x^4}{5!} \right|$$

$$\max \textit{error} \le \left| \frac{(1)^4}{5!} \right| = 0.0083$$

$$0.0083 < \frac{1}{100}$$

The Maclaurin series for the function f is given by $f(x) = \sum_{n=0}^{\infty} \left(-\frac{x}{4}\right)^n$. What is the value of $f(3)$?

$f(3) = \sum_{n=0}^{\infty} \left(-\frac{3}{4}\right)^n$ *is a Geometric series with* $r = -\frac{3}{4}$

which converges to a sum of $\frac{a}{1-r} = \frac{1}{1-\left(-\frac{3}{4}\right)} = \frac{4}{7}$

So $f(3) = \frac{4}{7}$.