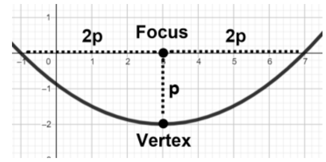
<u>AP Calculus BC – Study Guide: Unit 8 – Parametric Equations, Polar Coordinates, and Vectors</u>

Conic sections…

Convert to standard form and sketch:

$$x^2-6x-8y-7=0$$

<u>Parabola</u>:

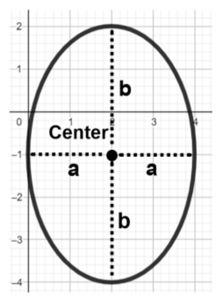


$$x^2-6x-8y-7=0$$
$$x^2-6x+\underline{\underline{9}}=8y+7+\underline{\underline{9}}$$
$$(x-3)^2=8y+16$$
$$(x-3)^2=8(y+2) \qquad (x-h)^2=4p(y-k)$$
$$(h,k)=vertex \qquad p=distance\ vertex\ to\ focus\ and\ directrix$$

$$9x^2+4y^2-36x+8y+4=0$$

<u>Ellipse</u> :

$$9x^2+4y^2-36x+8y+4=0$$
$$9(x^2-4x\quad)+4(y^2+2y\quad)=-4$$
$$9(x^2-4x+\underline{\underline{4}})+4(y^2+2y+\underline{\underline{1}})=-4+\underline{\underline{36}}+\underline{\underline{4}}$$
$$9(x-2)^2+4(y+1)^2=36$$
$$\frac{(x-2)^2}{4}+\frac{(y+1)^2}{9}=1 \qquad \frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$
$$(h,k)=center \quad a=dist.\ center\ to\ vertex\ in\ x$$
$$b=dist.\ center\ to\ curve\ in\ y$$



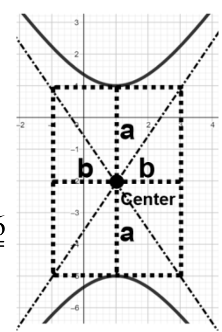
<u>Circle</u> :

special case of ellipse with a=b:

$$\frac{(x-2)^2}{4}+\frac{(y+1)^2}{4}=1$$
$$(x-2)^2+(y+1)^2=4$$

$$9x^2-4y^2-18x-16y+29=0$$

<u>Hyperbola</u> :



$$9x^2-4y^2-18x-16y+29=0$$
$$9(x^2-2x\quad)-4(y^2+4y\quad)=-29$$
$$9(x^2-4x+\underline{\underline{1}})+4(y^2+2y+\underline{\underline{4}})=-4+\underline{\underline{9}}+-\underline{\underline{16}}$$
$$9(x-1)^2-4(y+2)^2=-36$$
$$4(y+2)^2-9(x-1)^2=36$$
$$\frac{(y+2)^2}{9}-\frac{(x+2)^2}{4}=1 \qquad \frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$$
$$(h,k)=center \quad a=dist.\ center\ to\ vertex\ on\ transverse\ axis$$
$$b=dist.\ center\ to\ edge\ of\ asymptote\ box$$

Different representational forms of relationships…

<u>*Rectangular*</u>: $y = f(x)$ <u>*Parametric*</u>: $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$

Convert to rectangular form and sketch:

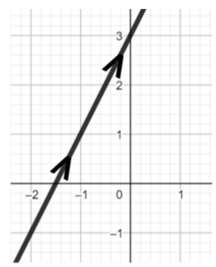
$$\begin{cases} x = 2t \\ y = 4t + 3 \end{cases}$$

$$\begin{cases} x = 1 + 2\cos t \\ y = -2 + 3\sin t \end{cases}$$

Convert to rectangular form:

$$\begin{cases} x = 2t \\ y = 4t + 3 \end{cases} \quad \textit{eliminate parameter by substitution}$$

$$t = \frac{1}{2}x, \quad y = 4\left(\frac{1}{2}x\right) + 3, \quad y = 2x + 3$$



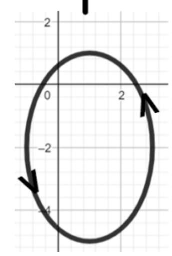
(include direction arrows)

$$\begin{cases} x = 1 + 2\cos t \\ y = -2 + 3\sin t \end{cases} \quad \textit{use } \sin^2 t + \cos^2 t = 1$$

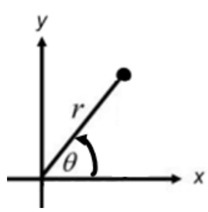
$$\cos t = \frac{x-1}{2}, \quad \sin t = \frac{y+2}{3}$$

$$\frac{(x-1)^2}{4} + \frac{(y+2)^2}{9} = 1 \qquad (\textit{ellipse})$$

t	(x,y)
0	(3,-2)
$\frac{\pi}{2}$	(1,1)



<u>*Polar*</u>:



Formulas for converting polar - rectangular:

$x =$

$y =$

$x^2 + y^2 =$

$\tan\theta =$

Formulas for converting polar - rectangular:

$x = r\cos\theta$

$y = r\sin\theta$

$x^2 + y^2 = r^2$

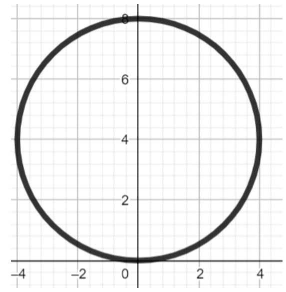
$\tan\theta = \frac{y}{x}$

Convert to rectangular form and sketch:

$r = 8\sin\theta$

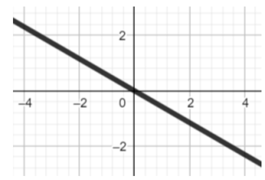
Convert to rectangular form:

$$r = 8\sin\theta$$
$$r^2 = 8r\sin\theta$$
$$x^2 + y^2 = 8y$$
$$x^2 + y^2 - 8y = 0$$
$$x^2 + (y-4)^2 + 16$$

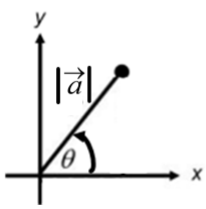


$\theta = \dfrac{5\pi}{6}$

$$\theta = \frac{5\pi}{6}$$
$$\tan\theta = \tan\left(\frac{5\pi}{6}\right)$$
$$\frac{y}{x} = \frac{\sin\left(\frac{5\pi}{6}\right)}{\cos\left(\frac{5\pi}{6}\right)} = \frac{\left(\frac{1}{2}\right)}{\left(\frac{-\sqrt{3}}{2}\right)} = \frac{1}{-\sqrt{3}}$$
$$y = \frac{1}{-\sqrt{3}}x$$



<u>*Vectors*</u>:



Position: $\vec{a} = \langle a_x, a_y \rangle,\ a\ distance\ of\ |\vec{a}|\ in\ direction\ \theta$

Formulas for vector $\left(\vec{a} = \langle a_x, a_y \rangle\right)$:

$magnitude\ of\ a = |\vec{a}| =$

$components:$

$a_x =$ $\qquad\qquad$ $a_y =$

Formulas for vectors:

$$magnitude\ of\ a = |\vec{a}| = \sqrt{(a_x)^2 + (a_y)^2}$$

$components:$

$$a_x = |\vec{a}|\cos\theta \qquad a_y = |\vec{a}|\sin\theta$$

Vectors are equal if…

Vectors are equal if their component values are equal.

Find the vector from $(1,3)\ to\ (9,4)$

$$\vec{a} = \langle 9-1,\ 4-3 \rangle = \langle 8,1 \rangle$$

<u>Vector-valued functions</u>:

Input is a parameter (typically, *t*), output is a vector

Example: position vector: $\vec{r}(t) = \langle x(t),\ y(t) \rangle$

<u>Vector-valued functions</u>:

Example: position vector: $\vec{r}(t) = \langle x(t),\ y(t) \rangle$

Arithmetic operations and properties for different representations…

Multiplication by a constant…

$3\langle 6,-3\rangle =$ $\qquad$ $3\langle 6,-3\rangle = \langle 18,-9\rangle$

$\lim_{x\to 2} 3f(x) =$ $\qquad$ $\lim_{x\to 2} 3f(x) = 3\lim_{x\to 2} f(x)$

$\frac{d}{dx}\left[3f(x)\right] =$ $\qquad$ $\frac{d}{dx}\left[3f(x)\right] = 3\frac{d}{dx}\left[f(x)\right]$

$\int 3f(x)\,dx =$ $\qquad$ $\int 3f(x)\,dx = 3\int f(x)\,dx$

$\sum_{n=1}^{\infty} 3a_n =$ $\qquad$ $\sum_{n=1}^{\infty} 3a_n = 3\sum_{n=1}^{\infty} a_n$

In general, multiplication of objects other than numbers is not straightforward (derivative of function multiplied requires product rule, integral requires integration by parts, multiplication of a vector by another vector not defined for this class, cannot multiply two series in summation form.)

Addition/subtraction…

$\langle 8,1\rangle - \langle 2,5\rangle =$ $\qquad$ $\langle 8,1\rangle - \langle 2,5\rangle = \langle 8-2,\ 1-5\rangle = \langle 6,-4\rangle$

$\lim_{x\to c}\left(x^3 - \frac{1}{x}\right) =$ $\qquad$ $\lim_{x\to c}\left(x^3 - \frac{1}{x}\right) = \lim_{x\to c} x^3 - \lim_{x\to c}\frac{1}{x}$

$\frac{d}{dx}\left[x^3 - \sin(x)\right] =$ $\qquad$ $\frac{d}{dx}\left[x^3 - \sin(x)\right] = 3x^2 - \cos(x)$

$\int \left(x^3 - \cos(x)\right) dx =$ $\qquad$ $\int \left(x^3 - \cos(x)\right) dx = \frac{1}{4}x^4 - \sin(x) + C$

$\sum_{n=1}^{\infty}(a_n \pm b_n) =$ $\qquad$ $\sum_{n=1}^{\infty}(a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n$

PEMDAS still applies…

$2\langle 8,1\rangle - 3\langle 2,5\rangle =$

$2\langle 8,1\rangle - 3\langle 2,5\rangle$

$\langle 16,2\rangle - \langle 6,15\rangle$ *multiplication before addition*

$\langle 16-6,\ 2-15\rangle = \langle 10,-13\rangle$

For vectors things like limits, derivatives, or integrals apply separately to each term:

$\lim_{t\to 4}\left(\langle t^2,\ t^3\rangle\right)=$ $\lim_{t\to 4}\left(\langle t^2,\ t^3\rangle\right)=\left\langle \lim_{t\to 4}t^2, \lim_{t\to 4}t^3 \right\rangle$

$\frac{d}{dx}\left[\langle t^2,\ t^3\rangle\right]=$ $\frac{d}{dx}\left[\langle t^2,\ t^3\rangle\right]=\langle 2t,\ 3t^2\rangle$

$\int\langle t^2,\ t^3\rangle\, dt=$ $\int\langle t^2,\ t^3\rangle\, dt=\left\langle \frac{1}{3}t^3+C_1,\ \frac{1}{4}t^4+C_2\right\rangle=\left\langle \frac{1}{3}t^3,\ \frac{1}{4}t^4\right\rangle+\vec{C}$

For limits:

$\lim_{x\to c}\left[f(x)\right]^n=$ $\lim_{x\to c}\left[f(x)\right]^n=\left[\lim_{x\to c}f(x)\right]^n$

$\lim_{x\to c}\left[\sqrt[n]{f(x)}\right]=$ $\lim_{x\to c}\left[\sqrt[n]{f(x)}\right]=\sqrt[n]{\lim_{x\to c}f(x)}$

Derivatives in parametric form $\begin{cases}x=t^2-3t\\ y=\sin(t)\end{cases}$:

$\frac{dy}{dx}=$ $\frac{dy}{dx}=\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}=\frac{\cos(t)}{2t-3}$

$\frac{d^2y}{dx^2}=$ $\frac{d^2y}{dx^2}=\frac{\frac{d}{dx}\left[\frac{\cos(t)}{2t-3}\right]}{(2t-3)}=\frac{\frac{(2t-3)(-\sin(t))-\cos(t)2}{(2t-3)^2}}{2}$

Derivatives in polar form $r=4\sin(\theta)$:

$\frac{dy}{dx}=$

$x=r\cos\theta=4\sin\theta\cos\theta$ $y=r\sin\theta=4(\sin\theta)^2$

$\frac{dx}{d\theta}=(4\sin\theta)(-\sin\theta)+\cos\theta(4\cos\theta)=-4\sin^2\theta+4\cos^2\theta$

$\frac{dy}{dx}$ is the slope of the tangent line on the x-y plane.

$\frac{dy}{d\theta}=8\sin\theta\cos\theta$

$\frac{dy}{dx}=\frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)}=\frac{8\sin\theta\cos\theta}{-4\sin^2\theta+4\cos^2\theta}$

Horizontal tangents occur when… Horizontal tangents occur when… $\frac{dy}{dx}=0$

Vertical tangents occur when… Vertical tangents occur when… $\frac{dy}{dx}$ is undefined

Intersections are always system solutions
(find the intersections):

$$\begin{cases} y = x^2 - 6 \\ y = -x \end{cases}$$

$$\begin{cases} y = x^2 - 6 \\ y = -x \end{cases}$$

$$\begin{array}{lll} x^2 - 6 = -x & x = -3, & x = 2 \\ x^2 + x - 6 = 0 & y = -(-3) & y = -(2) \\ (x+3)(x-2) = 0 & (-3, 3) & (2, -2) \end{array}$$

$$\begin{cases} r = 3(1 + \sin\theta) \\ r = 3(1 - \sin\theta) \end{cases}$$

$$\begin{cases} r = 3(1 + \sin\theta) \\ r = 3(1 - \sin\theta) \end{cases}$$

$$\begin{array}{lll} 3(1+\sin\theta) = 3(1-\sin\theta) & \theta = 0, & \theta = \pi \\ \sin\theta = -\sin\theta & r = 3(1+\sin 0) & r = 3(1+\sin\pi) \\ 2\sin\theta = 0 & r = 3 & r = 3 \\ \sin\theta = 0 & (r,\theta):\ (3,0) & (3,\pi) \end{array}$$

$\textit{must also graph and check for } r = 0\ (\textit{is an intersection here}):$

$$\textit{for } r = 3(1+\sin\theta) \rightarrow 0 = 3(1+\sin\theta),\ \sin\theta = -1,\ \theta = \frac{3\pi}{2}$$

$$\textit{for } r = 3(1-\sin\theta) \rightarrow 0 = 3(1-\sin\theta),\ \sin\theta = 1,\ \theta = \frac{\pi}{2}$$

$$\textit{so } \left(0, \frac{3\pi}{2}\right) \textit{ and } \left(0, \frac{\pi}{2}\right) \ (\textit{coincident, but not a 'collision'} - \textit{different } \theta)$$

$$\begin{cases} x_1 = 3\sin t \\ y_1 = 2\cos t \end{cases} \quad \begin{cases} x_2 = 3 + \cos t \\ y_2 = 1 + \sin t \end{cases} \quad 0 \le t < 2\pi$$

$$\begin{cases} x_1 = 3\sin t \\ y_1 = 2\cos t \end{cases} \quad \begin{cases} x_2 = 3 + \cos t \\ y_2 = 1 + \sin t \end{cases} \quad 0 \le t < 2\pi$$

$$\begin{array}{ll} x_1 = x_2 & y_1 = y_2 \\ 3\sin t = 3 + \cos t & 2\cos t = 1 + \sin t \\ \textit{by calculator graph}: & \textit{by calculator graph}: \\ \textit{at } t = 1.5708,\ t = 2.2143 & \textit{at } t = 0.6435,\ t = 4.7123 \end{array}$$

$$(\textit{these are intersections, but not 'collisions'} - \textit{different } t)$$

Tangent lines…

Rectangular:

For $(x-2)^2+(y+3)^2=4$

(a) *Write the equation of the tangent line at* $\left(1,-3+\sqrt{3}\right)$

(b) *Where does this curve have horizontal tangents?*

(c) *Where does this curve have vertical tangents?*

For $(x-2)^2+(y+3)^2=4$

(a) *Write the equation of the tangent line at* $\left(1,-3+\sqrt{3}\right)$

$$m=\frac{dy}{dx}\left[\text{use implicit differentiation if needed}\right]:$$

$$2(x-2)(1)+2(y+3)\left(\frac{dy}{dx}\right)=0,\quad \frac{dy}{dx}=\frac{-x+2}{y+3}=\frac{-(1)+2}{\left(-3+\sqrt{3}\right)+3}=\frac{1}{\sqrt{3}}$$

$$\left(y-\left(-3+\sqrt{3}\right)\right)=\frac{1}{\sqrt{3}}(x-1)$$

(b) *Where does this curve have horizontal tangents?*

$$\text{where } \frac{dy}{dx}=0 \ (\text{numerator}=0),\ -x+2=0,\quad \text{at } x=2\ (2 \text{ points})$$

(c) *Where does this curve have vertical tangents?*

$$\text{where } \frac{dy}{dx}=DNE \ (\text{denominator}=0),\ y+3=0,\quad \text{at } y=-3\ (2 \text{ points})$$

Parametric:

For $\begin{cases} x=2t-\pi\sin t \\ y=2-\pi\cos t \end{cases}$

(a) *Write the equation of the tangent line at* $t=\frac{2\pi}{3}$

(b) *Where does this curve have horizontal tangents?*

(c) *Where does this curve have vertical tangents?*

For $\begin{cases} x=2t-\pi\sin t \\ y=2-\pi\cos t \end{cases}$

(a) *Write the equation of the tangent line at* $t=\frac{2\pi}{3}$

$$m=\frac{dy}{dx}=\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}=\left.\frac{\pi\sin t}{2-\pi\cos t}\right|_{t=\frac{2\pi}{3}}=\frac{\pi\left(\frac{\sqrt{3}}{2}\right)}{2-\pi\left(-\frac{1}{2}\right)}=\frac{\frac{\sqrt{3}\pi}{2}}{\frac{2}{1}+\frac{\pi}{2}}=\frac{\sqrt{3}\pi}{4+\pi}=0.76193$$

$$x=2\left(\frac{2\pi}{3}\right)-\pi\sin\left(\frac{2\pi}{3}\right)=\frac{4\pi}{3}-\pi\left(\frac{\sqrt{3}}{2}\right)=1.4681$$

$$y=2-\pi\cos\left(\frac{2\pi}{3}\right)=2-\pi\left(-\frac{1}{2}\right)=3.5708$$

$$(y-3.5708)=0.76193(x-1.4681)$$

(b) *Where does this curve have horizontal tangents?*

$$\text{where } \frac{dy}{dx}=0 \ (\text{numerator}=0)\ \ \pi\sin t=0$$

$t=0, t=\pi$ (*and other values, use calculator*)

(c) *Where does this curve have vertical tangents?*

$$\text{where } \frac{dy}{dx}=DNE \ (\text{denominator}=0)\ \ 2-\pi\cos t=0$$

$t=-0.8807, t=0.8807$ (*and other values, use calculator*)

Polar:

$For\ r = 4\sin\theta$

$(a)\ Write\ the\ equation\ of\ the\ tangent\ line\ at\ \theta = \frac{\pi}{3}$

$(b)\ Where\ does\ this\ curve\ have\ horizontal\ tangents?$

$(c)\ Where\ does\ this\ curve\ have\ vertical\ tangents?$

$For\ r = 4\sin\theta$

$(a)\ Write\ the\ equation\ of\ the\ tangent\ line\ at\ \theta = \frac{\pi}{3}$

$$m = \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dy}{d\theta}\right)}$$

$$and\ \ x = r\cos\theta = 4\sin\theta\cos\theta, \quad y = r\sin\theta = 4(\sin\theta)^2$$

$$\frac{dx}{d\theta} = 4\sin\theta(-\sin\theta) + \cos\theta(4\cos\theta) \quad \frac{dy}{d\theta} = 8\sin\theta\cos\theta$$

$$\frac{dx}{d\theta} = 4\left(\cos^2\theta - \sin^2\theta\right) \quad \frac{dy}{d\theta} = 8\sin\theta\cos\theta$$

$$m = \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dy}{d\theta}\right)} = \frac{8\sin\theta\cos\theta}{4\left(\cos^2\theta - \sin^2\theta\right)}\Bigg|_{t=\frac{\pi}{3}} = \frac{8\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right)}{4\left(\left(\frac{1}{2}\right)^2 - \left(\frac{\sqrt{3}}{2}\right)^2\right)}$$

$$= \frac{2\sqrt{3}}{-1} = -2\sqrt{3} = -3.464$$

$$x = 4\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{3}\right) = 4\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right) = \sqrt{3} = 1.732$$

$$y = 4(\sin\theta)^2 = 4\left(\frac{\sqrt{3}}{2}\right)^2 = 3$$

$$(y - 1.732) = -3.464(x - 3)$$

$(b)\ Where\ does\ this\ curve\ have\ horizontal\ tangents?$

$$where\ \frac{dy}{dx} = 0\ \ (numerator = 0)\ \ 8\sin\theta\cos\theta = 0$$

$$\theta = 0, t = \frac{\pi}{2}\ \ (and\ other\ values, use\ calculator)$$

$(c)\ Where\ does\ this\ curve\ have\ vertical\ tangents?$

$$where\ \frac{dy}{dx} = DNE\ \ (denominator = 0)\ \ 4\left(\cos^2\theta - \sin^2\theta\right) = 0$$

$$t = -0.7854, t = 0.7854\ \ (and\ other\ values, use\ calculator)$$

Position, Velocity (speed), Acceleration…

In 1D:

An object moves in one direction with position x given by $x(t)=t^3-4t^2+3$.

(a) Find velocity as function of time.
(b) What acceleration as a function of time.
(c) What is the position of the particle at *t* = 2?
(d) What is the speed of the particle at *t* = 2?

In 1D:

$x(t)=t^3-4t^2+3$

$(a)\quad v(t)=x'(t)=3t^2-8t$

$(b)\quad a(t)=v'(t)=6t-8$

$(c)\quad x(2)=(2)^3-4(2)^2+3=-5\quad (\textit{include units if given in problem})$

$(d)\quad \textit{speed}=|v(2)|=\left|3(2)^2-8(2)\right|=|-4|=4$

An object is launched upward with an initial velocity of $30\ m/s$ from an initial height of $10\ m$ in gravity field with $a(t)=-9.8\,m/s^2$.

(a) Find velocity as a function of time.
(b) Find height as a function of time.
(c) At what time does the object reach maximum height and what is the max height?
(d) At what time does the object hit the ground?

$a(t)=-9.8$

$(a)\quad v(t)=\int a(t)\,dt=\int(-9.8)\,dt=-9.8t+C_1$

$v(0)=30,\ so\ 30=-9.8(0)+C_1,\ C_1=30$

$v(t)=-9.8t+30$

$(b)\quad x(t)=\int v(t)\,dt=\int(-9.8t+30)\,dt=-4.9t^2+30t+C_2$

$x(0)=10,\ so\ 10=-4.9(0)^2+30(0)+C_2,\ C_2=10$

$x(t)=-4.9t^2+30t+10$

$(c)\quad \textit{Max height when } v=0:\ -9.8t+30=0,\ t=3.06122\text{ sec}$

$x(3.06122)=55.91837\ m$

$(d)\quad \textit{On ground when } x=0:\ -4.9t^2+30t+10=0$

$$at\ t=\frac{-30\pm\sqrt{(30)^2-4(-4.9)(10)}}{2(-4.9)}=\cancel{-0.3169},\ 6.439\text{ sec}$$

In 2D (vector/parametric):

An object moves in the xy-plane with:
a velocity vector $\vec{v}(t)=\langle t^3-5t^2,\cos t\rangle$
…or could be given as parametric equations:

$$\begin{cases}x(t)=t^3-5t^2\\ y(t)=\cos t\end{cases}$$

(a) Find the position vector if $\vec{x}(0)=\langle 3,0\rangle$.
(b) Find the acceleration vector.
(c) What is the position, velocity, and acceleration of the object at *t* = 2?
(d) What is the speed of the object at *t* = 2?

In 2D (vector/parametric):

$\vec{v}(t)=\langle t^3-5t^2,\ \cos t\rangle$

$(a)\quad \vec{r}(t)=\left\langle\int(t^3-5t^2)dt,\ \int(\cos t)dt\right\rangle=\left\langle\frac{1}{4}t^4+C_1,\ \sin t+C_2\right\rangle$

$\vec{v}(0)=\langle 3,0\rangle\quad so\quad \langle 3,0\rangle=\left\langle\frac{1}{4}(0)^4+C_1,\ \sin(0)+C_2\right\rangle=\langle C_1,\ C_2\rangle$

$C_1=3,\ C_2=0,\quad \vec{r}(t)=\left\langle\frac{1}{4}t^4+3,\ \sin t\right\rangle$

$(b)\quad \vec{a}(t)=\left\langle\frac{d}{dx}\left[t^3-5t^2\right],\ \frac{d}{dx}[\cos t]\right\rangle=\langle 3t^2-10t,\ -\sin t\rangle$

$(c)\quad \vec{r}(2)=\left\langle\frac{1}{4}(2)^4+3,\ \sin(2)\right\rangle=\langle 7, 0.9093\rangle$

$\vec{v}(2)=\left\langle(2)^3-5(2)^2,\ \cos(2)\right\rangle=\langle -12,-0.4161\rangle$

$\vec{a}(2)=\left\langle 3(2)^2-10(2),\ -\sin(2)\right\rangle=\langle -8,\ -0.9093\rangle$

$(d)\ \textit{speed}=\left|\vec{v}(2)\right|=\sqrt{(-12)^2+(-0.9093)^2}=12.0344$

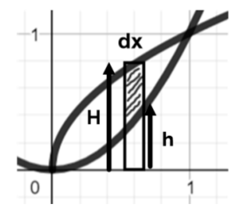
NOTE: Polar is similar to vector / parametric, the parameter is just θ instead of t, with $x=r\cos\theta,\ y=r\sin\theta$.

Applications of integrals…

<u>Area between curves (rectangular)</u>:

Find the area enclosed by $f(x)=x^2$ and $g(x)=\sqrt{x}$.

$$A=\int_a^b H\,dx-\int_a^b h\,dx$$
$$=\int_0^1 \sqrt{x}\,dx-\int_0^1 x^2\,dx$$
$$=\int_0^1\left(\sqrt{x}-x^2\right)dx=0.333$$

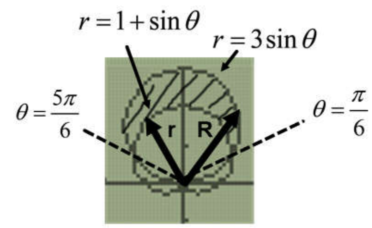


<u>Area between curves (polar)</u>:

Find the area inside $r=3\sin\theta$ and outside $r=1+\sin\theta$.

intersections :

$$3\sin\theta=1+\sin\theta$$
$$2\sin\theta=1$$
$$\sin\theta=\frac{1}{2}$$
$$\theta=\frac{\pi}{6},\ \theta=\frac{5\pi}{6}$$



$$A=\frac{1}{2}\int_\alpha^\beta R^2\,d\theta-\frac{1}{2}\int_\alpha^\beta r^2\,d\theta$$
$$=\frac{1}{2}\int_{\pi/6}^{5\pi/6}(3\sin\theta)^2\,d\theta-\frac{1}{2}\int_{\pi/6}^{5\pi/6}(1+\sin\theta)^2\,d\theta$$
$$=\frac{1}{2}\int_{\pi/6}^{5\pi/6}\left[(3\sin\theta)^2-(1+\sin\theta)^2\right]d\theta=3.142$$

<u>Arclength (rectangular)</u>:

If $f(x)=\frac{x^3}{6}+\frac{1}{2x}$ find the length of this curve for $\frac{1}{2}\le x\le 1$.

$$arclength=\int_a^b\sqrt{1+\left[f'(x)\right]^2}\,dx$$

$$f(x)=\frac{1}{6}x^3+\frac{1}{2}x^{-1},\quad f'(x)=\frac{1}{2}x^2-\frac{1}{2}x^{-2}$$

$$=\int_{1/2}^{1}\sqrt{1+\left[\frac{1}{2}x^2-\frac{1}{2}x^{-2}\right]^2}\,dx=0.646$$

<u>Arclength (parametric)</u>:
Find the arclength of the curve $x=6t^2,\ y=2t^3$ over the interval $1\le t\le 4$.

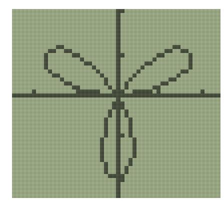
$$arclength=\int_a^b\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

$$x=6t^2,\quad \frac{dx}{dt}=12t,\quad y=2t^3,\quad \frac{dy}{dt}=6t^2$$

$$arclength=\int_1^4\sqrt{(12t)^2+\left(6t^2\right)^2}\,dt=156.525$$

<u>Arclength (polar)</u>:
Find the arclength of one petal of $r=2\sin(3\theta)$.

$$arclength=\int_\alpha^\beta\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$$



$one\ petal\ when\ r=0:$

$2\sin(3\theta)=0\quad substitute:\phi=3\theta$

$2\sin\phi=0,\quad \sin\phi=0,\quad \phi=0,\pi$

$back\ substitute:\qquad 3\theta=0\quad 3\theta=\pi$

$\theta=0,\quad \theta=\frac{\pi}{3}$

$$r=2\sin(3\theta),\ \frac{dr}{d\theta}=6\cos(3\theta)$$

$$arclength=\int_0^{\pi/3}\sqrt{\left(2\sin(3\theta)\right)^2+\left(6\cos(3\theta)\right)^2}\,d\theta=4.455$$

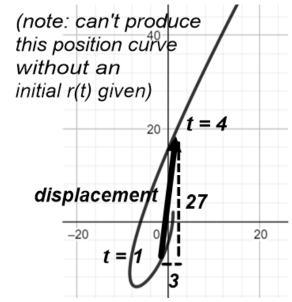
Displacement vs. total distance:

The velocity of a particle is given by $\vec{v}(t) = \langle 3t^2 - 8t,\ 3t^2 - 12 \rangle$. Find:

(a) The displacement of the particle from $t=1$ to $t=4$.

(b) The total distance traveled by the particle from $t=1$ to $t=4$.

Displacement vs. total distance:



$$(a)\ \text{total displacement} = \vec{r}(4) - \vec{r}(1)$$

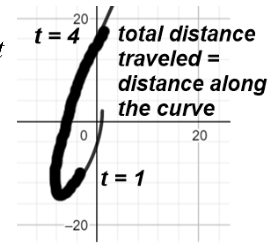
$$\int_1^4 \vec{r}'(t)\,dt = \int_1^4 \vec{v}(t)\,dt = \vec{r}(4) - \vec{r}(1)$$

$$\int_1^4 \langle 3t^2 - 8t,\ 3t^2 - 12 \rangle\,dt = \vec{r}(4) - \vec{r}(1)$$

$$\left\langle \int_1^4 3t^2 - 8t\,dt,\ \int_1^4 3t^2 - 12\,dt \right\rangle = \vec{r}(4) - \vec{r}(1)$$

$$\left\langle \left[t^3 - 4t^2\right]_1^4,\ \left[t^3 - 12t\right]_1^4 \right\rangle = \vec{r}(4) - \vec{r}(1)$$

$$\langle 3,\ 27 \rangle = \vec{r}(4) - \vec{r}(1)$$



$$(b)\ \text{total distance traveled} = \int_a^b \left|\vec{v}(t)\right|\,dt$$

$$\left|\vec{v}(t)\right| = \sqrt{(3t^2 - 8t)^2 + (3t^2 - 12)^2}$$

total distance traveled

$$= \int_1^4 \sqrt{(3t^2 - 8t)^2 + (3t^2 - 12)^2}\ dt$$

41.655