Calc III - Ch 15 Part 2 - Required Practice

Name: Key

15.6

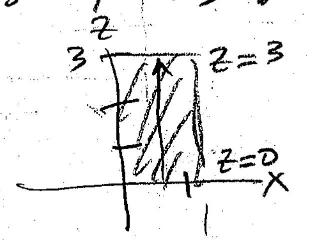
#1. Evaluate the given integral over the specified region using the three specified orders of integration.

$$\iiint_E xyz^2 \, dV$$

$$E = \{(x,y,z) \mid 0 \le x \le 1,\ -1 \le y \le 2,\ 0 \le z \le 3\}$$

(i) …integrating first with respect to y, then z, and then x.

floor: $y=-1$, ceiling: $y=2$



$$\int_0^1 \int_0^3 \int_{-1}^2 xyz^2 \, dy\, dz\, dx$$

$$\int_{-1}^2 xyz^2 dy = xz^2\left[\tfrac{1}{2}y^2\right]_{-1}^2$$

$$= xz^2\left(\tfrac{1}{2}\left((2)^2-(1)^2\right)\right) = \tfrac{3}{2}xz^2$$

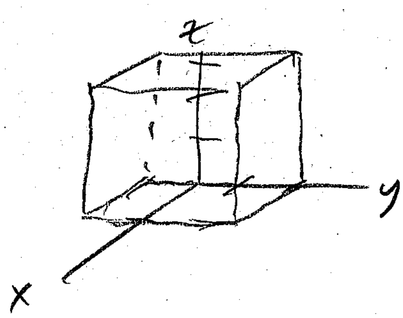
$$\int_0^3 \tfrac{3}{2}xz^2 dz = \tfrac{3}{2}x\left[\tfrac{1}{3}z^3\right]_0^3$$

$$= \tfrac{1}{2}x\left[(3)^3-(0)^3\right] = \tfrac{27}{2}x$$

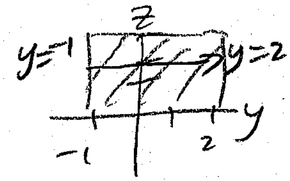
$$\int_0^1 \tfrac{27}{2}x\,dx = \tfrac{27}{2}\left(\tfrac{1}{2}x^2\right]_0^1$$

$$= \tfrac{27}{4}\left[(1)-(0)\right]$$

$$= \boxed{\tfrac{27}{4}}$$



(ii) …integrating first with respect to ~~z~~ x then y, and then z.



$$\int_0^3 \int_{-1}^2 \int_0^1 xyz^2 \, dx\, dy\, dz$$

$$\int_0^1 xyz^2 dx = yz^2\left[\tfrac{1}{2}x^2\right]_0^1 = yz^2\tfrac{1}{2}\left((1)^2-(0)^2\right)$$

$$= \tfrac{1}{2}yz^2$$

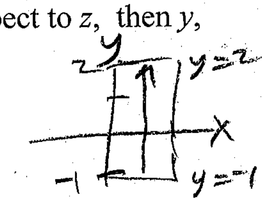
$$\int_{-1}^2 \tfrac{1}{2}yz^2 dy = \tfrac{1}{2}z^2\left[\tfrac{1}{2}y^2\right]_{-1}^2 = \tfrac{1}{4}z^2\left[(2)^2-(-1)^2\right]$$

$$= \tfrac{3}{4}z^2$$

$$\int_0^3 \tfrac{3}{4}z^2 dz = \tfrac{3}{4}\left[\tfrac{1}{3}z^3\right]_0^3 = \tfrac{1}{4}\left[(3)^3-(0)^3\right]$$

$$= \boxed{\tfrac{27}{4}}$$

(iii) …integrating first with respect to z, then y, and then x.



$$\int_0^1 \int_{-1}^2 \int_0^3 xyz^2 \, dz\, dy\, dx$$

$$\int_0^3 xyz^2 dz = xy\left[\tfrac{1}{3}z^3\right]_0^3 = \tfrac{1}{3}xy\left[(3)^3-(0)^3\right]$$

$$= 9xy$$

$$\int_{-1}^2 9xy\,dy = 9x\left[\tfrac{1}{2}y^2\right]_{-1}^2 = \tfrac{9}{2}x\left[(2)^2-(-1)^2\right]$$

$$= \tfrac{27}{2}x$$

$$\int_0^1 \tfrac{27}{2}x\,dx = \tfrac{27}{2}\left[\tfrac{1}{2}x^2\right]_0^1 = \tfrac{27}{4}\left((1)^2-(0)^2\right)$$

$$= \boxed{\tfrac{27}{4}}$$

#2. Evaluate the iterated integral

$$\int_0^1\int_0^z\int_0^y ze^{-y^2}\,dx\,dy\,dz$$

$\int_0^y ze^{-y^2}dx = ze^{-y^2}[x]_0^y$

$= ze^{-y^2}[(y)-(0)] = zye^{-y^2}$

$\int_0^z zye^{-y^2}dy = z\underline{\int_0^z ye^{-y^2}dy}$

u-sub:

$u=-y^2$ $\quad y\,dy=-\frac{1}{2}du$ $\quad y=0\to u=0$

$\frac{du}{dy}=-2y$ $\quad y=z\to u=-z^2$

$z\int_0^{-z^2} e^u\left(-\frac{1}{2}du\right)$

$-\frac{1}{2}z\left[e^u\right]_0^{-z^2} = -\frac{1}{2}z\left[e^{(-z^2)}-e^{(0)}\right]$

$=-\frac{1}{2}ze^{(-z^2)}+\frac{1}{2}z$

$\int_0^1\left(-\frac{1}{2}ze^{(-z^2)}\right)dz+\frac{1}{2}\int_0^1 z\,dz$

u-sub:

$u=-z^2$ $\quad -z\,dz=\frac{1}{2}du$

$\frac{du}{dz}=-2z$ $\quad z=0\to u=0$ $\quad z=1\to u=-1$

$\int_0^{-1}\left(\frac{1}{2}\right)\left(\frac{1}{2}du\right)e^u du+\frac{1}{2}\int_0^1 z\,dz$

$\frac{1}{4}\left[e^u\right]_0^{-1}+\frac{1}{4}\left[z^2\right]_0^1$

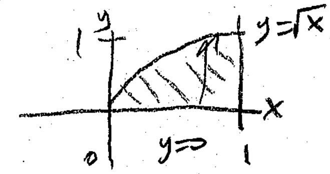
$\frac{1}{4}\left[e^{-1}-e^0\right]+\frac{1}{4}\left((1)^2-(0)^2\right)$

$\frac{1}{4e}-\frac{1}{4}+\frac{1}{4}$

$=\boxed{\frac{1}{4e}}$

#3. Evaluate the integral $\iiint_E 6xy\,dV$ where E lies under the plane $z=1+x+y$ and above the region in the xy-plane bounded by the curves $y=\sqrt{x}$, $y=0$, and $x=1$.

projection (xy)



floor: $z=0$, ceiling: $z=1+x+y$

$\int_0^1\int_0^{\sqrt{x}}\int_0^{1+x+y} 6xy\,dz\,dy\,dx$

$\int_0^{1+x+y}6xy\,dz=6xy[z]_0^{1+x+y}$

$=6xy(1+x+y)=6xy+6x^2y+6xy^2$

$6\int_0^{\sqrt{x}}(xy+x^2y+xy^2)\,dy$

$=6\left[\frac{1}{2}xy^2+\frac{1}{2}x^2y^2+\frac{1}{3}xy^3\right]_0^{\sqrt{x}}$

$=3x(\sqrt{x})^2+3x^2(\sqrt{x})^2+2x(\sqrt{x})^3-(0)$

$=3x^2+3x^3+2x^{5/2}$

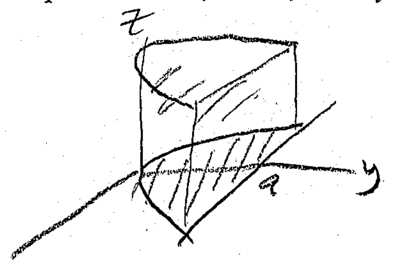
$\int_0^1(3x^2+3x^3+2x^{5/2})\,dx$

$=\left[x^3+\frac{3}{4}x^4+2\left(\frac{2}{7}\right)x^{7/2}\right]_0^1$

$=(1)^3+\frac{3}{4}(1)^4+\frac{4}{7}(1)^{7/2}-[0]$

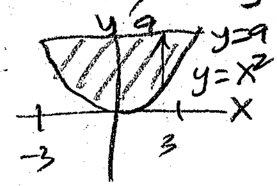
$=\boxed{\frac{65}{28}}$

#4. Set up (but do not evaluate) a triple integral to find the volume of the solid bounded by the cylinder $y = x^2$
and the planes $z = 0,\ z = 4,$ and $y = 9$.



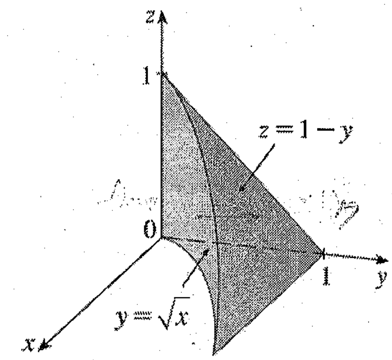
floor: $z=0$, ceiling: $z=4$

x-y



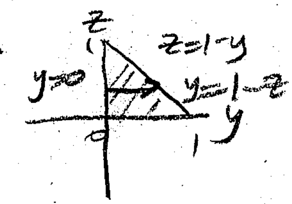
$$\boxed{\int_{-3}^{3}\int_{x^2}^{9}\int_{0}^{4} (1)\,dz\,dy\,dx}$$

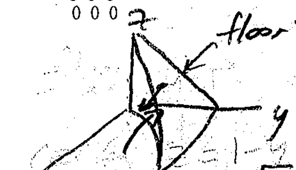
#5. The figure shows the region of integration for the integral $\int_0^1\int_{\sqrt{x}}^1\int_0^{1-y} f(x,y,z)\,dz\,dy\,dx$.



Rewrite this integral (but do not evaluate) as an equivalent iterated integral in the specified integration orders....

(i) $\int_0^?\int_0^?\int_0^? f(x,y,z)\,dx\,dy\,dz$.



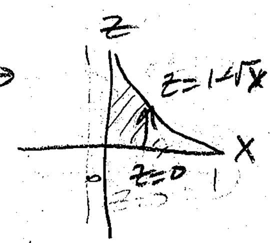


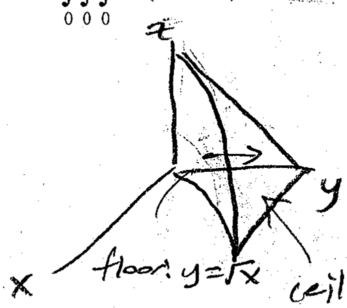
floor: $x=0$

ceiling: $y=\sqrt{x}$, $x=y^2$

$$\boxed{\int_0^1\int_0^{1-z}\int_0^{y^2} f(x,y,z)\,dx\,dy\,dz}$$

(ii) $\int_0^?\int_0^?\int_0^? f(x,y,z)$ ~~dx dz dy~~ dy dz dx.





intersection:
$\begin{cases} z=1-y \\ y=\sqrt{x} \end{cases}$
$z=1-\sqrt{x}$

floor: $y=\sqrt{x}$

ceiling: $z=1-y$
$y=1-z$

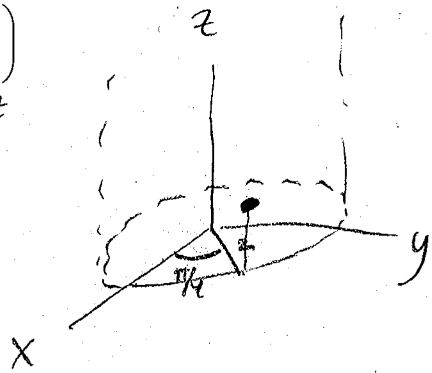
$$\boxed{\int_0^1\int_0^{1-\sqrt{x}}\int_{\sqrt{x}}^{1-z} f(x,y,z)\,dy\,dz\,dx}$$

15.7

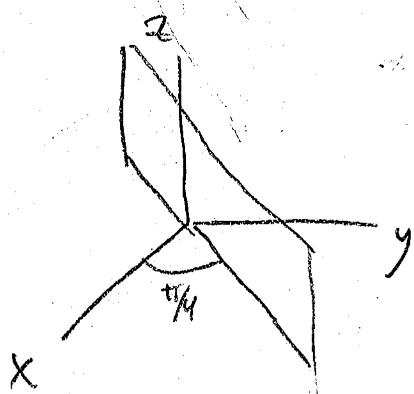
#1. Plot the point whose cylindrical coordinates are given:

$\left(2, \frac{\pi}{4}, 1\right)$

$r\ \theta\ z$



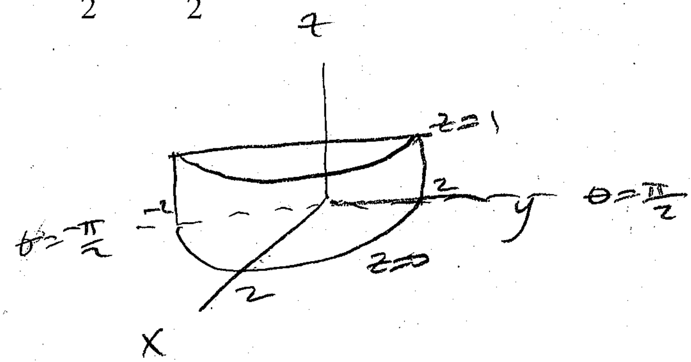
#2. Sketch and describe in words the surface whose equation is given: $\theta = \frac{\pi}{4}$



a plane

#3. Sketch the solid described by the inequalities:

$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}, \quad 0 \le r \le 2, \quad 0 \le z \le 1$



#4. Sketch the solid whose volume is given by the integral: $\int_0^4 \int_0^{2\pi} \int_r^4 r\, dz\, d\theta\, dr$

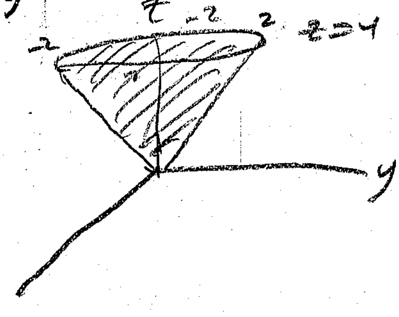
$z=r$ to $z=4$, $\theta=0$ to $\theta=2\pi$, $r=0$ to $r=4$

$z=r$

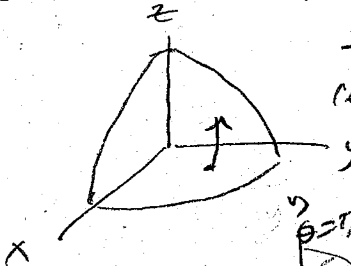
$z=\sqrt{x^2+y^2}$

$z^2=x^2+y^2$

Cone



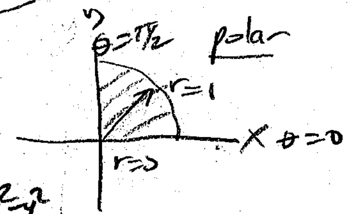
#5. Evaluate $\iiint_E \left(x^3+xy^2\right) dV$, where E is the solid in the first octant that lies beneath the paraboloid $z=1-x^2-y^2$.



floor: $z=0$

ceiling: $z=1-x^2-y^2$

projection:



$$\int_0^{\pi/2}\int_0^1\int_0^{1-x^2-y^2} (x^3+xy^2)\, r\, dz\, dr\, d\theta$$

$1-x^2-y^2 \qquad x^3+xy^2$

$1-(x^2+y^2) \qquad x(x^2+y^2)$

$1-r^2 \qquad r\cos\theta(r^2)=r^3\cos\theta$

$$\int_0^{\pi/2}\int_0^1\int_0^{1-r^2} r^4\cos\theta\, dz\, dr\, d\theta$$

$$\int_0^{1-r^2} r^4\cos\theta\, dz = r^4\cos\theta\,[z]_0^{1-r^2}$$

$$= r^4\cos\theta\,(1-r^2)$$

$$\int_0^1 (r^4\cos\theta)\,dr - \int_0^1 r^6\cos\theta\, dr$$

$$\cos\theta\,\tfrac{1}{5}[r^5]_0^1 - \tfrac{1}{7}\cos\theta\,[r^7]_0^1$$

$$\tfrac{1}{5}\cos\theta\,[(1)^5-(0)^5] - \tfrac{1}{7}\cos\theta\,((1)^7-(0)^7)$$

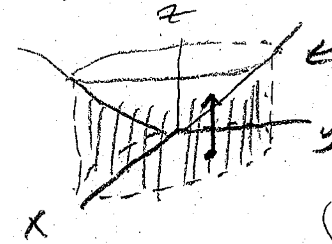
$$= \tfrac{1}{5}\cos\theta - \tfrac{1}{7}\cos\theta = \tfrac{2}{35}\cos\theta$$

$$\int_0^{\pi/2} \tfrac{2}{35}\cos\theta\, d\theta = \tfrac{2}{35}[\sin\theta]_0^{\pi/2}$$

$$= \tfrac{2}{35}\left[\underset{1}{\sin \tfrac{\pi}{2}} - \underset{1}{\sin 0}\right].$$

$$= \boxed{\tfrac{2}{35}}$$

#6. Evaluate $\iiint_E x^2\, dV$, where E is the solid that lies within the cylinder $x^2+y^2=1$, above the plane $z=0$, and below the cone $z^2=4x^2+4y^2$.



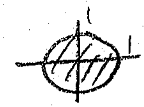
← intersection:
$\begin{cases} x^2+y^2=1 \\ z^2=4x^2+4y^2 \end{cases}$

$z^2=4(1),\ \underline{z=2}$

floor: $z=0$

ceiling: $z^2=4x^2+4y^2=4r^2$, $z=2r$

projection:



$r=0$ to $r=1$

$\theta=0$ to 2π

integrand x^2 $(r\cos\theta)^2$

$$\int_0^{2\pi}\int_0^1\int_0^{2r} (r\cos\theta)^2\, r\, dz\, dr\, d\theta$$

$$\int_0^{2r} r^3\cos^2\theta\, dz = r^3\cos^2\theta\,[z]_0^{2r}$$

$$= r^3\cos^2\theta\,[2r] = 2r^4\cos^2\theta$$

$$\int_0^1 2r^4\cos^2\theta\, dr = 2\cos^2\theta\,\tfrac{1}{5}[r^5]_0^1 \quad (1^5-0^5)$$

$$= \tfrac{2}{5}\cos^2\theta$$

$$\tfrac{2}{5}\int_0^{2\pi} \cos^2\theta\, d\theta = \tfrac{2}{5}\int_0^{2\pi}\left(\tfrac{1}{2}+\tfrac{1}{2}\cos 2\theta\right)d\theta$$

$$= \tfrac{1}{5}[\theta]_0^{2\pi} + \tfrac{1}{5}\left[\tfrac{1}{2}\sin(2\theta)\right]_0^{2\pi}$$

$$= \tfrac{1}{5}(2\pi-0) + \tfrac{1}{10}\left[\underset{0}{\sin 4\pi} - \underset{0}{\sin 0}\right]$$

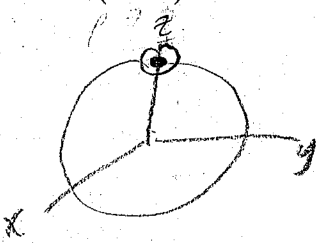
$$= \boxed{\tfrac{2\pi}{5}}$$

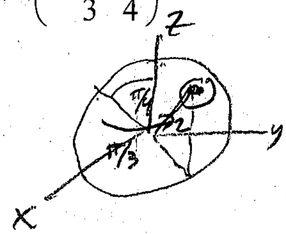
15.8

#1. Plot the point whose ~~cylindrical~~ spherical coordinates are given and find the rectangular coordinates of the point:

(i) $(1,0,0)$ $\quad \rho\ \theta\ \phi$

(ii) $\left(2,\frac{\pi}{3},\frac{\pi}{4}\right)$ $\quad \rho\ \theta\ \phi$





(i)

$x=\rho\sin\phi\cos\theta =$
$=(1)\sin 0\cos 0=(1)(0)(1)=0$

$y=\rho\sin\phi\sin\theta =$
$=(1)\sin 0\sin 0=(1)(0)(0)=0$

$z=\rho\cos\phi=(1)\cos 0=1$

$\boxed{(0,0,1)}$

(ii)

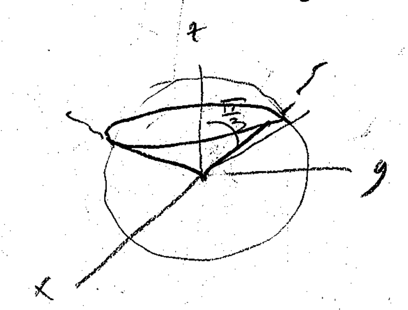
$x=\rho\sin\phi\cos\theta$
$=(2)\sin\frac{\pi}{4}\cos\frac{\pi}{3}$
$=2\left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)=\frac{\sqrt{2}}{2}$

$y=\rho\sin\phi\sin\theta$
$=(2)\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)=\frac{\sqrt{6}}{2}$

$z=\rho\cos\phi=$
$=(2)\cos\frac{\pi}{4}=2\left(\frac{\sqrt{2}}{2}\right)=\sqrt{2}$

$\boxed{\left(\frac{\sqrt{2}}{2},\frac{\sqrt{6}}{2},\sqrt{2}\right)}$

#2. Sketch and describe in words the surface whose equation is given: $\phi=\frac{\pi}{3}$



a cone

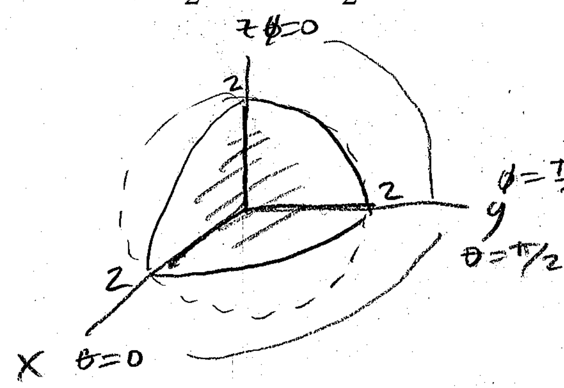
#3. Identify the surface whose equation is given: $\rho=\sin\theta\sin\phi$

$\rho=\sin\theta\sin\phi$
$\rho(\rho)=\rho(\sin\theta\sin\phi)$
$\rho^2=\rho\sin\phi\sin\theta$
$x^2+y^2+z^2=y$
$x^2+y^2-y+\frac{1}{4}+z^2=0+\frac{1}{4}$
$x^2+\left(y-\frac{1}{2}\right)^2+z^2=\frac{1}{4}$

a sphere centered at $\left(0,\frac{1}{2},0\right)$ with radius $\sqrt{\frac{1}{4}}=\frac{1}{2}$

#4. Sketch the solid described by the given inequalities:

$\rho\le 2,\quad 0\le\phi\le\frac{\pi}{2},\quad 0\le\theta\le\frac{\pi}{2}$



#5. Write the equation in spherical coordinates:

(i) $z^2 = x^2 + y^2 = r^2$

$(\rho\cos\phi)^2 = (\rho\sin\phi)^2$

$\rho^2\cos^2\phi = \rho^2\sin^2\phi$

$\rho^2\cos^2\phi - \rho^2\sin^2\phi = 0$

$\rho^2(\cos^2\phi - \sin^2\phi) = 0$

either $\rho = 0$ or $\cos^2\phi - \sin^2\phi = 0$

$\cos^2\phi = \sin^2\phi$

$\frac{\sin^2\phi}{\cos^2\phi} = 1, \tan^2\phi = 1$

$\tan\phi = \pm 1$

$\boxed{\rho = 0}$ or $\boxed{\phi = \frac{\pi}{4}, \phi = \frac{3\pi}{4}}$

(ii) $x^2 + z^2 = 9$

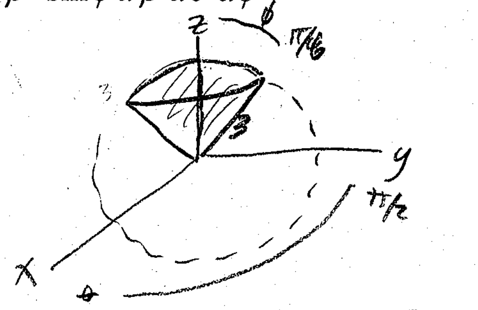
$(\rho\sin\phi\cos\theta)^2 + (\rho\cos\phi)^2 = 9$

$\rho^2\sin^2\phi\cos^2\theta + \rho^2\cos^2\phi = 9$

$\boxed{\rho^2(\sin^2\phi\cos^2\theta + \cos^2\phi) = 9}$

#6. Sketch the solid whose volume is given by the integral and evaluate the integral.

$$\int_0^{\pi/6}\int_0^{\pi/2}\int_0^3 \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$



$\int_0^3 \rho^2\sin\phi \, d\rho = \frac{1}{3}\sin\phi\left[\rho^3\right]_0^3$

$= \frac{1}{3}\sin\phi\left[(3)^3 - (0)^3\right] = 9\sin\phi$

$\int_0^{\pi/2} 9\sin\phi \, d\theta = 9\sin\phi\left[\theta\right]_0^{\pi/2}$

$= 9\sin\phi\left[\pi/2 - 0\right] = \frac{9\pi}{2}\sin\phi$

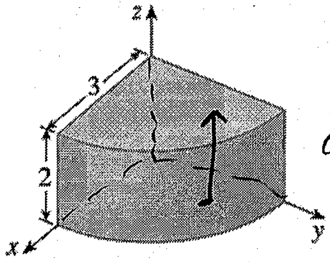
$\int_0^{\pi/6} \frac{9\pi}{2}\sin\phi \, d\phi = \frac{9\pi}{2}\left[-\cos\phi\right]_0^{\pi/6}$

$= \frac{9\pi}{2}\left[-\cos\pi/6 - (-\cos 0)\right]$

$= \frac{9\pi}{2}\left[-\left(\frac{\sqrt{3}}{2}\right) + (1)\right]$

$= \boxed{\frac{9\pi}{2}\left[1 - \frac{\sqrt{3}}{2}\right]}$

#7. Set up the triple integral of an arbitrary continuous function $f(x,y,z)$ in cylindrical or spherical coordinates over the solid shown.

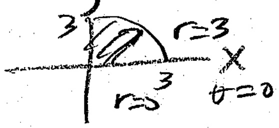


cylindrical

floor: $z=0$

ceiling: $z=2$

projection:



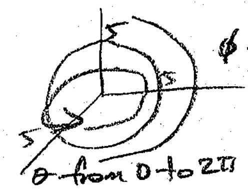
$$\int_0^{\pi/2}\int_0^{3}\int_0^{2} f(x,y,z)\, r\,dz\,dr\,d\theta$$

change to cylindrical

$x = r\cos\theta$
$y = r\sin\theta$
$z = z$

$$\boxed{\int_0^{\pi/2}\int_0^{3}\int_0^{2} f(r\cos\theta, r\sin\theta, z)\, r\,dz\,dr\,d\theta}$$

#8. Evaluate using spherical coordinates $\iiint\limits_B \left(x^2+y^2+z^2\right)^2 dV$, where B is the ball with center at the origin and radius 5.



ϕ from 0 to π

θ from 0 to 2π

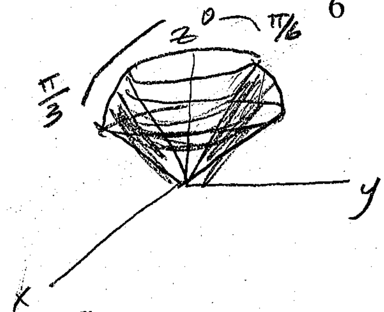
$(x^2+y^2+z^2)^2$
$(\rho^2)^2 = \rho^4$

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{5} (\rho^4)\,\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

$$\int_0^5 \rho^6 \sin\phi\, d\rho = \frac{1}{7}\sin\phi\left[\rho^7\right]_0^5$$
$$= \frac{1}{7}\sin\phi\left[5^7 - 0^7\right] = \frac{78125}{7}\sin\phi$$

$$\int_0^{\pi} \frac{78125}{7}\sin\phi\, d\phi = \frac{78125}{7}\left[-\cos\phi\right]_0^{\pi}$$
$$= \frac{78125}{7}\left[-\cos\pi - (-\cos 0)\right]$$
$$= \frac{78125}{7}\left(-(-1)+1\right) = \frac{156250}{7}$$
$$\int_0^{2\pi} \frac{156250}{7}\, d\theta = \frac{156250}{7}\left[\theta\right]_0^{2\pi}$$
$$= \boxed{\frac{312500\pi}{7}}$$

#9. Find the volume of the part of the ball $\rho \le 4$ that lies between the cones $\phi = \dfrac{\pi}{6}$ and $\phi = \dfrac{\pi}{3}$.



$$\int_0^{2\pi}\int_{\pi/6}^{\pi/3}\int_0^{4} (1)\rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$\int_0^4 \rho^2 \sin\phi \, d\rho = \sin\phi \frac{1}{3}[\rho^3]_0^4$$

$$= \frac{1}{3}\sin\phi\,[4^3 - 0^3] = \frac{64}{3}\sin\phi$$

$$\int_{\pi/6}^{\pi/3} \frac{64}{3}\sin\phi \, d\phi = \frac{64}{3}(-\cos\phi)\Big|_{\pi/6}^{\pi/3}$$

$$-\frac{64}{3}(\cos \pi/3 - \cos \pi/6) = -\frac{64}{3}\left(\frac{1}{2} - \frac{\sqrt{3}}{2}\right)$$

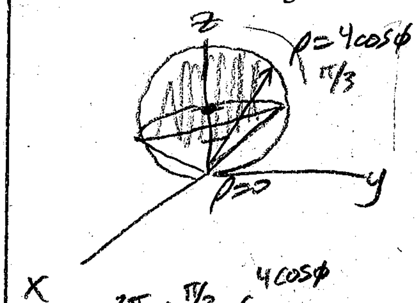
$$= \frac{64}{3}\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right) = \frac{32}{3}(1-\sqrt{3})$$

$$\int_0^{2\pi} \frac{32}{3}(1-\sqrt{3})\, d\theta = \frac{32}{3}(1-\sqrt{3})[\theta]_0^{2\pi}$$

$$= \frac{32}{3}(1-\sqrt{3})\,2\pi$$

$$= \boxed{\frac{64\pi}{3}(1-\sqrt{3})}$$

#10. Find the volume of the solid that lies above the cone $\phi = \dfrac{\pi}{3}$ and below the sphere $\rho = 4\cos\phi$



$$\rho^2 = 4\rho\cos\phi$$

$$x^2+y^2+z^2 = 4z$$

$$x^2+y^2+z^2-4z+4 = 0+4$$

$$x^2+y^2+(z-2)^2 = 4$$

$$\int_0^{2\pi}\int_0^{\pi/3}\int_0^{4\cos\phi} (1)\rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$\int_0^{4\cos\phi} \rho^2 \sin\phi \, d\rho = \frac{1}{3}\sin\phi\,[\rho^3]_0^{4\cos\phi}$$

$$= \frac{1}{3}\sin\phi\,((4\cos\phi)^3 - 0) = \frac{64}{3}\sin\phi\cos^3\phi$$

$$\int_0^{\pi/3} \frac{64}{3}\sin\phi\cos^3\phi\, d\phi$$

$u = \cos\phi$ $\quad$ $\phi = 0 \to u = 1$

$\dfrac{du}{d\phi} = -\sin\phi$ $\quad$ $\phi = \pi/3 \to u = \frac{1}{2}$

$\sin\phi\, d\phi = -du$

$$-\frac{64}{3}\int_1^{1/2} u^3\, du$$

$$= \frac{64}{3}\int_{1/2}^{1} u^3\, du = \frac{64}{3}\cdot\frac{1}{4}[u^4]_{1/2}^{1}$$

$$= \frac{64}{12}\left[1^4 - \left(\tfrac{1}{2}\right)^4\right] = 5$$

$$\int_0^{2\pi} 5\, d\theta = 5[\theta]_0^{2\pi} = \boxed{10\pi}$$

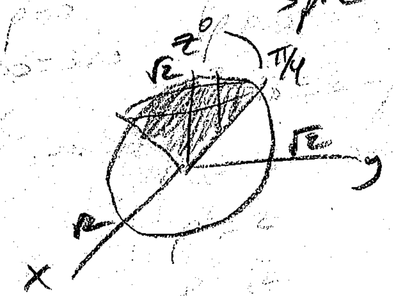
#11. Evaluate the integral by changing to spherical coordinates: $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} xy \, dz \, dy \, dx$

$z=\sqrt{x^2+y^2}$ to $z=\sqrt{2-x^2-y^2}$

$z^2=x^2+y^2$ $z^2=2-x^2-y^2$

cone $x^2+y^2+z^2=2$

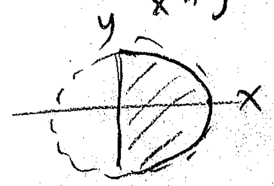
sphere



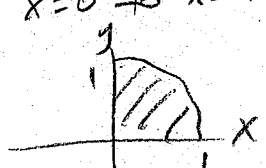
$y=0$ to $y=\sqrt{1-x^2}$

$y^2=1-x^2$

$x^2+y^2=1$



$x=0$ to $x=1$

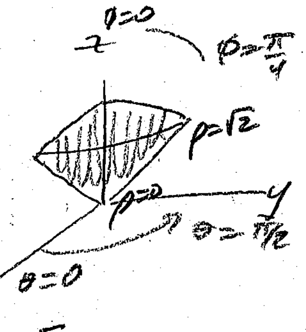


only 1st octant

integrand: xy

$(\rho\sin\phi\cos\theta)(\rho\sin\phi\sin\theta)$

$\rho^2\sin^2\phi\sin\theta\cos\theta$



$$\int_0^{\pi/2}\int_0^{\pi/4}\int_0^{\sqrt{2}} (\rho^2\sin^2\phi\sin\theta\cos\theta)\rho^2\sin\phi \, d\rho\, d\phi\, d\theta$$

$$\int_0^{\sqrt{2}} \rho^4\sin^3\phi\sin\theta\cos\theta\, d\rho = \frac{1}{5}\sin^3\phi\sin\theta\cos\theta\left[\rho^5\right]_0^{\sqrt{2}}$$

$$=\frac{1}{5}\sin^3\phi\sin\theta\cos\theta\left((\sqrt{2})^5-0\right)=\frac{4\sqrt{2}}{5}\sin^3\phi\sin\theta\cos\theta$$

$$\frac{4\sqrt{2}}{5}\sin\theta\cos\theta\int_0^{\pi/4}\sin^3\phi\, d\phi$$

$$\int_0^{\pi/4}\sin^2\phi\sin\phi\, d\phi$$

$$\int_0^{\pi/4}(1-\cos^2\phi)\sin\phi\, d\phi$$

$$\int_0^{\pi/4}\sin\phi\, d\phi-\int_0^{\pi/4}\cos^2\phi\sin\phi\, d\phi$$

$u=\cos\phi$

$\phi=0\to u=1$

$\phi=\pi/4\to u=\frac{\sqrt{2}}{2}$

$\frac{du}{d\phi}=-\sin\phi$

$\sin\phi\, d\phi=-du$

$$\int_0^{\pi/4}\sin\phi\, d\phi+\int_1^{\sqrt{2}/2}u^2\,du$$

$$\left[-\cos\phi\right]_0^{\pi/4}+\frac{1}{3}\left[u^3\right]_1^{\sqrt{2}/2}$$

$$\left(-\cos\tfrac{\pi}{4}+\cos 0\right)+\frac{1}{3}\left(\left(\frac{\sqrt{2}}{2}\right)^3-1\right)$$

$$\left(-\frac{\sqrt{2}}{2}+1\right)+\frac{1}{3}\left(\frac{2\sqrt{2}}{8}-1\right)$$

$$=\left(-\frac{5}{12}\sqrt{2}+\frac{2}{3}\right)\frac{4\sqrt{2}}{5}\sin\theta\cos\theta=\left(-\frac{2}{3}+\frac{8\sqrt{2}}{15}\right)\sin\theta\cos\theta$$

$$\left(\frac{8\sqrt{2}}{15}-\frac{2}{3}\right)\int_0^{\pi/2}\sin\theta\cos\theta\, d\theta$$

$u=\sin\theta$ $\theta=0\to u=0$

$\frac{du}{d\theta}=\cos\theta$ $\theta=\pi/2\to u=1$

$\cos\theta\, d\theta=du$

$$\left(\frac{8\sqrt{2}}{15}-\frac{2}{3}\right)\int_0^1 u\, du$$

$$\left(\frac{8\sqrt{2}}{15}-\frac{2}{3}\right)\frac{1}{2}\left[u^2\right]_0^1=\boxed{\frac{1}{2}\left(\frac{8\sqrt{2}}{15}-\frac{2}{3}\right)}$$