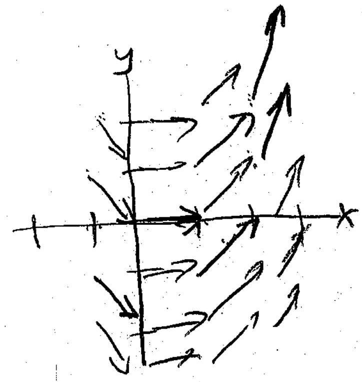
Calc III - Ch 16 - Required Practice

Name: Key

16.1 and 16.2 day 1

#1. Sketch the vector field for $\vec{F}(x,y) = \langle 1, x \rangle$

(x,y)	$\vec{F} = \langle 1, x \rangle$
$(0,0)$	$\langle 1, 0 \rangle$
$(1,0)$	$\langle 1, 1 \rangle$
$(1,-1)$	$\langle 1, 1 \rangle$
$(-1,1)$	$\langle 1, -1 \rangle$
$(-1,-1)$	$\langle 1, -1 \rangle$
$(2,2)$	$\langle 1, 2 \rangle$
$(2,1)$	$\langle 1, 2 \rangle$



#2. Evaluate the line integral, where C is the given curve: $\int_C y^3\, ds \quad C: x = t^3,\ y = t,\ 0 \le t \le 2$

Scalar: $= \int_a^b f(x(t), y(t))\, |\vec{r}\,'|\, dt$

$\vec{r} = \langle t^3, t \rangle$

$\vec{r}\,' = \langle 3t^2, 1 \rangle$

$|\vec{r}\,'| = \sqrt{9t^4 + 1}$

$\int_0^2 (t)^3 \sqrt{9t^4+1}\, dt$ $\quad u = 9t^4 + 1 \quad \frac{du}{dt} = 36t^3 \quad du = 36t^3 dt \quad t^3 dt = \frac{1}{36} du$ $\quad t = 0 \to u = 1 \quad t = 2 \to u = 145$

$\frac{1}{36} \int_1^{145} u^{1/2}\, dt$

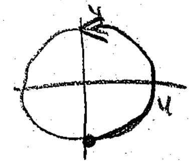
$\frac{1}{36} \frac{2}{3} \left[u^{3/2} \right]_1^{145}$

$\frac{1}{54} \left[(145)^{3/2} - (1)^{3/2} \right]$

$\boxed{\frac{1}{54} \left[145\sqrt{145} - 1 \right]}$

#3. Evaluate the line integral, where C is the given curve:

$\int_C xy^4\, ds \quad$ *C is the right half of circle* $x^2 + y^2 = 16$



$\vec{r}(t) = \langle 4\cos t, 4\sin t \rangle$

$-\frac{\pi}{2} \le t \le \frac{\pi}{2}$

$\vec{r}\,' = \langle -4\sin t, 4\cos t \rangle$

$|\vec{r}\,'| = \sqrt{16\sin^2 t + 16\cos^2 t} = 4$

$\int_{-\pi/2}^{\pi/2} (4\cos t)(4\sin t)^4 (4)\, dt$

$4096 \int_{-\pi/2}^{\pi/2} \sin^4 t \cos t\, dt$ $\quad u = \sin t \quad \frac{du}{dt} = \cos t \quad \cos t\, dt = du \quad t = -\pi/2 \to u = -1 \quad t = \pi/2 \to u = 1$

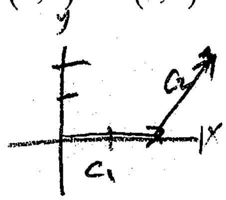
$4096 \int_{-1}^{1} u^4\, du$

$4096 \left(\frac{1}{5}\right) \left[u^5 \right]_{-1}^{1}$

$\frac{4096}{5} \left[1^5 - (-1)^5 \right]$

$= \boxed{\frac{8192}{5}}$

#4. Evaluate the line integral, where C is the given curve: $\int_C xy\,dx + (x-y)\,dy$ where C consists of line segments from $(0,0)$ to $(2,0)$ and from $(2,0)$ to $(3,2)$.



C_1: use x as parameter; C_2: use x as parameter

$\vec{r} = \langle t, 0 \rangle$ $\quad$ $\vec{r} = \langle t, 2t-4 \rangle$

$0 \le t \le 2$ $\quad$ $2 \le t \le 3$

$\frac{dx}{dt} = 1 \quad \frac{dy}{dt} = 0$ $\quad$ $\frac{dx}{dt} = 1 \quad \frac{dy}{dt} = 2$

$dx = dt \quad dy = 0$ $\quad$ $dx = dt \quad dy = 2dt$

$$\int_{C_1} xy\,dx + (x-y)\,dy + \int_{C_2} xy\,dx + (x-y)\,dy$$

$$\int_0^2 (t)(0)\,dt + (t-0)(0) +$$

$$+ \int_2^3 (t)(2t-4)\,dt + (t-(2t-4))\,2\,dt$$

$$(2t^2-4t) \quad (2t-4t+8)$$

$$\int_0^2 0\,dt + \int_2^3 (2t^2 - 6t + 8)\,dt$$

$$0 + \left[\tfrac{2}{3}t^3 - 3t^2 + 8t\right]_2^3$$

$$[0] + \left(\left[\tfrac{2}{3}(3)^3 - 3(3)^2 + 8(3)\right] - \left[\tfrac{2}{3}(2)^3 - 3(2)^2 + 8(2)\right]\right)$$

$$= 18 - 27 + 24 - \tfrac{16}{3} + 12 - 16$$

$$= \boxed{\tfrac{17}{3}}$$

#5. Evaluate the line integral, where C is the given curve: $\int_C x e^{yz}\,ds$ where C is the line segment from $(0,0,0)$ to $(1,2,3)$.

$$\vec{r}(t) = (1-t)\vec{r_0} + t\vec{r_1}$$
$$= (1-t)\langle 0,0,0 \rangle + t\langle 1,2,3 \rangle$$
$$= \langle 0,0,0 \rangle + \langle t, 2t, 3t \rangle \qquad 0 \le t \le 1$$
$$= \langle t, 2t, 3t \rangle$$

$$\vec{r}\,' = \langle 1,2,3 \rangle$$
$$|\vec{r}\,'| = \sqrt{1^2+2^2+3^2} = \sqrt{14}$$

$$\int_0^1 (t) e^{(2t)(3t)} \sqrt{14}\,dt$$

$$\sqrt{14}\int_0^1 t e^{6t^2}\,dt \qquad u = 6t^2$$

$$\frac{du}{dt} = 12t \qquad du = 12t\,dt \qquad t\,dt = \tfrac{1}{12}du$$

$t = 0 \to u = 0$

$t = 1 \to u = 6$

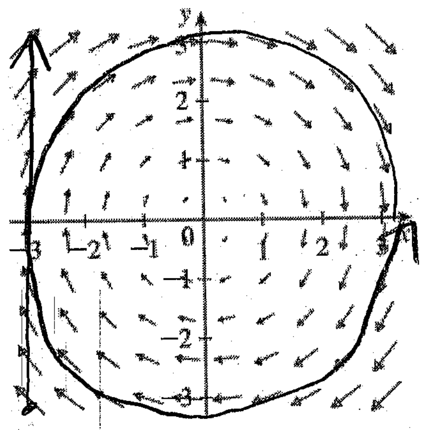
$$\frac{\sqrt{14}}{12}\int_0^6 e^u\,du$$

$$\frac{\sqrt{14}}{12}\left[e^u\right]_0^6 = \frac{\sqrt{14}}{12}\left[e^6 - e^0\right]$$

$$\boxed{\frac{\sqrt{14}}{12}(e^6 - 1)}$$

16.2 day 2

#1. Let $\vec{F}$ be the vector field shown in the figure.
(i) If C_1 is the vertical line segment from $(-3,\,-3)$ to $(-3,\,3)$, determine whether $\int_{C_1} \vec{F} \cdot d\vec{r}$ is positive, negative, or zero.
(ii) If C_2 is the counterclockwise-oriented circle with radius 3 and center at the origin, determine whether $\int_{C_2} \vec{F} \cdot d\vec{r}$ is positive, negative, or zero.



(i) $\int_{C_1} \vec{F} \cdot d\vec{r}$ would be $\boxed{\text{positive}}$
(arrows in direction of path)

(ii) $\int_{C_2} \vec{F} \cdot d\vec{r}$ would be $\boxed{\text{negative}}$
(arrows in opposite direction of path)

#2. Evaluate the line integral $\int_{C_1} \vec{F} \cdot d\vec{r}$ where C is given by the vector function $\vec{r}(t)$

$\vec{F}(x,y) = \langle xy,\ 3y^2 \rangle$

$\vec{r}(t) = \langle 11t^4,\ t^3 \rangle \quad 0 \le t \le 1$

$\vec{r}'(t) = \langle 44t^3, 3t^2 \rangle$

$\vec{F} = \langle (11t^4)(t^3),\ 3(t^3)^2 \rangle$

$\vec{F} = \langle 11t^7, 3t^6 \rangle$

$\vec{F} \cdot \vec{r}' = \langle 11t^7, 3t^6 \rangle \cdot \langle 44t^3, 3t^2 \rangle$

$= (11t^7)(44t^3) + (3t^6)(3t^2)$

$= 484t^{10} + 9t^8$

$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 (484t^{10} + 9t^8)\,dt$

$= \left[\frac{484}{11} t^{11} + t^9\right]_0^1$

$= \left(\frac{484}{11}(1)^{11} + (1)^9\right) - (0)$

$= \boxed{45}$

#3. Find the work done by the force field $\vec{F}(x,y) = \langle x\sin y,\ y \rangle$ on a particle that moves along the parabola $y = x^2$ from $(-1,1)$ to $(2,4)$.

Use x as parameter:

$\vec{r} = \langle t, t^2 \rangle \quad -1 \le t \le 2$

$\vec{r}\,' = \langle 1, 2t \rangle$

$\vec{F} = \langle (t)\sin(t^2), (t^2) \rangle$

$\vec{F} = \langle t\sin(t^2), t^2 \rangle$

$\vec{F} \cdot \vec{r}\,' = \langle t\sin(t^2), t^2 \rangle \cdot \langle 1, 2t \rangle$

$= (t\sin(t^2))(1) + (t^2)(2t)$

$= t\sin(t^2) + 2t^3$

$\int_{-1}^{2} (t\sin(t^2) + 2t^3)\,dt$

$\int_{-1}^{2} t\sin(t^2)\,dt + 2\int_{-1}^{2} t^3\,dt$

$u = t^2 \quad t=-1 \to u = 1$

$\frac{du}{dt} = 2t \quad t = 2 \to u = 4$

$du = 2t\,dt$

$t\,dt = \frac{1}{2}du$

$2\left(\frac{1}{4}\right)[t^4]_{-1}^{2}$

$\frac{1}{2}(2^4 - (-1)^4)$

$\frac{15}{2}$

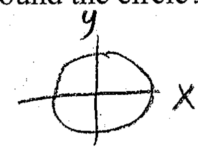
$\frac{1}{2}\int_{1}^{4} \sin u\,du$

$-\frac{1}{2}[\cos u]_{1}^{4}$

$-\frac{1}{2}(\cos 4 - \cos 1) + \frac{15}{2}$

$\boxed{= \frac{1}{2}[15 - \cos 4 + \cos 1]}$

#4. Show that a constant force field does zero work on a particle that moves once uniformly around the circle $x^2 + y^2 = 1$.



$\vec{r} = \langle \cos t, \sin t \rangle$

$0 \le t \le 2\pi$

$\vec{r}\,' = \langle -\sin t, \cos t \rangle$

$\vec{F}_{constant} = \langle a, b \rangle$

$\vec{F} \cdot \vec{r}\,' = \langle a, b \rangle \cdot \langle -\sin t, \cos t \rangle$

$= (a)(-\sin t) + (b)(\cos t)$

$= b\cos t - a\sin t$

$b\int_{0}^{2\pi} \cos t\,dt - a\int_{0}^{2\pi} \sin t\,dt$

$b[\sin t]_{0}^{2\pi} - a[-\cos t]_{0}^{2\pi}$

$b(\sin 2\pi - \sin 0) + a[\cos 2\pi - \cos 0]$

$b(0-0) + a(1-1)$

$= \boxed{0}$

16.3

#1. Determine whether or not $\vec{F}$ is a conservative vector field. If it is, find a function f such that $\vec{F} = \nabla f$.

(i) $\vec{F}(x,y) = \langle \overset{P}{2x-3y}, \overset{Q}{-3x+4y-8} \rangle$

$\frac{\partial P}{\partial y} = -3 \quad \frac{\partial Q}{\partial x} = -3 \quad =$ yes, conservative

$f_x = 2x-3y$

$f = \int (2x-3y)\,dx = x^2 - 3yx + g(y)$

$f_y = -3x + g'(y) \overset{\text{must}}{=} -3x+4y-8$

$g'(y) = 4y-8$

$g(y) = \int (4y-8)\,dy = 2y^2 - 8y + C$

$\boxed{f(x,y) = x^2 - 3xy + 2y^2 - 8y + C}$

(ii) $\vec{F}(x,y) = \langle \overset{P}{ye^x + \sin y}, \overset{Q}{e^x + x\cos y} \rangle$

$\frac{\partial P}{\partial y} = e^x + \cos y \quad \frac{\partial Q}{\partial x} = e^x + \cos y$

$=$ yes, conservative

$f_x = ye^x + \sin y$

$f = \int (ye^x + \sin y)\,dx = ye^x + x\sin y + g(y)$

$f_y = e^x + x\cos y + g'(y) \overset{\text{must}}{=} e^x + x\cos y$

$g'(y) = 0$

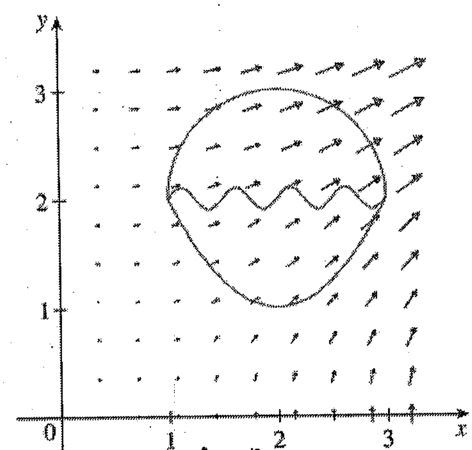
$g(y) = \int 0\,dy = 0 + C$

$\boxed{f(x,y) = ye^x + x\sin y + C}$

#2. The figure shows the vector field $\vec{F}(x,y) = \langle 2xy,\ x^2 \rangle$ and three curves that start at (1,2) and end at (3,2).

(i) Explain why $\int_C \vec{F} \cdot d\vec{r}$ has the same value for all three curves.

(ii) What is this common value?



(i) $\vec{F} = \langle \overset{P}{2xy}, \overset{Q}{x^2} \rangle \quad \frac{\partial P}{\partial y} = 2x, \quad \frac{\partial Q}{\partial x} = 2x$

field is conservative so line integrals are path independent.

(ii) $f_x = 2xy$

$f = \int 2xy\,dx = x^2y + g(y)$

$f_y = x^2 + g'(y) \overset{\text{must}}{=} x^2$

$g'(y) = 0 \quad g(y) = \int 0\,dy = C$

$f(x,y) = x^2y + C$

so $\int_C \vec{F} \cdot d\vec{r} = \left[x^2y\right]_{(1,2)}^{(3,2)}$

$((3)^2(2)) - ((1)^2(2))$

$18 - 2$

$= \boxed{16}$

#3. Find a function f such that $\vec{F} = \nabla f$ and use it to evaluate $\int_C \vec{F} \cdot d\vec{r}$ along the given curve C.

$\vec{F}(x,y) = \langle xy^2,\ x^2 y \rangle$

$C:\ \vec{r}(t) = \left\langle t + \sin\left(\frac{\pi}{2}t\right),\ t + \cos\left(\frac{\pi}{2}t\right) \right\rangle \quad 0 \le t \le 1$

conservative? $\frac{\partial P}{\partial y} = 2xy \qquad \frac{\partial Q}{\partial x} = 2xy$

yes

$f_x = xy^2$

$f = \int xy^2\,dx = \frac{1}{2}x^2y^2 + g(y)$

$f_y = x^2y + g'(y) \overset{\text{must}}{=} x^2y$

$g'(y) = 0 \quad g(y) = \int 0\,dy = C$

$f(x,y) = \frac{1}{2}x^2y + C$

endpoints:

$\vec{r}(0) = \langle 0 + \sin(0),\ 0 + \cos(0) \rangle$

$= \langle 0, 1 \rangle \quad (0,1)$

$\vec{r}(1) = \langle 1 + \sin\frac{\pi}{2},\ 1 + \cos\frac{\pi}{2} \rangle$

$= \langle 2, 1 \rangle \quad (2,1)$

$\int_C \vec{F} \cdot d\vec{r} = \left[\frac{1}{2}x^2y\right]_{(0,1)}^{(2,1)}$

any path

$\left[\frac{1}{2}(2)^2(1)\right] - \left(\frac{1}{2}(0)^2(1)\right)$

$2 - 0$

$= \boxed{2}$

#4. Show that the line integral is independent of path and evaluate the integral.

$\int_C \overbrace{\tan y\,dx}^{P} + \overbrace{x\sec^2 y\,dy}^{Q}$

C is any path from $(1,0)$ to $\left(2, \frac{\pi}{4}\right)$

$\frac{\partial P}{\partial y} = \sec^2 y \qquad \frac{\partial Q}{\partial x} = \sec^2 y \quad \Rightarrow$, so conservative

$f_x = \tan y$

$f = \int \tan y\,dx = x\tan y + g(y)$

$f_y = x\sec^2 y + g'(y) \overset{\text{must}}{=} x\sec^2 y$

$g'(y) = 0 \quad g(y) = \int 0\,dy = C$

$f(x,y) = x\tan y + C$

so $\int_C \tan y\,dx + x\sec^2 y\,dy$ (any path)

$= \left[x\tan y\right]_{(1,0)}^{(2,\pi/4)}$

$\left[(2)\tan\left(\frac{\pi}{4}\right)\right] - \left[(1)\tan(0)\right]$

$2(1) - 1(0)$

$= \boxed{2}$

#5. Find the work done by the force field $\vec{F}$ in moving an object from P to Q.

$\vec{F}(x,y) = \langle 2y^{3/2},\ 3x\sqrt{y} \rangle$

$P(1,1)$, $Q(2,4)$

conservative? $\frac{\partial P}{\partial y} = 3y^{1/2}$ $\frac{\partial Q}{\partial x} = 3\sqrt{y}$

$=$, yes

$f_x = 2y^{3/2}$

$f = \int 2y^{3/2}\,dx = 2xy^{3/2} + g(y)$

$f_y = 3\sqrt{y}\,x + g'(y) \overset{\text{must}}{=} 3x\sqrt{y}$

$g'(y) = 0$, $g(y) = \int 0\,dy = c$

$f(x,y) = 2xy^{3/2} + c$

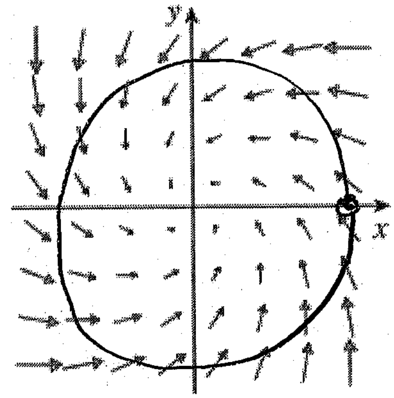
$W = \int_C \vec{F} \cdot d\vec{r} = \left[2xy^{3/2}\right]_{(1,1)}^{(2,4)}$

$\left[2(2)(4)^{3/2}\right] - \left[2(1)(1)^{3/2}\right]$

$32 - 2$

$= \boxed{30}$

#6. Is the vector field shown in the figure conservative? Explain.



try a closed path...
always in direction of arrows
so $\oint_C \vec{F} \cdot d\vec{r} \neq 0$ (must be positive)

therefore
this field is not conservative.

16.4

#1. Evaluate the line integral (i) directly and (ii) using Green's Theorem.

$$\oint_C (x-y)\,dx + (x+y)\,dy \qquad \vec{F} = \langle \underset{P}{x-y},\ \underset{Q}{x+y} \rangle$$

C is the circle with center at the origin, radius 2.

conservative? $\frac{\partial P}{\partial y} = -1 \quad \frac{\partial Q}{\partial x} = 1 \quad \neq$ (no)

(i) <u>directly</u> $\vec{r} = \langle 2\cos t, 2\sin t \rangle$

$0 \le t \le 2\pi$

$\vec{r}\,' = \langle -2\sin t, 2\cos t \rangle$

$\vec{F}(r) = \langle 2\cos t - 2\sin t,\ 2\cos t + 2\sin t \rangle$

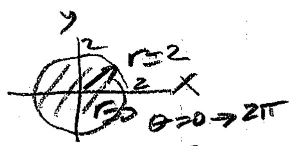
$\vec{F} \cdot \vec{r}\,' = (2\cos t - 2\sin t)(-2\sin t) + (2\cos t + 2\sin t)(2\cos t)$

$= \cancel{-4\cos t\sin t} + 4\sin^2 t + 4\cos^2 t + \cancel{4\sin t\cos t}$

$= 4(\sin^2 t + \cos^2 t) = 4$

$\int_0^{2\pi} (4)\,dt = 4[t]_0^{2\pi} = \boxed{8\pi}$

(ii) <u>Green's</u>



$\iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_0^{2\pi}\int_0^{2} (1-(-1))\,r\,dr\,d\theta$

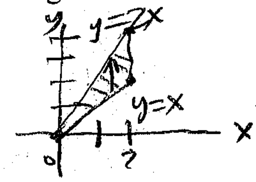
$\int_0^2 2r\,dr = [r^2]_0^2 = [2^2 - 0^2] = 4$

$\int_0^{2\pi} 4\,d\theta = 4[\theta]_0^{2\pi} = \boxed{8\pi}$

#2. Use Green's Theorem to evaluate the line integral along the given positively oriented curve.

$$\oint_C xy^2\,dx + 2x^2y\,dy \qquad \vec{F} = \langle \underset{P}{xy^2},\ \underset{Q}{2x^2y} \rangle$$

C is the triangle with vertices (0,0), (2,2) and (2,4).



conservative? $\frac{\partial P}{\partial y} = 2xy, \quad \frac{\partial Q}{\partial x} = 4xy \quad \neq$, no

$= \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$

$\int_0^2 \int_x^{2x} (4xy - 2xy)\,dy\,dx$

$\int_0^2 \int_x^{2x} 2xy\,dy\,dx$

$\int_x^{2x} 2xy\,dy = x[y^2]_x^{2x}$

$= x((2x)^2 - (x)^2) = x(4x^2 - x^2)$

$= 3x^3$

$\int_0^2 3x^3\,dx = \frac{3}{4}[x^4]_0^2$

$= \frac{3}{4}((2)^4 - (0)^4)$

$= \boxed{12}$

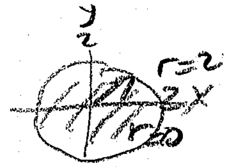
#3. Use Green's Theorem to evaluate the line integral along the given positively oriented curve.

$\oint_C y^3\,dx - x^3\,dy$ $\vec{F} = \langle y^3, -x^3 \rangle$

C is the circle $x^2 + y^2 = 4$.

conservative? $\frac{\partial f}{\partial y} = 3y^2$ $\frac{\partial g}{\partial x} = -3x^2$

$\neq$, no



$\iint \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA$

$\int_0^{2\pi}\int_0^2 (-3x^2 - 3y^2)\, r\,dr\,d\theta$ $-3x^2-3y^2$

to polar $-3(x^2+y^2)$

$-3r^2$

$\int_0^{2\pi}\int_0^2 (-3r^2)\, r\,dr\,d\theta$

$-3\int_0^2 r^3\,dr = -\frac{3}{4}\left[r^4\right]_0^2 = -\frac{3}{4}\left[2^4 - 0^4\right]$

$= -12$

$\int_0^{2\pi} -12\,d\theta = -12\left[\theta\right]_0^{2\pi} = \boxed{-24\pi}$

"positively oriented curve" means path is counterclockwise around circle.

$\ominus 24\pi$ means overall opposing direction of force arrow on this path

#4. Use Green's Theorem to evaluate $\int_C \vec{F} \cdot d\vec{r}$.

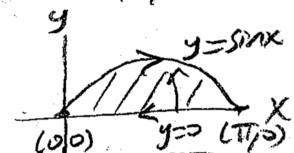
(Check the orientation of the curve before applying the theorem)

$\vec{F}(x,y) = \left\langle \sqrt{x} + y^3,\ x^2 + \sqrt{y} \right\rangle$

C consists of the arc of the curve $y = \sin x$ from $(0,0)$ to $(\pi,0)$ and the line segment from $(\pi,0)$ to $(0,0)$. closed path, conservative?

$\frac{\partial f}{\partial y} = 3y^2$, $\frac{\partial g}{\partial x} = 2x$ $\neq$, no

$\iint \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA$



(– direction)

$\int_0^{\pi}\int_0^{\sin x} (2x - 3y^2)\,dy\,dx$

$\int_0^{\sin x} (2x - 3y^2)\,dy$

$= \left[2xy - y^3\right]_0^{\sin x}$

$= (2x(\sin x) - (\sin x)^3) - (0)$

$2\int_0^{\pi} x\sin x\,dx - \int_0^{\pi} \sin^3 x\,dx$

by parts:

$u = x$ $dv = \sin x\,dx$

$\frac{du}{dx} = 1$ $\int dv = \int \sin x\,dx$

$du = dx$ $v = -\cos x$

$-x\cos x + \int \cos x\,dx$

$2\left[-x\cos x + \sin x\right]_0^{\pi}$

$\int_0^{\pi} \sin^2 x \sin x\,dx$

$\int_0^{\pi} (1 - \cos^2 x)\sin x\,dx$

$\int_0^{\pi} \sin x\,dx - \int_0^{\pi} \cos^2 x \sin x\,dx$

$u = \cos x$

$\frac{du}{dx} = -\sin x$

$+\int_1^{-1} u^2\,du$

$2\left[-x\cos x + \sin x\right]_0^{\pi} - \left[-\cos x\right]_0^{\pi} + \left(\frac{1}{3}u^3\right]_1^{-1}$

$(-2\pi\cos\pi + 2\sin\pi) + (\cos\pi - \cos 0) - \left[\frac{1}{3}(-1)^3 - \frac{1}{3}(1)^3\right)$

$-(-2(0)\cos 0 + 2\sin 0)$

$(2\pi) + (-1-1) - \frac{1}{3}(-1-1)$

$2\pi - \frac{4}{3}$ but – curve orientation, so...

$\boxed{\frac{4}{3} - 2\pi}$

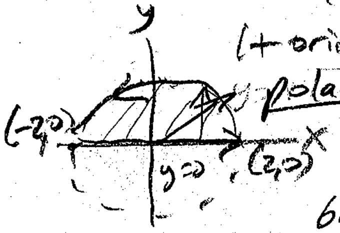
#5. A particle starts at the point $(-2,0)$, moves along the x-axis to $(2,0)$, and then along the semicircle $y=\sqrt{4-x^2}$ to the starting point. Use Green's theorem to find the work done on this particle by the force field
$\vec{F}(x,y)=\langle x,\ x^3+3xy^2\rangle$.

$\vec{F}$ conservative? $\frac{\partial P}{\partial y}=0$ $\quad \frac{\partial Q}{\partial x}=3x^2+3y^2$

$\neq$, no

(+ orientation)

polar



Green's

$W=\int_C \vec{F}\cdot\vec{r}\,'dr = \oint\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dA$

integrand: $3x^2+3y^2-0=3(x^2+y^2)=3r^2$

$\int_0^{\pi}\int_0^{2}(3r^2)\,r\,dr\,d\theta$

$3\int_0^2 r^3\,dr=\frac{3}{4}\left[r^4\right]_0^2=\frac{3}{4}(2^4-0)=12$

$\int_0^{\pi}12\,d\theta=12(\theta)_0^{\pi}=\boxed{12\pi}$

16.5

#1. Find (i) the curl and (ii) the divergence of the vector field $\vec{F}(x,y,z)=\langle xyz,\ 0,\ -x^2y\rangle$

(i) $\operatorname{curl}\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} + & - & + \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & 0 & -x^2y \end{vmatrix}$

$= \langle -x^2-0,\ -(-2xy-xy),\ 0-xz\rangle$

$= \langle -x^2,\ 2xy+xy,\ -xz\rangle$

$= \boxed{\langle -x^2,\ 3xy,\ -xz\rangle}$

(ii) $\operatorname{div}\vec{F} = \nabla\cdot\vec{F}$

$= \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\rangle \cdot \langle xyz, 0, -x^2y\rangle$

$= \frac{\partial}{\partial x}[xyz] + \frac{\partial}{\partial y}[0] + \frac{\partial}{\partial z}[-x^2y]$

$= yz + 0 + 0$

$= \boxed{yz}$

#2. Find (i) the curl and (ii) the divergence of the vector field $\vec{F}(x,y,z)=\langle \ln x,\ \ln(xy),\ \ln(xyz)\rangle$

(i) $\operatorname{curl}\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} + & - & + \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \ln(x) & \ln(xy) & \ln(xyz) \end{vmatrix}$

$= \langle \frac{1}{xyz}(xz) - \frac{1}{xy}0,\ -\left(\frac{1}{xyz}(yz) - 0\right),\ \frac{1}{xy}(y) - 0\rangle$

$= \langle \frac{xz}{xyz},\ \frac{-yz}{xyz},\ \frac{y}{xy}\rangle$

$= \boxed{\langle \frac{1}{y},\ \frac{-1}{x},\ \frac{1}{x}\rangle}$

(ii) $\operatorname{div}\vec{F} = \nabla\cdot\vec{F}$

$\frac{\partial}{\partial x}[\ln x] + \frac{\partial}{\partial y}[\ln(xy)] + \frac{\partial}{\partial z}[\ln(xyz)]$

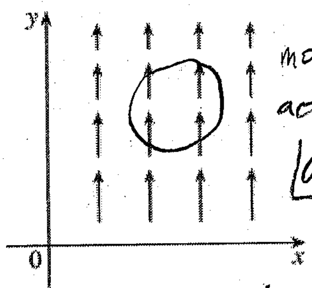
$\frac{1}{x} + \frac{1}{xy}(x) + \frac{1}{xyz}(xy)$

$= \boxed{\frac{1}{x} + \frac{1}{y} + \frac{1}{z}}$

#3. The vector field $\vec{F}$ is shown in the xy-plane and looks the same in all other horizontal planes (its z-component is zero).

(i) Is $div\,\vec{F}$ positive, negative, or zero? Explain.

(ii) Determine whether $curl\,\vec{F} = \vec{0}$. If not, in which direction does $curl\,\vec{F}$ point?



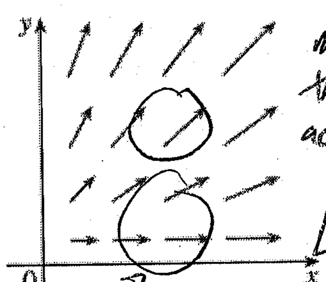
more going in than out
acting as a sink
$\boxed{div \vec{F} < 0}$

No tendency to rotate,
$\boxed{curl \vec{F} = \vec{0}}$

#4. The vector field $\vec{F}$ is shown in the xy-plane and looks the same in all other horizontal planes (its z-component is zero).

(i) Is $div\,\vec{F}$ positive, negative, or zero? Explain.

(ii) Determine whether $curl\,\vec{F} = \vec{0}$. If not, in which direction does $curl\,\vec{F}$ point?



More going out than in,
acting as a source
$\boxed{div \vec{F} > 0}$

no motion in ← direction
so no rotation
$\boxed{curl \vec{F} = \vec{0}}$

#5. Let f be a scalar field and $\vec{F}$ a vector field. State whether each expression is meaningful. If not, explain why. If so, state whether it is a scalar field or a vector field.

(i) $curl\,\vec{F}$ meaningful = $\boxed{\text{vector}}$ ($\perp$ to $\vec{F}$, axis of rotation)

(ii) $div\,\vec{F}$ meaningful = $\boxed{\text{scalar}}$ (source/sink)

(iii) $\nabla\vec{F}$ $\boxed{\text{not meaningful}}$
gradient is taken on scalar fields:
$\nabla f = \langle f_x, f_y, f_z \rangle$

(iv) $div\left(\nabla f\right)$ meaningful
div (vector) = $\boxed{\text{scalar}}$

(v) $curl\left(curl\,\vec{F}\right)$ meaningful
curl (vector) = $\boxed{\text{vector}}$

(vi) $(\nabla f) x \left(div\,\vec{F}\right)$
(vector) x not possible
Can't take div of vector field
$\boxed{\text{Not meaningful}}$

16.6 day 1

#1. Determine whether the points P and Q lie on the given surface.

$\vec{r}(u,v) = \langle 2u+3v,\ 1+5u-v,\ 2+u+v \rangle$

$P(7,10,4)$, $Q(5,22,5)$

P: $\begin{cases} 2u+3v=7 \\ 1+5u-v=10 \\ 2+u+v=4 \end{cases} \rightarrow \begin{cases} 2u+3v=7 \\ 5u-v=9 \\ u+v=2 \end{cases}$

$\left[\begin{array}{cc|c} 2 & 3 & 7 \\ 5 & -1 & 9 \\ 1 & 1 & 2 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$ $u=0$, $v=0$, $0=1$

0 would have to equal 1

so system has no solution

there is no u,v where $\vec{r}(u,v) = \langle 7,10,4 \rangle$

so $\boxed{P \text{ is } \underline{\text{not}} \text{ on this surface}}$

Q: $\begin{cases} 2u+3v=5 \\ 1+5u-v=22 \\ 2+u+v=5 \end{cases} \rightarrow \begin{cases} 2u+3v=5 \\ 5u-v=21 \\ u+v=3 \end{cases}$

$\left[\begin{array}{cc|c} 2 & 3 & 5 \\ 5 & -1 & 21 \\ 1 & 1 & 3 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right]$ $u=4$, $v=-1$, $0=0$ ok :)

$\vec{r}(4,-1) = \langle 5,22,5 \rangle$

so $\boxed{Q \text{ } \underline{\text{is}} \text{ on this surface}}$

#2. Identify the surface with the given vector equation.

$\vec{r}(u,v) = \langle \underline{u+v},\ \underline{3-v},\ \underline{1+4u+5v} \rangle$

all linear terms,

this is a $\boxed{\text{plane}}$

more detail... let's find 3 points on the plane by selecting values u,v. (choose any u,v):

$\vec{r}(0,0) = \langle 0+0, 3-0, 1+4(0)+5(0) \rangle = \langle 0,3,1 \rangle$

$\vec{r}(0,1) = \langle 0+1, 3-1, 1+4(0)+5(1) \rangle = \langle 1,2,6 \rangle$

$\vec{r}(1,0) = \langle 1+0, 3-0, 1+4(1)+5(0) \rangle = \langle 1,3,5 \rangle$

make 2 vectors on the plane:

$\vec{v_1} = \langle 1-0, 2-3, 6-1 \rangle = \langle 1,-1,5 \rangle$

$\vec{v_2} = \langle 1-0, 3-3, 5-1 \rangle = \langle 1,0,4 \rangle$

$\vec{n} = \vec{v_1} \times \vec{v_2} = \begin{vmatrix} + & - & + \\ 1 & -1 & 5 \\ 1 & 0 & 4 \end{vmatrix}$

$\vec{n} = \langle -4-0, -(4-5), 0+1 \rangle = \langle -4, 1, 1 \rangle$

$\vec{r_0} = \langle 0,3,1 \rangle$

$ax+by+cz = \vec{n} \cdot \vec{r_0}$

$-4x+1y+1z = \langle -4,1,1 \rangle \cdot \langle 0,3,1 \rangle$

$= (-4)(0)+(1)(3)+(1)(1)$

$\boxed{-4x+y+z=4}$ ⟸ this specific plane

#3. Find a parametric representation for the surface: the plane that passes through the point $(1,2,-3)$ and contains the vectors $\langle 1,1,-1\rangle$ and $\langle 1,-1,1\rangle$.

equation of plane:

$$\vec{n} = \langle 1,1,-1\rangle \times \langle 1,-1,1\rangle = \begin{vmatrix} + & - & + \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix}$$

$$\vec{n} = \langle 1-1, -(1+1), -1-1\rangle = \langle 0,-2,-2\rangle$$

$$\vec{r_0} = \langle 1,2,-3\rangle$$

$$ax+by+cz = \vec{n}\cdot\vec{r_0}$$

$$0x-2y-2z = \langle 0,-2,-2\rangle \cdot \langle 1,2,-3\rangle$$

$$= (0)(1)+(-2)(2)+(-2)(-3)$$

$$-2y-2z = 2$$

$$y+z=-1 \quad \text{or} \quad z=-1-y$$

to parametrize, use $x=u$

$y=v$

then $z=-1-y$

$z=-1-v$

$$\boxed{\vec{r}(u,v) = \langle u, v, -1-v\rangle \quad -\infty \le u \le \infty,\ -\infty \le v \le -\infty}$$

#4. Find a parametric representation for the surface: the lower half of the ellipsoid $2x^2+4y^2+z^2=1$

$$z^2 = 1-2x^2-4y^2$$

$$z = \pm\sqrt{1-2x^2-4y^2}$$

lower half: $z = -\sqrt{1-2x^2-4y^2}$

use x & y as parameters:

$x=u$

$y=v$

$z=-\sqrt{1-2u^2-4v^2}$

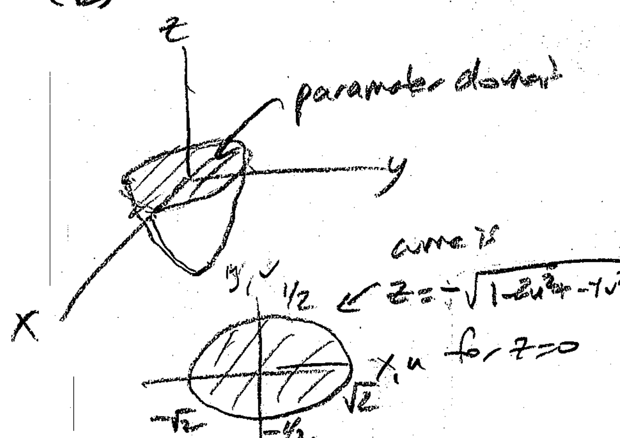
$$\boxed{\vec{r}(u,v) = \langle u, v, -\sqrt{1-2u^2-4v^2}\rangle}$$

to get ranges for the parameter surfaces other than planes you need to sketch:

$$2x^2+4y^2+z^2=1$$

$$\frac{x^2}{(\frac{1}{2})} + \frac{y^2}{(\frac{1}{4})} + \frac{z^2}{(1)} = 1$$

$$\frac{x^2}{(\frac{1}{\sqrt{2}})^2} + \frac{y^2}{(\frac{1}{2})^2} + \frac{z^2}{(1)^2} = 1$$



$$-\sqrt{1-2u^2-4v^2} = 0$$

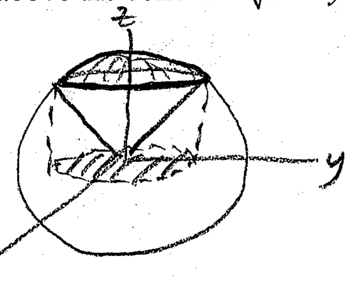
$$1-2u^2-4v^2=0$$

$$2u^2+4v^2=1$$

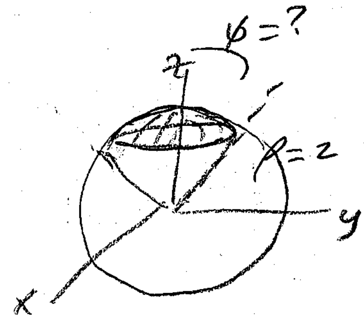
$$\boxed{D = \{(u,v) \mid 2u^2+4v^2 \le 1\}}$$

↑ don't need to include unless specifically asked for

#5. Find a parametric representation for the surface: the part of the sphere $x^2+y^2+z^2=4$ that lies above the cone $z=\sqrt{x^2+y^2}$.



—or—



could use x, y as parameters...

$x = u$

$y = v$

$z = \sqrt{u^2+v^2}$

$$\boxed{\begin{array}{l}\vec{r}(u,v) = \langle u, v, \sqrt{u^2+v^2}\rangle \\ \text{or could write as} \\ \vec{r}(x,y) = \langle x, y, \sqrt{x^2+y^2}\rangle\end{array}}$$

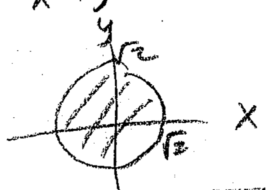
for parameter domain, set by intersection:

$$\begin{cases} x^2+y^2+z^2=4 \\ z=\sqrt{x^2+y^2}\end{cases}$$

$x^2+y^2+(\sqrt{x^2+y^2})^2=4$

$2x^2+2y^2=4$

$x^2+y^2=2$



$$\boxed{D=\{(x,y)\mid x^2+y^2\le 2\}}$$

Could use spherical coordinates, since this is part of a sphere...

to find ϕ, intersection $\begin{cases} x^2+y^2+z^2=4 \\ z=\sqrt{x^2+y^2}\end{cases}$

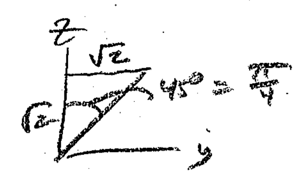
$x^2+y^2+(\sqrt{x^2+y^2})^2=4$

$2x^2+2y^2=4$

$x^2+y^2=2$

this occurs at $z=\sqrt{x^2+y^2}=\sqrt{2}$

and radius of circle $=\sqrt{2}$



then the parameters are ϕ & θ: (with $\rho=2$ constant on the surface)

$x = 2\sin\phi\cos\theta$

$y = 2\sin\phi\sin\theta$

$z = 2\cos\phi$

$$\boxed{\vec{r}(u,v)=\vec{r}(\phi,\theta)=\langle 2\sin\phi\cos\theta, 2\sin\phi\sin\theta, 2\cos\phi\rangle}$$

parameter domain:

$$\boxed{0\le\phi\le\frac{\pi}{4},\quad 0\le\theta\le 2\pi}$$

16.6 day 2

#1. Find an equation of the tangent plane to the given parametric surface at the specified point.

$\vec{r}(u,v) = \langle u+v,\ 3u^2,\ u-v\rangle$

$(2, 3, 0)$ $\quad 3u^2 = 3 \quad \begin{cases} u+v=2 \\ u-v=0 \end{cases}$

$u^2 = 1 \qquad 2u = 2$

$u = \pm 1 \qquad u = 1,\ v = 1$

$\vec{r}_u = \langle 1, 6u, 1\rangle = \langle 1, 6, 1\rangle$

$\vec{r}_v = \langle 1, 0, -1\rangle = \langle 1, 0, -1\rangle$

$(1,1)$

$\vec{n} = \vec{r}_u \times \vec{r}_v = \begin{vmatrix} + & - & + \\ 1 & 6 & 1 \\ 1 & 0 & -1 \end{vmatrix}$

$= \langle -6-0, -(-1-1), 0-6\rangle$

$= \langle -6, 2, -6\rangle$

$\vec{r}_0 = \langle 2, 3, 0\rangle$

$ax + by + cz = \vec{n}\cdot\vec{r}_0$

$-6x + 2y - 6z = \langle -6, 2, -6\rangle \cdot \langle 2, 3, 0\rangle$

$= (-6)(2) + (2)(3) + (-6)(0)$

$\cancel{-6x + 2y - 6z} = -6$

$\boxed{3x - y + 3z = 3}$

#2. Find the area of the surface: the part of the plane $3x + 2y + z = 6$ that lies in the first octant.

$A = \iint_D |\vec{r}_u \times \vec{r}_v|\, dA$

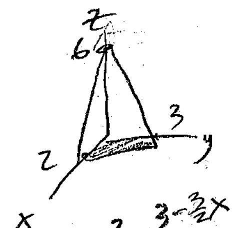
parametrize surface: $\vec{r}(x,y) = \langle x, y, 6-3x-2y\rangle$

$\vec{r}_u = \vec{r}_x = \langle 1, 0, -3\rangle$

$\vec{r}_v = \vec{r}_y = \langle 0, 1, -2\rangle$

$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} + & - & + \\ 1 & 0 & -3 \\ 0 & 1 & -2 \end{vmatrix} = \langle 0+3, -(-2-0), 1-0\rangle = \langle 3, 2, 1\rangle$

$|\vec{r}_u \times \vec{r}_v| = \sqrt{3^2+2^2+1^2} = \sqrt{14}$



parameter domain:

rectangular

$y = 0$ to $y = 3 - \frac{3}{2}x$

$x = 0$ to 2

$\int_0^2 \int_0^{3-\frac{3}{2}x} \sqrt{14}\, dy\, dx \qquad \int_0^{3-3/2x} \sqrt{14}\, dy = \sqrt{14}\,[y]_0^{3-\frac{3}{2}x}$

$= \sqrt{14}(3 - \frac{3}{2}x) - (0)$

$3\sqrt{14}\int_0^2 (1 - \frac{1}{2}x)\, dx = 3\sqrt{14}\left[x - \frac{1}{4}x^2\right]_0^2$

$= 3\sqrt{14}\left(2 - \frac{1}{4}(2^2)\right) = \boxed{3\sqrt{14}}$

-or- $A = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$ (simpler formula)

$\frac{\partial z}{\partial x} = -3, \quad \frac{\partial z}{\partial y} = -2$

$A = \int_0^2 \int_0^{3-\frac{3}{2}x} \sqrt{1 + (-3)^2 + (-2)^2}\, dy\, dx$

$\int_0^2 \int_0^{3-\frac{3}{2}x} \sqrt{14}\, dy\, dx \ldots = \boxed{3\sqrt{14}}$

#3. Find the area of the surface: the part of the surface $z = xy$ that lies within the cylinder $x^2 + y^2 = 1$.

using: $\iint_D |\vec{r_u} \times \vec{r_v}|\, dA$

Surface: $\vec{r} = \langle x, y, xy \rangle$

$\vec{r_u} = \vec{r_x} = \langle 1, 0, y \rangle$, $\vec{r_v} = \vec{r_y} = \langle 0, 1, x \rangle$

$$\vec{r_x} \times \vec{r_y} = \begin{vmatrix} + & - & + \\ 1 & 0 & y \\ 0 & 1 & x \end{vmatrix}$$

$= \langle 0-y, -(x-0), 1-0 \rangle = \langle -y, -x, 1 \rangle$

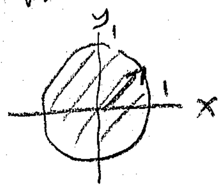
$|\vec{r_x} \times \vec{r_y}| = \sqrt{y^2+x^2+1} = \sqrt{r^2+1}$

parameter domain:

polar

$r = 0$ to $r = 1$

$\theta = 0$ to $\theta = 2\pi$



$$A = \int_0^{2\pi} \int_0^1 \sqrt{r^2+1}\, r\,dr\,d\theta$$

$$\int_0^1 \sqrt{r^2+1}\, r\,dr \qquad u = r^2+1 \quad \frac{du}{dr} = 2r \quad r\,dr = \tfrac{1}{2}du \qquad r=0 \to u=1 \quad r=1 \to u=2$$

$$\frac{1}{2}\int_1^2 u^{1/2}\,du = \frac{1}{2}\left(\frac{2}{3}\right)\left[u^{3/2}\right]_1^2 = \frac{1}{3}\left[2^{3/2} - 1^{3/2}\right]$$

$$= \frac{2\sqrt{2}-1}{3}$$

$$\int_0^{2\pi} \frac{1}{3}(2\sqrt{2}-1)\,d\theta = \frac{2\sqrt{2}-1}{3}\left[\theta\right]_0^{2\pi} = \boxed{\frac{2\pi}{3}(2\sqrt{2}-1)}$$

using $\iint \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$

$\frac{\partial z}{\partial x} = y \quad \frac{\partial z}{\partial y} = x$

Integrand: $\sqrt{1+y^2+x^2} = \sqrt{1+r^2}$

(same integral)

#4. Find the area of the surface: the part of the hyperbolic paraboloid $z = y^2 - x^2$ that lies between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

parameter domain

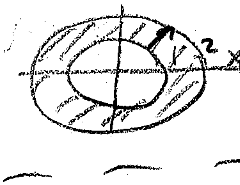
surface: $\vec{r} = \langle x, y, y^2 - x^2 \rangle$

parameter domain:

polar:

$r = 1$ to $r = 2$

$\theta = 0$ to $\theta = 2\pi$



using $\iint_D |\vec{r_u} \times \vec{r_v}|\, dA$

$\vec{r_u} = \vec{r_x} = \langle 1, 0, -2x \rangle \quad \vec{r_v} = \vec{r_y} = \langle 0, 1, 2y \rangle$

$$\vec{r_x} \times \vec{r_y} = \begin{vmatrix} + & - & + \\ 1 & 0 & -2x \\ 0 & 1 & 2y \end{vmatrix} = \langle 0+2x, -(2y-0), 1-0 \rangle = \langle 2x, -2y, 1 \rangle$$

$|\vec{r_u} \times \vec{r_v}| = \sqrt{4x^2+4y^2+1} = \sqrt{4r^2+1}$

$$\int_0^{2\pi} \int_1^2 \sqrt{4r^2+1}\, r\,dr\,d\theta$$

— or —

using $\iint \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$

$\frac{\partial z}{\partial x} = -2x, \quad \frac{\partial z}{\partial y} = 2y$

Integrand: $\sqrt{1+(-2x)^2+(2y)^2} = \sqrt{4r^2+1}$

$$\int_0^{2\pi} \int_1^2 \sqrt{4r^2+1}\, r\,dr\,d\theta$$

$$\int_1^2 \sqrt{4r^2+1}\, r\,dr \qquad u = 4r^2+1 \quad \frac{du}{dr} = 8r \quad r\,dr = \tfrac{1}{8}du \qquad r=1 \to u=5 \quad r=2 \to u=17$$

$$\frac{1}{8}\int_5^{17} u^{1/2}\,du = \frac{1}{8}\left(\frac{2}{3}\right)\left[u^{3/2}\right]_5^{17}$$

$$= \frac{1}{12}\left[17^{3/2} - 5^{3/2}\right] = \frac{1}{12}\left(17\sqrt{17} - 5\sqrt{5}\right)$$

$$\int_0^{2\pi} \frac{1}{12}\left(17\sqrt{17} - 5\sqrt{5}\right)d\theta = \frac{1}{12}\left(17\sqrt{17} - 5\sqrt{5}\right)\left[\theta\right]_0^{2\pi}$$

$$= \boxed{\frac{\pi}{6}\left(17\sqrt{17} - 5\sqrt{5}\right)}$$

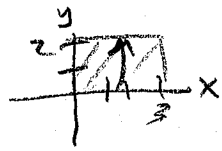
16.7 day 1

#1. Evaluate the surface integral $\iint_S x^2yz\,dS$ (scalar)

S is the part of the plane $z = 1+2x+3y$ that lies above the rectangle $0 \le x \le 3$, $0 \le y \le 2$.

surface: $\vec{r} = \langle x, y, 1+2x+3y \rangle$

parameter domain:
rectangular
$y=0$ to $y=2$
$x=0$ to $x=3$



choosing $\iint f(x,y,z)\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA$

$\frac{\partial z}{\partial x} = 2, \quad \frac{\partial z}{\partial y} = 3$

$$\int_0^3\int_0^2 x^2y(1+2x+3y)\sqrt{1+(2)^2+(3)^2}\,dy\,dx$$

$$\sqrt{14}\int_0^3\int_0^2 (x^2y+2x^3y+3x^2y^2)\,dy\,dx$$

$$\int_0^2 (x^2y+2x^3y+3x^2y^2)\,dy = \left[\tfrac{1}{2}x^2y^2+x^3y^2+x^2y^3\right]_0^2$$
$$= \left(\tfrac{1}{2}x^2(2)^2+x^3(2)^2+x^2(2)^3\right)-[0] = 2x^2+4x^3+8x^2 = 10x^2+4x^3$$

$$\sqrt{14}\int_0^3 (10x^2+4x^3)\,dx = \sqrt{14}\left[\tfrac{10}{3}x^3+x^4\right]_0^3$$
$$= \sqrt{14}\left(\tfrac{10}{3}(3)^3+(3)^4\right)-[0]$$
$$= \boxed{171\sqrt{14}}$$

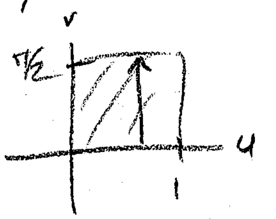
#2. Evaluate the surface integral $\iint_S yz\, dS$ (scalar)

S is the surface with parametric equations
$x = u^2, \quad y = u\sin v, \quad z = u\cos v$

$0 \le u \le 1, \quad 0 \le v \le \dfrac{\pi}{2}.$

Surface: $\vec{r} = \langle u^2, u\sin v, u\cos v \rangle$

parameter domain:



rectangular
$v=0$ to $v=\pi/2$
$u=0$ to $u=1$

must use $\iint f(x,y,z)\, |\vec{r}_u \times \vec{r}_v|\, dA$ $\quad \vec{r}_u = \langle 2u, \sin v, \cos v \rangle, \quad \vec{r}_v = \langle 0, u\cos v, -u\sin v \rangle$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} + & - & + \\ 2u & \sin v & \cos v \\ 0 & u\cos v & -u\sin v \end{vmatrix} = \langle -u\sin^2 v - u\cos^2 v, -(-2u^2\sin v - 0), 2u^2\cos v - 0 \rangle$$
$$= \langle -u, 2u^2\sin v, 2u^2\cos v \rangle$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{u^2 + 4u^4\cos^2 v + 4u^4\sin^2 v} = \sqrt{u^2 + 4u^4} = \sqrt{u^2}\sqrt{1+4u^2} = u\sqrt{1+4u^2}$$

$$\int_0^1 \int_0^{\pi/2} (u\sin v)(u\cos v)\, u\sqrt{1+4u^2}\, dv\, du = \int_0^1 u^3\sqrt{1+4u^2}\, du \int_0^{\pi/2} \sin v\cos v\, dv$$

$h = \sin v$
$\frac{dh}{dv} = \cos v$
$\cos v\, dv = dh$
$v=0 \to h=0$
$v=\pi/2 \to h=1$

$g = 1+4u^2 \quad g=0 \to u=1$
$\frac{dg}{du} = 8u \quad g=1 \to u=5$
$u\,du = \frac{1}{8}dg$
$u^2 = \frac{1}{4}(g-1)$

$\frac{1}{8}\int_1^5 \frac{1}{4}(g-1)g^{1/2}\, dg \qquad \int_0^1 h\, dh$

$\frac{1}{32}\int_1^5 (g^{3/2} - g^{1/2})\, dg \qquad \frac{1}{2}[h^2]_0^1$

$\left(\frac{1}{32}\left[\frac{2}{5}g^{5/2} - \frac{2}{3}g^{3/2}\right]_1^5\right)\left(\frac{1}{2}[1^2 - 0]\right)$

$\left(\frac{1}{32}\left[\frac{2}{5}(5)^{5/2} - \frac{2}{3}(5)^{3/2}\right] - \frac{1}{32}\left[\frac{2}{5}(1)^{5/2} - \frac{2}{3}(1)^{3/2}\right]\right)\left(\frac{1}{2}\right)$

$\left(\frac{1}{80}25\sqrt{5} - \frac{1}{48}5\sqrt{5} - \frac{1}{80} + \frac{1}{48}\right)\left(\frac{1}{2}\right) =$

$\frac{5}{32}\sqrt{5} - \frac{5}{96}\sqrt{5} + \frac{1}{240}$

$\boxed{\frac{5}{48}\sqrt{5} + \frac{1}{240}}$

#3. Evaluate the surface integral $\iint_S (x^2z + y^2z)\, dS$ (scalar)

S is the hemisphere $x^2 + y^2 + z^2 = 4$, $z \geq 0$.

Surface: $\vec{r} = \langle x, y, \sqrt{4-x^2-y^2} \rangle$ ← very difficult derivatives

take advantage of spherical surface: use spherical coordinates: $x^2+y^2+z^2=4$, $\rho^2=4$ is $\rho=2$

with $\rho = 2$...

$\vec{r}(\phi,\theta) = \langle 2\sin\phi\cos\theta, 2\sin\phi\sin\theta, 2\cos\phi \rangle$

then must use $\iint f(x,y,z)\,|\vec{r}_u \times \vec{r}_v|\, dA$

$\vec{r}_u = \vec{r}_\phi = \langle 2\cos\phi\cos\theta, 2\cos\phi\sin\theta, -2\sin\phi \rangle$

$\vec{r}_v = \vec{r}_\theta = \langle -2\sin\phi\sin\theta, 2\sin\phi\cos\theta, 0 \rangle$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \overset{+}{} & \overset{-}{} & \overset{+}{} \\ 2\cos\phi\cos\theta & 2\cos\phi\sin\theta & -2\sin\phi \\ -2\sin\phi\sin\theta & 2\sin\phi\cos\theta & 0 \end{vmatrix}$$

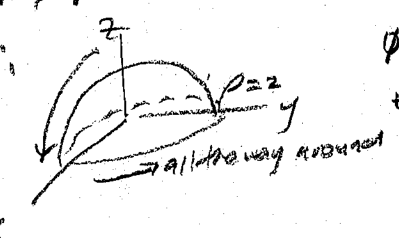
$= \langle 0 + 4\sin^2\phi\cos\theta, -(0 - 4\sin^2\phi\sin\theta), 4\sin\phi\cos\phi\cos^2\theta + 4\sin\phi\cos\phi\sin^2\theta \rangle$

$= \langle 4\sin^2\phi\cos\theta, 4\sin^2\phi\sin\theta, 4\sin\phi\cos\phi \rangle$

$|\vec{r}_u \times \vec{r}_v| = \sqrt{16\sin^4\phi\cos^2\theta + 16\sin^4\phi\sin^2\theta + 16\sin^2\phi\cos^2\phi} = \sqrt{16\sin^4\phi + 16\sin^2\phi\cos^2\phi}$

$= \sqrt{16}\sqrt{\sin^2\phi}\sqrt{\sin^2\phi + \cos^2\phi} = 4\sin\phi$

parameter domain:



$\phi = 0$ to $\phi = \pi/2$

$\theta = 0$ to $\theta = 2\pi$

integrand: $(x^2z + y^2z)\, 4\sin\phi$ (note: do not add $\rho^2\sin\phi$ — not integrating volume)

$= r^2 z\, 4\sin\phi = (2\sin\phi)^2(2\cos\phi)\, 4\sin\phi = 32\sin^3\phi\cos\phi$

$u = \sin\phi$, $\frac{du}{d\phi} = \cos\phi$, $\cos\phi\, d\phi = du$

$\phi = 0 \to u = 0$, $\phi = \pi/2 \to u = 1$

$32\int_0^{2\pi}\int_0^{\pi/2} \sin^3\phi\cos\phi\, d\phi\, d\theta = 32\int_0^{2\pi} d\theta \int_0^{\pi/2} \sin^3\phi\cos\phi\, d\phi$

$= 32\int_0^{2\pi} d\theta \int_0^1 u^3\, du = 32\left[\theta\right]_0^{2\pi}\left[\frac{1}{4}u^4\right]_0^1$

$= \boxed{64\pi}$

16.7 day 2

#1. Evaluate the surface integral $\iint_S \vec{F} \, dS$ vector field

(find the flux of $\vec{F}$ across S):

$$\vec{F}(x,y,z) = \langle \overset{P}{xy},\ \overset{Q}{yz},\ \overset{R}{zx} \rangle$$

S is the part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the square $0 \le x \le 1,\ \ 0 \le y \le 1$, and has upward orientation.

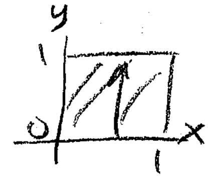
surface: $\vec{r}(x,y) = \langle x, y, 4-x^2-y^2 \rangle$

parameter domain

rectangular

$y = 0$ to 1

$x = 0$ to 1



Two ways to get integrand...

(1) $-P\frac{\partial z}{\partial x} - Q\frac{\partial z}{\partial y} + R$

$\frac{\partial z}{\partial x} = -2x,\ \frac{\partial z}{\partial y} = -2y$

$-(xy)(-2x) - (yz)(-2y) + (zx)$

$2x^2y + 2y^2z + xz$

$2x^2y + (2y^2+x)z$

$2x^2y + (2y^2+x)(4-x^2-y^2)$

$2x^2y + 8y^2 - 2x^2y^2 + 2y^4 + 4x - x^3 - xy^2$

(2) $\vec{F} \cdot (\vec{r}_u \times \vec{r}_v)$ $\quad \vec{r}_u = \vec{r}_x = \langle 1, 0, -2x \rangle$

$\vec{r}_v = \vec{r}_y = \langle 0, 1, -2y \rangle$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} + & - & + \\ 1 & 0 & -2x \\ 0 & 1 & -2y \end{vmatrix} = \langle 0+2x, -(-2y-0), 1-0 \rangle$$

$= \langle 2x, 2y, 1 \rangle$

$\vec{F} \cdot (\vec{r}_u \times \vec{r}_v) = \langle xy, yz, zx \rangle \cdot \langle 2x, 2y, 1 \rangle$

$= (xy)(2x) + (yz)(2y) + (zx)(1)$

$= 2x^2y + 2zy^2 + zx = 2x^2y + z(2y^2+x)$

$= 2x^2y + (4-x^2-y^2)(2y^2+x)$

$= 2x^2y + 8y^2 - 2x^2y^2 + 2y^4 + 4x - x^3 - xy^2$

now the integral:

$$\int_0^1 \int_0^1 (2x^2y + 8y^2 - 2x^2y^2 + 2y^4 + 4x - x^3 - xy^2)\,dy\,dx$$

$\int_0^1 (2x^2y + 8y^2 - 2x^2y^2 + 2y^4 + 4x - x^3 - xy^2)\,dy = \left[x^2y^2 + \frac{8}{3}y^3 - \frac{2}{3}x^2y^3 + \frac{2}{5}y^5 + 4xy - x^3y - \frac{1}{3}xy^3\right]_0^1$

$= (x^2(1)^2 + \frac{8}{3}(1)^3 - \frac{2}{3}x^2(1)^3 + \frac{2}{5}(1)^5 + 4x(1) - x^3(1) - \frac{1}{3}x(1)^3) - (0)$

$= x^2 + \frac{8}{3} - \frac{2}{3}x^2 + \frac{2}{5} + 4x - x^3 - \frac{1}{3}x = \frac{1}{3}x^2 + \frac{46}{15} + \frac{11}{3}x - x^3$

$\int_0^1 (\frac{1}{3}x^2 + \frac{46}{15} + \frac{11}{3}x - x^3)\,dx = \left[\frac{1}{9}x^3 + \frac{46}{15}x + \frac{11}{6}x^2 - \frac{1}{4}x^4\right]_0^1$

$= (\frac{1}{9}(1)^3 + \frac{46}{15}(1) + \frac{11}{6}(1)^2 - \frac{1}{4}(1)^4) - (0) = \boxed{\frac{857}{180}}$

#2. Evaluate the surface integral $\underbrace{\iint_S \vec{F} \cdot dS}_{\text{(vector field)}}$

(find the flux of $\vec{F}$ across S):

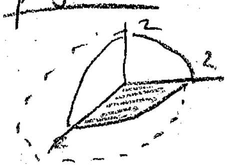
$\vec{F}(x,y,z) = \langle \overset{P}{x}, \overset{Q}{-z}, \overset{R}{y} \rangle$

S is the part of the sphere $x^2+y^2+z^2=4$ in the first octant, with orientation toward the origin.

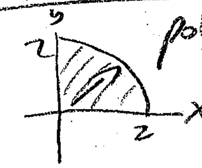
could use x, y for parameters, or spherical here we'll try xy...

surface $\vec{r}(x,y) = \langle x, y, \sqrt{4-(x^2+y^2)} \rangle$

projection:



parameter domain:



polar: $r=0$ to $r=2$

$\theta = 0$ to $\theta = \frac{\pi}{2}$

Two ways to get the integrand, but we'll choose the $-P\frac{\partial z}{\partial x} - Q\frac{\partial z}{\partial y} + R$ method here:

$\frac{\partial z}{\partial x} = \frac{1}{2}(4-x^2-y^2)^{-1/2}(-2x) = \frac{-x}{\sqrt{4-x^2-y^2}}$ $\quad \frac{\partial z}{\partial y} = \frac{1}{2}(4-x^2-y^2)^{-1/2}(-2y) = \frac{-y}{\sqrt{4-x^2-y^2}}$

$-(x)\left(\frac{-x}{\sqrt{4-x^2-y^2}}\right) - (-z)\left(\frac{-y}{\sqrt{4-x^2-y^2}}\right) + (y) = \frac{x^2 - yz}{\sqrt{4-x^2-y^2}} + y = \frac{x^2 - y(\sqrt{4-x^2-y^2})}{\sqrt{4-x^2-y^2}} + y$

convert to polar: $\frac{(r\cos\theta)^2 - (r\sin\theta)\sqrt{4-r^2}}{\sqrt{4-r^2}} + r\sin\theta$

$\frac{(r\cos\theta)^2}{\sqrt{4-r^2}} \cancel{- r\sin\theta} \cancel{+ r\sin\theta} = (r\cos\theta)^2(4-r^2)^{-1/2}$

now the integral...

$\int_0^{\pi/2}\int_0^2 (r\cos\theta)^2(4-r^2)^{-1/2}\, r\,dr\,d\theta = \int_0^{\pi/2}\cos^2\theta\,d\theta \int_0^2 r^3(4-r^2)^{-1/2}\,dr$

$u = 4-r^2$ $\quad r=0 \to u=4$

$\frac{du}{dr} = -2r$ $\quad r=2 \to u=0$

$r\,dr = -\frac{1}{2}du$ $\quad r^2 = 4-u$

$\left(\int_0^{\pi/2}\left(\frac{1}{2}+\frac{1}{2}\cos(2\theta)\right)d\theta\right)\left(-\frac{1}{2}\int_4^0 (4-u)u^{-1/2}\,du\right)$

$\left[\frac{1}{2}\theta + \frac{1}{4}\sin(2\theta)\right]_0^{\pi/2} \frac{1}{2}\int_0^4 \left(4u^{-1/2} - u^{1/2}\right)du$

$\left[\frac{1}{2}\theta + \frac{1}{4}\sin(2\theta)\right]_0^{\frac{\pi}{2}} \frac{1}{2}\left[8u^{1/2} - \frac{2}{3}u^{3/2}\right]_0^4$

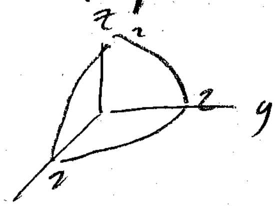
$\left(\frac{1}{2}\left(\frac{\pi}{2}\right) + \frac{1}{4}\sin(\pi)\right) - (0)\Big)\left(\frac{1}{2}\right)\left(8\sqrt{4} - \frac{2}{3}4^{3/2}\right)$

$\left(\frac{\pi}{4}\right)\left(\frac{1}{2}\right)\left(16 - \frac{16}{3}\right)$

$= \frac{4\pi}{3}$ but... "with orientation toward the origin" is the negative direction (outward = positive)

so $= \boxed{-\frac{4\pi}{3}}$

#2 using spherical...

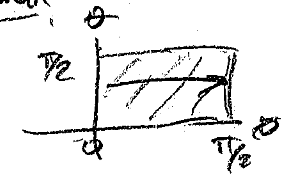


$\rho = 2$

$\phi = 0$ to $\phi = \pi/2$

$\theta = 0$ to $\theta = \pi/2$

parameter domain:



surface: $\vec{r}(\phi,\theta) = \langle 2\sin\phi\cos\theta, 2\sin\phi\sin\theta, 2\cos\phi \rangle$

must use $\iint \vec{F} \cdot (\vec{r_u} \times \vec{r_v})$

$\vec{r_u} = \vec{r_\phi} = \langle 2\cos\phi\cos\theta, 2\cos\phi\sin\theta, -2\sin\phi \rangle$ $\quad \vec{r_v} = \vec{r_\theta} = \langle -2\sin\phi\sin\theta, 2\sin\phi\cos\theta, 0 \rangle$

$$\vec{r_u} \times \vec{r_v} = \begin{vmatrix} + & - & + \\ 2\cos\phi\cos\theta & 2\cos\phi\sin\theta & -2\sin\phi \\ -2\sin\phi\sin\theta & 2\sin\phi\cos\theta & 0 \end{vmatrix}$$

$= \langle 0 + 4\sin^2\phi\cos\theta, -(0 - 4\sin^2\phi\sin\theta), 4\sin\phi\cos\phi\cos^2\theta + 4\sin\phi\cos\phi\sin^2\theta \rangle$

$= \langle 4\sin^2\phi\cos\theta, 4\sin^2\phi\sin\theta, 4\sin\phi\cos\phi \rangle$

$\vec{F}(\vec{r}) = \langle 2\sin\phi\cos\theta, -2\cos\phi, 2\sin\phi\sin\theta \rangle$

$\vec{F} \cdot (\vec{r_u} \times \vec{r_v}) = \langle 2\sin\phi\cos\theta, -2\cos\phi, 2\sin\phi\sin\theta \rangle \cdot \langle 4\sin^2\phi\cos\theta, 4\sin^2\phi\sin\theta, 4\sin\phi\cos\phi \rangle$

$= 8\sin^3\phi\cos^2\theta - 8\sin^2\phi\cos\phi\sin\theta + 8\sin^2\phi\cos\phi\sin\theta$

$= 8\sin^2\phi(\sin\phi\cos^2\theta - \cancel{\cos\phi\sin\theta} + \cancel{\cos\phi\sin\theta}) = 8\sin^3\phi\cos^2\theta$

now the integral...

$$\int_0^{\pi/2}\int_0^{\pi/2} 8\sin^3\phi\cos^2\theta\, d\phi\, d\theta = 8\int_0^{\pi/2}\sin^3\phi\, d\phi \int_0^{\pi/2}\cos^2\theta\, d\theta = 8\int_0^{\pi/2}\sin^2\phi\sin\phi\, d\phi \int_0^{\pi/2}\left(\tfrac{1}{2} + \tfrac{1}{2}\cos 2\theta\right)d\theta$$

$$= 8\int_0^{\pi/2}(1-\cos^2\phi)\sin\phi\, d\phi \int_0^{\pi/2}\left(\tfrac{1}{2}+\tfrac{1}{2}\cos 2\theta\right)d\theta = 8\left[\int_0^{\pi/2}\sin\phi\, d\phi - \int_0^{\pi/2}\cos^2\phi\sin\phi\, d\phi\right]\left[\int_0^{\pi/2}\tfrac{1}{2}\, d\theta + \int_0^{\pi/2}\tfrac{1}{2}\cos 2\theta\, d\theta\right]$$

$u = \cos\phi$ $\quad \phi = 0 \Rightarrow u = 1$

$\frac{du}{d\phi} = -\sin\phi$ $\quad \phi = \pi/2 \Rightarrow u = 0$

$-\sin\phi\, d\phi = du$

$-\int_0^1 u^2\, du$

$$= 8\left(\left[-\cos\phi\right]_0^{\pi/2} - \left[\tfrac{1}{3}u^3\right]_0^1\right)\left(\tfrac{1}{2}\theta + \tfrac{1}{4}\sin(2\theta)\right]_0^{\pi/2}$$

$= 8\left(-\cos\pi/2 + \cos 0 - \tfrac{1}{3}\right)\left(\tfrac{\pi}{4} + \tfrac{1}{4}\sin(\pi) - \tfrac{1}{4}\sin 0\right)$

$= 8\left(0 + 1 - \tfrac{1}{3}\right)\left(\tfrac{\pi}{4} + 0 - 0\right) = 8\left(\tfrac{2}{3}\right)\tfrac{1}{4}\pi = \tfrac{4\pi}{3}$

$\rightarrow \boxed{-\dfrac{4\pi}{3}}$

(orientation)

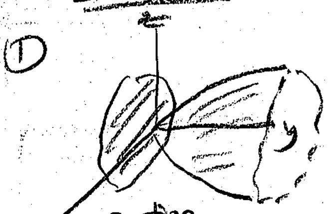
#3. Evaluate the surface integral $\iint_S \vec{F} \cdot dS$

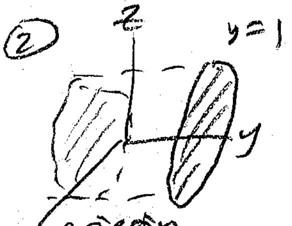
(find the flux of $\vec{F}$ across S):

$\vec{F}(x, y, z) = \langle 0,\ y,\ -z \rangle$

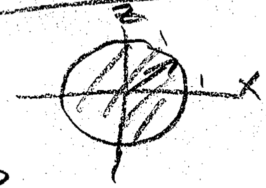
S consists of the paraboloid $y = x^2 + z^2,\ \ 0 \le y \le 1$ and the disk $x^2 + z^2 \le 1,\ \ y = 1$ with upward orientation. ?

2 surfaces

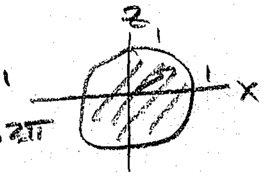
① 
x projection
$\vec{r}_1 = \langle x, x^2+z^2, z \rangle$

② 
x projection
$\vec{r}_2 = \langle x, 1, z \rangle$

parameter domains:



polar!
$r = 0$ to 1
$\theta = 0$ to 2π



paraboloid:

$\vec{r}_u = \vec{r}_x = \langle 1, 2x, 0 \rangle,\ \vec{r}_v = \vec{r}_z = \langle 0, 2z, 1 \rangle$

$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} + & - & + \\ 1 & 2x & 0 \\ 0 & 2z & 1 \end{vmatrix} = \langle 2x - 0, -(1-0), 2z - 0 \rangle$
$= \langle 2x, -1, 2z \rangle$

$\vec{F}(r) = \langle 0, x^2+z^2, -z \rangle \quad \vec{F} \cdot (\vec{r}_u \times \vec{r}_v) = \langle 0, x^2+z^2, -z \rangle \cdot \langle 2x, -1, 2z \rangle = -x^2 - z^2 - 2z^2$

to polar! $-(x^2+z^2) - 2(z)^2 = -r^2 - 2(r\sin\theta)^2 = -r^2(1 + 2\sin^2\theta) = -r^2(1 + 2(\frac{1}{2} - \frac{1}{2}\cos(2\theta)))$
$= -r^2(2 - \cos(2\theta))$

$\int_0^{2\pi}\int_0^1 (-r^2)(2 - \cos(2\theta))\, r\,dr\,d\theta = -\int_0^{2\pi}(2 - \cos(2\theta))\,d\theta \int_0^1 r^3\,dr$

$= -\left[2\theta - \frac{1}{2}\sin(2\theta)\right]_0^{2\pi}\left[\frac{1}{4}r^4\right]_0^1 = -((4\pi - 0) - (0 - 0))(\frac{1}{4} - 0) = (-\pi)$

disk:

$\vec{r}_u = \vec{r}_x = \langle 1, 0, 0 \rangle \quad \vec{r}_v = \vec{r}_z = \langle 0, 0, 1 \rangle$

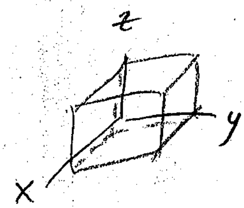
$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} + & - & + \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle 0-0, -(1-0), 0-0 \rangle = \langle 0, -1, 0 \rangle$ this is $\vec{n}$ pointing inward so will need to reverse orientation at end.

$\vec{F}(r) = \langle 0, 1, -z \rangle \quad \vec{F} \cdot (\vec{r}_u \times \vec{r}_v) = \langle 0, 1, -z \rangle \cdot \langle 0, -1, 0 \rangle = -1$

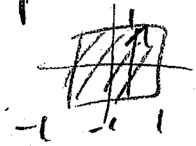
$\int_0^{2\pi}\int_0^1 (-1)\, r\,dr\,d\theta = -\int_0^{2\pi} 1\,d\theta \int_0^1 r\,dr = -[\theta]_0^{2\pi}\left[\frac{1}{2}r^2\right]_0^1 = -[2\pi][\frac{1}{2}] = -\pi \Rightarrow (\pi)$

total outward flux $= -\pi + \pi = \boxed{0}$

#4. Evaluate the surface integral $\iint_S \vec{F} \cdot d\vec{S}$



parameter domain for each



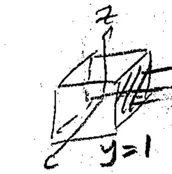
(find the flux of $\vec{F}$ across S):

$\vec{F}(x,y,z) = \langle x,\ 2y,\ 3z \rangle$

S is the cube with vertices $(\pm 1,\ \pm 1,\ \pm 1)$.

For each surface, $\vec{F} \cdot (\vec{r}_u \times \vec{r}_v) = \vec{F} \cdot \vec{n}$ outward normal on that plane

There are $\underline{6}$ surfaces...

① 

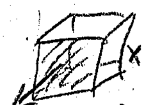
$\vec{n} = \langle 0,1,0 \rangle$

$\vec{F} = \langle x, 2, 3z \rangle$

$\vec{F} \cdot \vec{n} = \langle 0,1,0 \rangle \cdot \langle x, 2, 3z \rangle = 2$

$\int_{-1}^{1}\int_{-1}^{1} (2)\, dz\, dx = \int_{-1}^{1} [2z]_{-1}^{1}\, dx$

$= \int_{-1}^{1} 4\, dx = [4x]_{-1}^{1} = (8)$

③ 

$\vec{n} = \langle 1,0,0 \rangle$

$\vec{F} = \langle 1, 2y, 3z \rangle$

$\vec{F} \cdot \vec{n} = \langle 1, 2y, 3z \rangle \cdot \langle 1,0,0 \rangle = 1$

$\int_{-1}^{1}\int_{-1}^{1} 1\, dy\, dz = \int_{-1}^{1} [y]_{-1}^{1}\, dz$

$\int_{-1}^{1} 2\, dz = 2[z]_{-1}^{1} = (4)$

⑤ $z=1$ 

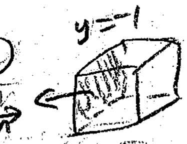
$\vec{n} = \langle 0,0,1 \rangle$

$\vec{F} = \langle x, 2y, 3 \rangle$

$\vec{F} \cdot \vec{n} = \langle x, 2y, 3 \rangle \cdot \langle 0,0,1 \rangle = 3$

$\int_{-1}^{1}\int_{-1}^{1} 3\, dx\, dy = \int_{-1}^{1} 3[x]_{-1}^{1}\, dy$

$= \int_{-1}^{1} 6\, dy = 6[y]_{-1}^{1} = (12)$

② 

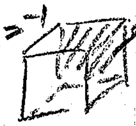
$\vec{F} = \langle x, -2, 3z \rangle$

$\vec{n} = \langle 0,-1,0 \rangle$

$\vec{F} \cdot \vec{n} = \langle x, -2, 3z \rangle \cdot \langle 0,-1,0 \rangle = 2$

$\int_{-1}^{1}\int_{-1}^{1} 2\, dz\, dx = \int_{-1}^{1} [2z]_{-1}^{1}\, dx$

$\int_{-1}^{1} 4\, dx = 4[x]_{-1}^{1} = (8)$

④ $x=-1$ 

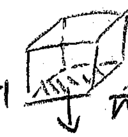
$\vec{n} = \langle -1,0,0 \rangle$

$\vec{F} = \langle -1, 2y, 3z \rangle$

$\vec{F} \cdot \vec{n} = \langle -1, 2y, 3z \rangle \cdot \langle -1,0,0 \rangle = 1$

$\int_{-1}^{1}\int_{-1}^{1} 1\, dy\, dz = \int_{-1}^{1} [y]_{-1}^{1}\, dz$

$\int_{-1}^{1} 2\, dz = 2[z]_{-1}^{1} = (4)$

⑥ 

$\vec{F} = \langle x, 2y, -3 \rangle$

$\vec{n} = \langle 0,0,-1 \rangle$

$\vec{F} \cdot \vec{n} = \langle x, 2y, -3 \rangle \cdot \langle 0,0,-1 \rangle = 3$

$\int_{-1}^{1}\int_{-1}^{1} 3\, dx\, dy = \int_{-1}^{1} 3[x]_{-1}^{1}\, dy$

$\int_{-1}^{1} 6\, dy = 6[y]_{-1}^{1} = (12)$

sum of outward fluxes: $8+8+4+4+12+12 = \boxed{48}$

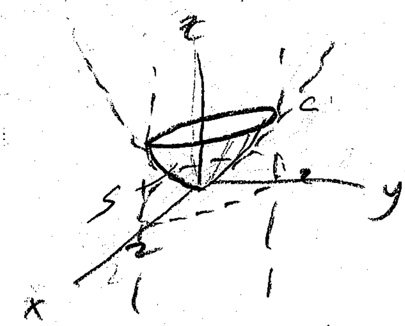
16.8

#1. Using Stokes' Theorem, write out and evaluate the single-integral which is equivalent to the surface integral which calculates $\iint_S \left(curl\ \vec{F} \right) \cdot dS$

where

$\vec{F}(x,y,z) = \langle x^2z^2,\ y^2z^2,\ xyz \rangle$

S is the part of the paraboloid $z = x^2 + y^2$ that lies insides the cylinder $x^2 + y^2 = 4$, oriented upward.



C is intersection: $\begin{cases} z = x^2+y^2 \\ x^2+y^2=4 \end{cases}$

$x^2+y^2=4$ at $z=4$

$\iint_S (\text{curl}\vec{F}) \cdot d\vec{S} = \oint_C \vec{F} \cdot d\vec{r} = \oint_C \vec{F}(\vec{r}) \cdot \vec{r}\,' dt$

parametrize C: $\vec{r}(t) = \langle 2\cos t, 2\sin t, 4 \rangle \quad (0 \le t \le 2\pi)$

$\vec{r}\,' = \langle -2\sin t, 2\cos t, 0 \rangle$

$\vec{F}(\vec{r}) = \langle (2\cos t)^2(4)^2,\ (2\sin t)^2(4)^2,\ (2\cos t)(2\sin t)(4) \rangle = \langle 64\cos^2 t, 64\sin^2 t, 16\cos t \sin t \rangle$

$\vec{F} \cdot \vec{r}\,' = \langle 64\cos^2 t, 64\sin^2 t, 16\cos t\sin t \rangle \cdot \langle -2\sin t, 2\cos t, 0 \rangle$

$= -128\cos^2 t \sin t + 128\sin^2 t\cos t = 128(\sin^2 t\cos t - \cos^2 t\sin t)$

$\oint_C \vec{F} \cdot \vec{r}\,' dt = 128 \int_0^{2\pi} \sin^2 t \cos t\, dt - 128 \int_0^{2\pi} \cos^2 t \sin t\, dt$

$u = \sin t$

$\frac{du}{dt} = \cos t dt$

$\cos t dt = du$

$t = 0 \rightarrow u = 0$

$t = 2\pi \rightarrow u = 0$

$u = \cos t$

$\frac{du}{dt} = -\sin t$

$\sin t dt = -du$

$t = 0 \rightarrow u = 1$

$t = 2\pi \rightarrow u = 1$

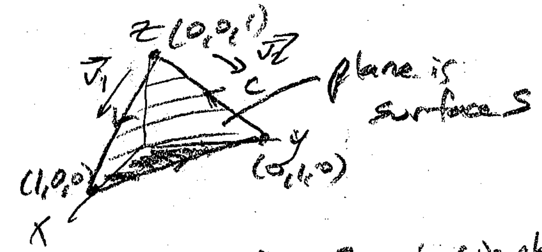
$128 \int_0^0 u^2 du + 128 \int_1^1 u^2 du$

$0 + 0 = \boxed{0}$

#2. Using Stokes' Theorem, write out and evaluate the double-integral which is equivalent to the line integral $\int_C \vec{F} \bullet d\vec{r}$ which sums the contributions of the field $\vec{F}$ along path C

$\vec{F}(x,y,z) = \langle x+y^2,\ y+z^2,\ z+x^2 \rangle$

C is the triangle with vertices (1,0,0), (0,1,0), and (0,0,1).



$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\text{curl}\,\vec{F}) \cdot d\vec{S}$

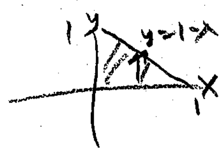
to parametrize plane, find 2 vectors in plane to give upward normal:

$\vec{v_1} = \langle 1-0, 0-0, 0-1 \rangle = \langle 1, 0, -1 \rangle$

$\vec{v_2} = \langle 0-0, 1-0, 0-1 \rangle = \langle 0, 1, -1 \rangle$

$\vec{n} = \vec{v_1} \times \vec{v_2} = \begin{vmatrix} + & - & + \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \langle 0+1, -(-1-0), 1-0 \rangle = \langle 1, 1, 1 \rangle$

parameter domain for surface:



rectangular

$y=0$ to $y=1-x$

$x=0$ to $x=1$

$ax+by+cz = \vec{n} \cdot \vec{r_0}$

$x+y+z = \langle 1,1,1 \rangle \cdot \langle 0,0,1 \rangle = 1$

$x+y+z=1,\quad z = 1-x-y$

$\vec{r}(x,y) = \langle x, y, 1-x-y \rangle$

$\text{curl}\,\vec{F} = \begin{vmatrix} + & - & + \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+y^2 & y+z^2 & z+x^2 \end{vmatrix}$

$= \langle 0-2z, -(2x-0), 0-2y \rangle$

$= \langle -2z, -2x, -2y \rangle$

$= \langle -2(1-x-y), -2x, -2y \rangle$

$\text{curl}\,\vec{F} \cdot d\vec{S} = \text{curl}\,\vec{F} \cdot \vec{n}\, dA$

$\vec{F} \cdot \vec{n} = \langle -2+2x+2y, -2x, -2y \rangle \cdot \langle 1,1,1 \rangle$

$= -2+2x+2y-2x-2y = -2$

$\iint (\text{curl}\,\vec{F}) \cdot d\vec{S} = \int_0^1 \int_0^{1-x} (-2)\, dy\, dx$

$\int_0^{1-x} (-2)\, dy = -2\,[y]_0^{1-x} = -2(1-x) - 0 = -2+2x$

$\int_0^1 (-2+2x)\, dx = [-2x+x^2]_0^1 = (-2(1)+(1)^2) - (0) = \boxed{-1}$

#3. Verify that Stokes' Theorem is true for the given vector field $\vec{F}$ and surface S by writing out and evaluating integrals for both sides of the Stokes' Theorem equation.

$\vec{F}(x,y,z) = \langle y^2,\ x,\ z^2 \rangle$

S is the part of the paraboloid $z = x^2 + y^2$ that lies below the plane $z = 1$, oriented upward.

$$\iint_S (\text{curl}\,\vec{F}) \cdot d\vec{S}$$

$$\text{curl}\,\vec{F} = \begin{vmatrix} + & - & + \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2 & x & z^2 \end{vmatrix}$$

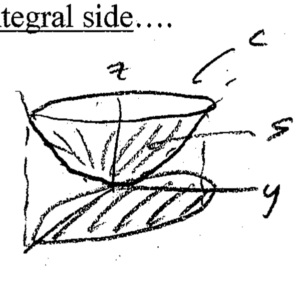
$$= \langle 0-0, -(0-0), 1-2y \rangle$$

$$= \langle 0, 0, 1-2y \rangle$$

$$(\text{curl}\,\vec{F}) \cdot \vec{n} = \langle 0,0,1-2y \rangle \cdot \langle -2x, -2y, 1 \rangle$$
$$= 1-2y$$

<u>Double-integral side</u>....

$z=1$



parametrize surface:

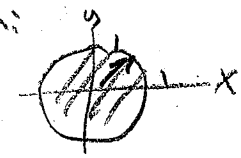
$\vec{r}(x,y) = \langle x, y, x^2+y^2 \rangle$

parameter domain:

polar:

$r=0$ to $r=1$

$\theta=0$ to $\theta=2\pi$



$(\text{curl}\,\vec{F}) \cdot d\vec{S} = (\text{curl}\,\vec{F}) \cdot \vec{n}\, dS$

need $\vec{n}$ for surface

$\vec{r}_u \times \vec{r}_x = \langle 1, 0, 2x \rangle$ $\quad \vec{r}_v = \vec{r}_y = \langle 0, 1, 2y \rangle$

$$\vec{n} = \vec{r}_u \times \vec{r}_v = \begin{vmatrix} + & - & + \\ 1 & 0 & 2x \\ 0 & 1 & 2y \end{vmatrix} = \langle 0-2x, -(2y-0), 1-0 \rangle$$
$$= \langle -2x, -2y, 1 \rangle$$

to polar: $1-2y = 1-2(r\sin\theta)$

$$\int_0^{2\pi}\int_0^1 (1-2r\sin\theta)\, r\,dr\,d\theta$$

$$\int_0^1 (r - 2r^2\sin\theta)\,dr = \int_0^1 r\,dr - 2\sin\theta \int_0^1 r^2\,dr = \left(\tfrac{1}{2}r^2 - 2\sin\theta\,\tfrac{1}{3}r^3\right)\Big|_0^1$$

$$= \left(\tfrac{1}{2} - \tfrac{2}{3}\sin\theta\right) - 0 = \tfrac{1}{2} - \tfrac{2}{3}\sin\theta$$

$$\int_0^{2\pi} \left(\tfrac{1}{2} - \tfrac{2}{3}\sin\theta\right) d\theta = \left[\tfrac{1}{2}\theta + \tfrac{2}{3}\cos\theta\right]_0^{2\pi}$$

$$= \left(\tfrac{1}{2}2\pi + \tfrac{2}{3}\cos 2\pi\right) - \left(\tfrac{1}{2}(0) + \tfrac{2}{3}\cos 0\right)$$

$$= \pi + \tfrac{2}{3}(1) - 0 - \tfrac{2}{3}(1)$$

$$= \boxed{\pi}$$

but.... could instead use an easier surface...

#3. Verify that Stokes' Theorem is true for the given vector field $\vec{F}$ and surface S by writing out and evaluating integrals for both sides of the Stokes' Theorem equation.

$\vec{F}(x,y,z)=\langle y^2,\ x,\ z^2\rangle$

S is the part of the paraboloid $z=x^2+y^2$ that lies below the plane $z=1$, oriented upward.

$$\iint_S (\text{curl}\,\vec{F})\cdot d\vec{r}$$

$$\text{curl}\,\vec{F} = \begin{vmatrix} + & - & + \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x & z^2 \end{vmatrix}$$

$= \langle 0-0,\ -(0-0),\ 1-2y\rangle$

$= \langle 0, 0, 1-2y\rangle$

$(\text{curl}\,\vec{F})\cdot\vec{n}$

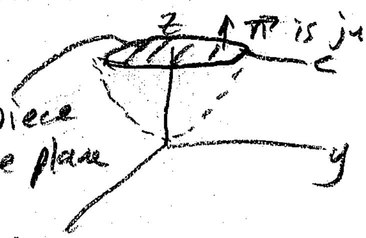
$\langle 0,0,1-2y\rangle \cdot \langle 0,0,1\rangle$

$= 1-2y$ to polar $\Rightarrow 1-2r\sin\theta$

$\int_0^{2\pi}\int_0^1 (1-2r\sin\theta)\, r\, dr\, d\theta$ --- Same integral we got with the original surface!

Double-integral side....

Stokes' says we can use any surface with the boundary curve...



this piece of the plane $z=1$ is also good, and a better choice :)

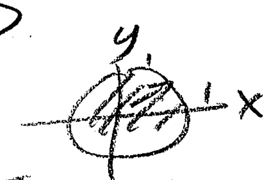
parametrize:

$\vec{r}(x,y)=\langle x, y, 1\rangle$

parameter domain:

polar: $r=0$ to $r=1$

~~$\theta=0$ to $\theta=2\pi$~~

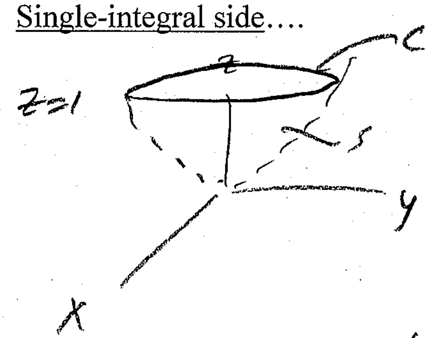


#3 (continued). Verify that Stokes' Theorem is true for the given vector field $\vec{F}$ and surface S by writing out and evaluating integrals for both sides of the Stokes' Theorem equation.

$\vec{F}(x,y,z) = \langle y^2,\ x,\ z^2 \rangle$

S is the part of the paraboloid $z = x^2 + y^2$ that lies below the plane $z = 1$, oriented upward.

Single-integral side....



$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C \vec{F}\cdot\vec{r}\,'dt$$

parametrize the curve: (radius = 1)

$\vec{r}(t) = \langle \cos t, \sin t, 1\rangle$

parameter-domain: $0 \le t \le 2\pi$ (to go around once)

$$\vec{r}\,' = \langle -\sin t, \cos t, 0\rangle$$

$$\vec{F}(\vec{r}) = \langle (\sin t)^2, (\cos t), (1)^2\rangle = \langle \sin^2 t, \cos t, 1\rangle$$

$$\vec{F}\cdot\vec{r}\,' = \langle \sin^2 t, \cos t, 1\rangle\cdot\langle -\sin t, \cos t, 0\rangle = -\sin^3 t + \cos^2 t + 0 = \cos^2 t - \sin^3 t$$

$$\int_0^{2\pi}\cos^2 t\,dt - \int_0^{2\pi}\sin^3 t\,dt = \int_0^{2\pi}\left(\tfrac{1}{2} + \tfrac{1}{2}\cos(2t)\right)dt - \int_0^{2\pi}\sin^2 t\sin t\,dt$$

$$= \int_0^{2\pi}\left(\tfrac{1}{2} + \tfrac{1}{2}\cos(2t)\right)dt - \int_0^{2\pi}(1-\cos^2 t)\sin t\,dt$$

$u = \cos u$, $\frac{du}{dt} = -\sin u$, $\sin t\,dt = -du$

$t = 0 \Rightarrow u = 1$

$t = 2\pi \Rightarrow u = 1$

$$= \int_0^{2\pi}\tfrac{1}{2}dt + \tfrac{1}{2}\int_0^{2\pi}\cos(2t)dt - \int_0^{2\pi}\sin t\,dt + \underbrace{\int_0^{2\pi}\cos^2 t\sin t\,dt}_{\int_1^1 u^2\,du\ (=0)}$$

$$= \left[\tfrac{1}{2}t\right]_0^{2\pi} + \tfrac{1}{4}\left(\sin(2t)\right]_0^{2\pi} - \left[-\cos t\right]_0^{2\pi}$$

$$= \tfrac{1}{2}(2\pi - 0) + \tfrac{1}{4}(\underbrace{\sin 4\pi - \sin 0}_{0}) + [\underbrace{\cos 2\pi}_{1} - \underbrace{\cos 0}_{1}])$$

$$= \boxed{\pi}$$ verified ✓

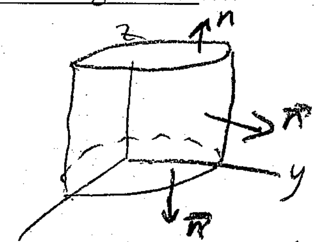
16.9

#1 Verify that the Divergence Theorem is true for the given vector field $\vec{F}$ on the region E by writing out and evaluating integrals for both sides of the Divergence Theorem equation.

$\vec{F}(x,y,z) = \langle xy,\ yz,\ zx \rangle$

E is the solid cylinder $x^2+y^2 \le 1,\ 0 \le z \le 1$.

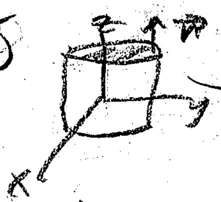
Double-integral side….



3 sides

$\iint \vec{F} \cdot d\vec{S} = \iint \vec{F} \cdot \vec{n}\, dA$ for each side

side (1)

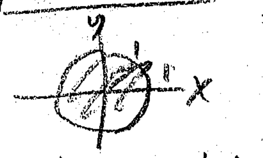


$\vec{n} = \langle 0,0,1 \rangle$, $z=1$

parametrize:

$\vec{r}(x,y) = \langle x, y, 1 \rangle$

parameter domain:



polar: $r = 0$ to 1

$\theta = 0$ to 2π

$\vec{F}(\vec{r}) = \langle xy, y(1), (1)x \rangle$

$= \langle xy, y, x \rangle$

$\vec{F} \cdot \vec{n} = \langle xy, y, x \rangle \cdot \langle 0,0,1 \rangle$

$= x$ to polar $= r\cos\theta$

$\int_0^{2\pi}\int_0^1 (r\cos\theta)\, r\, dr\, d\theta$

$\int_0^{2\pi} \cos\theta\, d\theta \int_0^1 r^2\, dr$

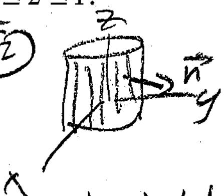
$[\sin\theta]_0^{2\pi} \left[\frac{1}{3}r^3\right]_0^1$

$(\sin 2\pi - \sin 0)\left[\frac{1}{3} - 0\right)$

$0\left(\frac{1}{3}\right)$

$= (0)$

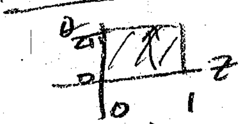
side (2)



use cylindrical to parametrize w/ $r=1$

$\vec{r}(\theta, z) = \langle \cos\theta, \sin\theta, z \rangle$

parameter domain



rectangular: $\theta = 0$ to 2π

$z = 0$ to 1

$\vec{F}(\vec{r}) = \langle (\cos\theta)(\sin\theta), (\sin\theta)z, z(\cos\theta) \rangle$

$= \langle \cos\theta\sin\theta, z\sin\theta, z\cos\theta \rangle$

$\vec{n} = \vec{r_u} \times \vec{r_v}$

$\vec{r_u} = \vec{r_\theta} = \langle -\sin\theta, \cos\theta, 0 \rangle$

$\vec{r_v} = \vec{r_z} = \langle 0, 0, 1 \rangle$

$\vec{n} = \vec{r_u} \times \vec{r_v} = \begin{vmatrix} + & - & + \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$

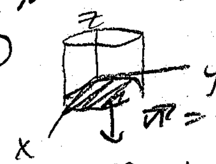
$\vec{n} = \langle \cos\theta - 0, -(-\sin\theta - 0), 0 - 0 \rangle$

$\vec{n} = \langle \cos\theta, \sin\theta, 0 \rangle$

$\vec{F} \cdot \vec{n} = \langle \cos\theta\sin\theta, z\sin\theta, z\cos\theta \rangle \cdot \langle \cos\theta, \sin\theta, 0 \rangle$

$= \cos^2\theta\sin\theta + z\sin^2\theta$

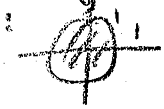
side (3)



$\vec{n} = \langle 0,0,-1 \rangle$

parametrize: $\vec{r}(x,y) = \langle x, y, 0 \rangle$

parameter domain:



polar:

$r = 0$ to 1

$\theta = 0$ to 2π

$\vec{F}(\vec{r}) = \langle xy, y(0), (0)x \rangle = \langle xy, 0, 0 \rangle$

$\vec{F} \cdot \vec{n} = \langle xy, 0, 0 \rangle \cdot \langle 0, 0, -1 \rangle = 0$

So $\int_0^{2\pi}\int_0^1 (0)\, r\, dr\, d\theta = (0)$

$\int_0^{2\pi}\int_0^1 (\cos^2\theta\sin\theta + z\sin^2\theta)\, dz\, d\theta$

$\cos^2\theta\sin\theta \int_0^1 1\, dr + \sin^2\theta \int_0^1 z\, dz$

$\cos^2\theta\sin\theta\, [r]_0^1 + \sin^2\theta\, \left[\frac{1}{2}z^2\right]_0^1$

$\cos^2\theta\sin\theta + \frac{1}{2}\sin^2\theta$

$\int_0^{2\pi} \cos^2\theta\sin\theta\, d\theta + \frac{1}{2}\int_0^{2\pi}\left(\frac{1}{2} - \frac{1}{2}\cos(2\theta)\right) d\theta$

$u = \cos\theta$

$\frac{du}{d\theta} = -\sin\theta$

$\sin\theta\, d\theta = -du$

$\theta = 0 \to u = 1$

$\theta = 2\pi \to u = 1$

$\int_1^1 u^2\, du$

$+ \frac{1}{4}\int_0^{2\pi} 1\, d\theta - \frac{1}{4}\int_0^{2\pi} \cos(2\theta)\, d\theta$

$\frac{1}{4}[\theta]_0^{2\pi} - \frac{1}{8}[\sin 2\theta]_0^{2\pi}$

$\frac{1}{4}(2\pi) - \frac{1}{8}(\sin 4\pi - \sin 0)$

$0 - 0$

$(0) + \frac{\pi}{2} + 0 = \frac{\pi}{2}$

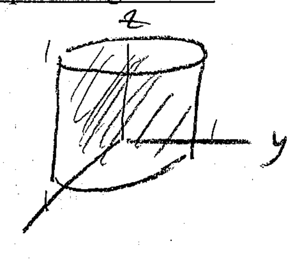
sum of 3 sides $= 0 + \frac{\pi}{2} + 0 = \boxed{\frac{\pi}{2}}$

#1(continued) Verify that the Divergence Theorem is true for the given vector field $\vec{F}$ on the region E by writing out and evaluating integrals for both sides of the Divergence Theorem equation.

$\vec{F}(x,y,z) = \langle xy,\ yz,\ zx \rangle$

E is the solid cylinder $x^2+y^2 \le 1,\quad 0 \le z \le 1.$

Triple-integral side....



$\iiint \operatorname{div}\vec{F}\,dV$

$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}[xy] + \frac{\partial}{\partial y}[yz] + \frac{\partial}{\partial z}[zx]$

$= y + z + x$

use cylindrical coordinates: r, θ, z

parameter domain: $r=0$ to $r=1$

$\theta = 0$ to $\theta = 2\pi$

$z = 0$ to $z = 1$

$\int_0^{2\pi}\int_0^1\int_0^1 (y+z+x)\, r\,dz\,dr\,d\theta$

to cylindrical: $y+z+x = (r\sin\theta) + z + (r\cos\theta)$

$\int_0^{2\pi}\int_0^1\int_0^1 (r\sin\theta + z + r\cos\theta)\, r\,dz\,dr\,d\theta$

$(r^2\sin\theta + r^2\cos\theta)\int_0^1 1\,dz + r\int_0^1 z\,dz = (r^2(\sin\theta+\cos\theta))[z]_0^1 + r[\frac{1}{2}z^2]_0^1$

$= r^2(\sin\theta+\cos\theta) + r(\frac{1}{2}(1)^2-(0)) = r^2(\sin\theta+\cos\theta) + \frac{1}{2}r$

$(\sin\theta+\cos\theta)\int_0^1 r^2\,dr + \frac{1}{2}\int_0^1 r\,dr = (\sin\theta+\cos\theta)[\frac{1}{3}r^3]_0^1 + \frac{1}{2}[\frac{1}{2}r^2]_0^1$

$= (\sin\theta+\cos\theta)(\frac{1}{3}-0) + \frac{1}{4}(1^2-0) = \frac{1}{3}(\sin\theta+\cos\theta) + \frac{1}{4}$

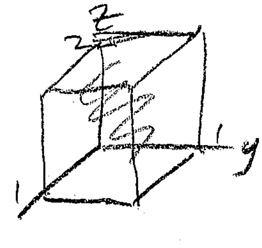
$\frac{1}{3}\int_0^{2\pi}(\sin\theta+\cos\theta)\,d\theta + \int_0^{2\pi}\frac{1}{4}\,d\theta$

$\frac{1}{3}[-\cos\theta+\sin\theta]_0^{2\pi} + \frac{1}{4}[\theta]_0^{2\pi} = \frac{1}{3}((-\cos 2\pi + \sin 2\pi) - (-\cos 0 + \sin 0)) + \frac{1}{4}(2\pi)$

$= \frac{1}{3}(-1+1) + \frac{\pi}{2} = \boxed{\frac{\pi}{2}}$ verified

#2. Using the Divergence Theorem, write out and evaluate the triple-integral which is equivalent to the surface integral $\iint_S \vec{F} \bullet dS$ which calculates the flux of $\vec{F}$ across S if

$\vec{F}(x, y, z) = \langle e^x \sin y,\ e^x \cos y,\ yz^2 \rangle$
S is the surface of the box bounded by the planes $x = 0,\ x = 1,\ y = 0,\ y = 1,\ z = 0,$ and $z = 2$.



rectangular: $x=0$ to $x=1$
$y=0$ to $y=1$
$z=0$ to $z=2$

$$\iiint \operatorname{div}\vec{F}\, dv$$

$$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}[e^x \sin y] + \frac{\partial}{\partial y}[e^x \cos y] + \frac{\partial}{\partial z}[yz^2]$$

$$= \underline{e^x \sin y} - \underline{e^x \sin y} + 2yz = 2yz$$

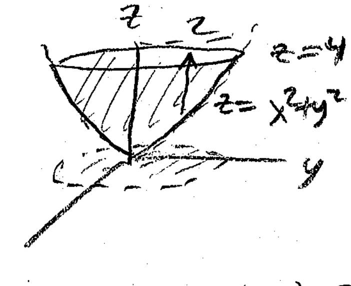
$$\int_0^1 \int_0^1 \int_0^2 2yz\, dz\, dy\, dx$$

$$\int_0^2 2yz\, dz = y[z^2]_0^2 = y(2^2 - 0) = 4y$$

$$\int_0^1 4y\, dy = 2y^2]_0^1 = 2(1^2 - 0) = 2$$

$$\int_0^1 2\, dx = 2(x)_0^1 = \boxed{2}$$

#3. Using the Divergence Theorem, write out and evaluate the triple-integral which is equivalent to the surface integral $\iint_S \vec{F} \bullet dS$ which calculates the flux of $\vec{F}$ across S if



$\vec{F}(x,y,z) = \langle \cos z + xy^2,\ xe^{-z},\ \sin y + x^2 z \rangle$

S is the surface of the solid bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 4$.

cylindrical coordinate: r, θ, z

Parameter domain:

floor: $z = x^2+y^2 = r^2$ (floor) to $z=4$ (ceiling)

then $r = 0$ to $r = 2$

then $\theta = 0$ to $\theta = 2\pi$

$\iiint \operatorname{div} \vec{F}\, dv$

$\operatorname{div} \vec{F} = \frac{\partial}{\partial x}[\cos z + xy^2] + \frac{\partial}{\partial y}[xe^{-z}] + \frac{\partial}{\partial z}[\sin y + x^2 z]$

$= y^2 + 0 + x^2$

$= x^2 + y^2 \Rightarrow$ to cylindrical

$= r^2$

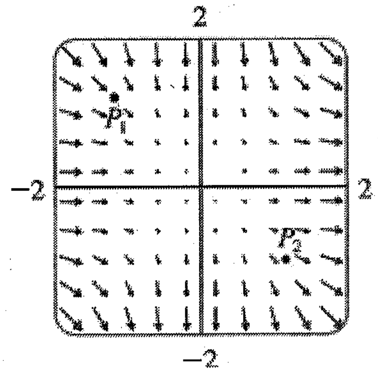
$\int_0^{2\pi} \int_0^{2} \int_{r^2}^{4} (r^2)\, r\, dz\, dr\, d\theta$

$\int_{r^2}^{4} r^3\, dz = r^3 [z]_{r^2}^{4} = r^3(4 - r^2) = 4r^3 - r^5$

$\int_0^2 (4r^3 - r^5)\, dr = \left(r^4 - \frac{1}{6} r^6\right)_0^2 = \left[2^4 - \frac{1}{6} 2^6\right] - 0 = \frac{16}{3}$

$\int_0^{2\pi} \frac{16}{3}\, d\theta = \frac{16}{3} [\theta]_0^{2\pi} = \boxed{\frac{32\pi}{3}}$

#4. A vector field $\vec{F}$ is shown. Determine whether is positive or negative at P_1 and P_2.



P_1 more going than leaving

so $\boxed{\text{div}\,\vec{F} \text{ at } P_1 < 0}$

P_2 more leaving than going in

so $\boxed{\text{div}\,\vec{F} \text{ at } P_2 > 0}$