DiffEq - Ch 3 - Required Practice

Name: Key

3.1

#1. The population of a community is known to increase at a rate proportional to the number of people present at time t. If an initial population P_0 has doubled in 5 years, how long will it take the population…
(a) …to triple?
(b) …to quadruple?

Deriving solution… $\frac{dP}{dt} = kP$

$\int \frac{1}{P}\,dP = \int k\,dt, \quad \ln|P| = kt + c_1$

$P = e^{kt+c_1} = e^{kt}e^{c_1} = Ce^{kt}$

$P(0) = P_0, \quad C = P_0, \quad P = P_0e^{kt}$

solution form: $P = P_0e^{kt}$

t	P
0	P_0
5	$2P_0$

$2P_0 = P_0e^{k(5)}$

$e^{5k} = 2$

$5k = \ln(2)$

$k = \frac{\ln 2}{5} \simeq 0.138629$

$P = P_0e^{0.138629t}$

(a) $3P_0 = P_0e^{0.138629t}$

$e^{0.138629t} = 3$

$0.138629t = \ln(3)$

$t = \frac{\ln(3)}{0.138629} = \boxed{7.9 \text{ yrs}}$

(b) $4P_0 = P_0e^{0.138629t}$

$t = \frac{\ln(4)}{0.138629} = \boxed{10.0 \text{ yrs}}$

#2. Suppose it is known that the population of the community in Problem #1 is 10,000 after 3 years.
(a) What was the initial population P_0?
(b) What will the population be in 10 years?
(c) How fast is the population growing at $t = 10$?

(a)

t	P
3	10000

$P = P_0e^{0.138629t}$

$10000 = P_0e^{0.138629(3)}$

$P = \frac{10000}{e^{0.138629(3)}} = 6597.548$

$\boxed{\text{about } 6598 \text{ people}}$

(b) $P = 6598e^{0.138629t}$

$P(10) = 6598e^{0.138629(10)}$

$= 26391.88$

$\boxed{26392 \text{ people}}$

(c) $\frac{dP}{dt} = kP$

$= 0.138629(26392)$

$= \boxed{3658.7 \text{ people/yr}}$

#3. Initially 100 milligrams of a radioactive substance was present. After 6 hours the mass had decreased by 3%. If the rate of decay is proportional to the amount of the substance present at time t, find the amount remaining after 24 hours.

solution form: $Q = Q_0 e^{kt} = 100 e^{kt}$

hrs t	mg Q
0	100
6	100(.97) = 97

$97 = 100e^{k(6)}$

$e^{6k} = \frac{97}{100}$

$6k = \ln\left(\frac{97}{100}\right)$, $k = \frac{\ln\left(\frac{97}{100}\right)}{6} = -0.0050765346$

$Q = 100e^{-0.0050765346t}$

$Q(24) = 100e^{-0.0050765346(24)}$

$= \boxed{88.53 \text{ mg}}$

#4. Determine the half-life of the radioactive substance described in Problem #3.

t	Q
0	100
t_{HL}	50

$50 = 100e^{-0.0050765346t}$

$-0.0050765346t = \ln\left(\frac{50}{100}\right)$

$t = \frac{\ln\left(\frac{50}{100}\right)}{-0.0050765346} = \boxed{136.5 \text{ hrs}}$

#5. When interest is compounded continuously, the amount of money increases at a rate proportional to the amount A present at time t, that is $\frac{dA}{dt} = rA$, where r is the annual rate of interest.

(a) Find the amount of money accrued at the end of 5 years when $5000 is deposited in an investment account drawing 5.75% annual interest compounded continuously.

(b) In how many years will the initial sum deposited have doubled?

solution form: $A = A_0 e^{rt}$

(a) $r = 0.0575$ $\quad A = 5000e^{0.0575t}$

$A(5) = 5000e^{0.0575(5)}$

$= \boxed{\$6665.45}$

(c) $10000 = 5000e^{0.0575t}$

$e^{0.0575t} = \frac{10000}{5000} = 2$

$0.0575t = \ln(2)$

$t = \frac{\ln(2)}{0.0575} = \boxed{12.055 \text{ yrs}}$

#6. A thermometer is removed from a room where the temperature if 70 °F and is taken outside, where the air temperature is 10 °F. After 30 seconds the thermometer reads 50 °F.
(a) What will the thermometer temperature reading by at $t = 1$ min?
(b) How long will it take for the thermometer reading to reach 15 °F?

Newton's law of cooling: $\frac{dT}{dt} = k(T - T_m)$

deriving solution form... $\int \frac{1}{T - T_m} dT = \int k\,dt$

$u = T - T_m$, $\frac{du}{dT} = 1$, $du = dT$

$\int \frac{1}{u} du = \int k\,dt$, $\ln u = kt + c_1$

$\ln(T - T_m) = kt + c_1$

$T - T_m = e^{(kt + c_1)} = e^{kt} e^{c_1} = Ce^{kt}$

$T = T_m + Ce^{kt}$ here, $T_m = 10°F$

t (min)	T (°F)
0	70°F
0.5	50

$T = 10 + Ce^{kt}$

$70 = 10 + Ce^{0}$

$C = 60$

$T = 10 + 60e^{kt}$

$50 = 10 + 60e^{k(0.5)}$

$40 = 60e^{0.5k}$

$0.5k = \ln\left(\frac{40}{60}\right)$

$k = \frac{\ln\left(\frac{40}{60}\right)}{0.5} = -0.8109302162$

$T(t) = 10 + 60e^{-0.8109302162t}$

(a) $T(1) = 10 + 60e^{-0.8109302162(1)}$

$= \boxed{36.7°F}$

(b) $15 = 10 + 60e^{-0.8109302162t}$

$5 = 60e^{-0.8109302162t}$

$-0.8109302162t = \ln(5/60)$

$t = \frac{\ln(5/60)}{-.8109302162} = \boxed{3.064 \text{ mins}}$

#7. A small metal bar, whose initial temperature was 20 °C, is dropped into a large container of boiling water.
(a) How long will it take the bar to reach 90 °C if it is known that its temperature increase 2°C in the first second?
(b) How long will it take for the bar to reach a temperature of 98 °C?

$T = T_m + Ce^{kt}$ boiling water, $T_m = 100°C$

$T_0 = 20°C$

$T = 100 + Ce^{kt}$

$20 = 100 + Ce^{k(0)}$, $C = 20 - 100 = -80$

$T = 100 - 80e^{kt}$

(a)

t (sec)	T (°C)
0	20
1	22

$22 = 100 - 80e^{k(1)}$

$-78 = -80e^{k}$

$e^{k} = \frac{-78}{-80}$

$k = \ln(78/80)$

$k = -0.0253178080$

$T(t) = 100 - 80e^{-0.025317808t}$

$90 = 100 - 80e^{-0.025317808t}$

$-10 = -80e^{-0.025317808t}$

$e^{-0.025317808t} = \frac{-10}{-80} = \frac{1}{8}$

$t = \frac{\ln\left(\frac{1}{8}\right)}{-0.025317808} = \boxed{82.13 \text{ seconds}}$

(b) $98 = 100 - 80e^{-0.025317808t}$

$-2 = -80e^{-0.025317808t}$

$e^{-0.025317808t} = \frac{-2}{-80} = \frac{1}{40}$

$t = \frac{\ln\left(\frac{1}{40}\right)}{-0.02531780\!8} = \boxed{145.7 \text{ seconds}}$

#8. A 30-Volt electromotive force is applied to an LR series circuit in which the inductance is 0.1 Henry and the resistance is 50 Ohms.
(a) Find the equation for current as a function of time $i(t)$ if $i(0)=0$.
(b) Determine the current at $t \to \infty$.

(a) $L\frac{di}{dt}+Ri=E$

$0.1\frac{di}{dt}+50i=30$

linear, $P(t)=500$

$\frac{di}{dt}+500i=300$

$I.F.=e^{\int 500dt}=e^{500t}$

$e^{500t}i=\int 300e^{500t}dt$

$e^{500t}i=\frac{300}{500}e^{500t}+c$

$\frac{e^{500t}i}{e^{500t}}=\frac{\frac{3}{5}e^{500t}}{e^{500t}}+\frac{c}{e^{500t}}$

$i(t)=\frac{3}{5}+Ce^{-500t}$

now, $i(0)=0$:

$0=\frac{3}{5}+Ce^{0}$, $C=-\frac{3}{5}$

$\boxed{i(t)=\frac{3}{5}-\frac{3}{5}e^{-500t}}$

(b)

$i(t)=\frac{3}{5}\underbrace{-\frac{3}{5}e^{-500t}}_{\text{transient term}\to 0 \text{ as } t\to\infty}$

$\boxed{i \to \frac{3}{5}\text{ Amp}}$

#9. A 100-Volt electromotive force is applied to an RC series circuit in which the resistance is 200 Ohms and the capacitance is 10^{-4} Farad.
(a) Find the charge on the capacitor as a function of time $q(t)$ if $q(0)=0$.
(b) Find the current as a function of time $i(t)$.

(a) $R\frac{dq}{dt}+\frac{1}{C}q=E$

$200\frac{dq}{dt}+\frac{1}{10^{-4}}q=100$

linear, $P(t)=50$

$\frac{dq}{dt}+50q=\frac{1}{2}$

$I.F.=e^{\int 50dt}=e^{50t}$

$e^{50t}q=\int \frac{1}{2}e^{50t}dt$

$e^{50t}q=\frac{1}{2}\left(\frac{1}{50}e^{50t}\right)+C$

$\frac{e^{50t}q}{e^{50t}}=\frac{\frac{1}{100}e^{50t}}{e^{50t}}+\frac{C}{e^{50t}}$

$q=\frac{1}{100}+Ce^{-50t}$

now, $q(0)=0$:

$0=\frac{1}{100}+Ce^{0}$, $C=-\frac{1}{100}$

so $\boxed{q(t)=\frac{1}{100}-\frac{1}{100}e^{-50t}}$

(b) $i=\frac{dq}{dt}$

$i=-\frac{1}{100}e^{-50t}(-50)$

$\boxed{i(t)=\frac{1}{2}e^{-50t}}$

#10. Suppose a small cannonball weighing 16 pounds is shot vertically upward with an initial velocity $v_0 = 300\ ft/s$. The answer to the question "How high does the cannonball go?" depends upon whether we take air resistance into account or not.

(a) Suppose air resistance is ignored. If the positive direction is upward, then a model for the state of the cannonball is given by:

$$\frac{d^2s}{dt^2} = -g$$

Since $\frac{ds}{dt} = v(t)$ the last differential equation is the same as:

$$\frac{dv}{dt} = -g$$

where we take $g = 32\ ft/s^2$.
Find the velocity of the cannonball at time t.

$\frac{dv}{dt} = -g$ (if v + upward)

$\frac{dv}{dt} = -32$ separable,

$\int dv = \int -32dt$

$v = -32t + C$ & $v(0) = 300$

$300 = -32(0) + C$, $C = 300$

$\boxed{v(t) = -32t + 300}$

(b) Use the result obtained in part (a) to determine the height $s(t)$ of the cannonball measured from ground level.

(c) Find the maximum height attained by the cannonball.

(b) $v = \frac{ds}{dt}$, so $s = \int v\,dt$

$s(t) = \int (-32t + 300)dt$

$s = -16t^2 + 300t + C$

assume cannonball started at ground level:

$s(0) = 0$

$0 = -16(0)^2 + 300(0) + C$, $C = 0$

$\boxed{s(t) = -16t^2 + 300t}$

(c) max height when $v = 0$

$0 = -32t + 300$

$32t = 300$

$t = \frac{300}{32} = 9.375$ sec

$s(9.375) = -16(9.375)^2 + 300(9.375)$

$\boxed{S_{max} = 1406.25\ ft}$

#11. Repeat problem #10, but this time, assume that air resistance is proportional to instantaneous velocity. Use the following differential equation:

$$m\frac{dv}{dt} = mg - kv$$

...and assume $k = 0.0025$.

(a) Find the velocity of the cannonball at time t.
(b) Use the result obtained in part (a) to determine the height $s(t)$ of the cannonball measured from ground level.
(c) Find the maximum height attained by the cannonball.

(a) $m\frac{dv}{dt} = mg - kv$ assumes v+ downward

change to v+ upward (kv term stays same, still opposed motion)

$m\frac{dv}{dt} = -mg - kv$ mass: $F = mg$

$16 \text{ lbs} = m\,32$

$\frac{dv}{dt} = -g - \frac{k}{m}v$ $m = \frac{16}{32} = \frac{1}{2}$ slug

$\frac{dv}{dt} = -32 - \frac{.0025}{1/2}v$

$\frac{dv}{dt} + .005v = -32$ linear, w/ $P(t) = .005$

I.F. $= e^{\int .005\,dt} = e^{.005t}$

$e^{.005t}v = \int -32e^{.005t}\,dt$

$e^{.005t}v = (-32)\left(\frac{1}{.005}\right)e^{.005t} + c$

$\frac{e^{.005t}v}{e^{.005t}} = \frac{-6400e^{.005t}}{e^{.005t}} + \frac{c}{e^{.005t}}$

$v(t) = -6400 + Ce^{-.005t}$

now, $v(0) = 300$

$300 = -6400 + Ce^{0}$, $C = 6700$

so $\boxed{v(t) = -6400 + 6700e^{-.005t}}$

(b) $\frac{ds}{dt} = v$, so $s = \int v\,dt$

$s = \int (-6400 + 6700e^{-.005t})\,dt$

$s = -6400t + \frac{6700}{-.005}e^{-.005t} + c$

assume start on ground: $s(0) = 0$

$0 = -6400(0) - 1340000e^{0} + c$

$C = 1340000$

$\boxed{s(t) = -6400t - 1340000e^{-.005t} + 1340000}$

(c) max height when $v = 0$

$v = -6400 + 6700e^{-.005t}$

$0 = -6400 + 6700e^{-.005t}$

$e^{-.005t} = \frac{6400}{6700} = \frac{64}{67}$

$-.005t = \ln\left(\frac{64}{67}\right)$

$t = \frac{\ln\left(\frac{64}{67}\right)}{-.005} = 9.1619$ sec

so: $s_{max} = s(9.1619)$

$= -6400(9.1619) - 1340000e^{-.005(9.1619)} + 1340000$

$= \boxed{1363.8 \text{ ft}}$

#12. Two large containers A and B of the same size are filled with different fluids. The fluids in containers A and B are maintained at 0 °C and 100 °C, respectively. A small metal bar, whose initial temperature is 100 °C, is lowered into container A. After 1 minute the temperature of the bar is 90 °C. After 2 minutes the bar is removed from container A and instantly transferred to container B. After 1 minute in container B the temperature of the bar rises 10 °C.
How long, measured from the start of the entire process, will it take for the bar to reach a temperature of 99.9 °C?

(phase 1)
Container A

min t	°C T
0	100
1	90

$T = 0 + Ce^{kt}$

$100 = 0 + Ce^{0}$

$C = 100$

$T = 0 + 100e^{kt}$

$90 = 0 + 100e^{k(1)}$

$e^{k} = \frac{90}{100}$, $k = \ln\left(\frac{90}{100}\right)$

$k = -0.1053605157$

$T = 0 + 100e^{-0.1053605157t}$

at 2 mins:

$T(2) = 0 + 100e^{-0.1053605159(2)}$

$= 80.9999 = \underline{81\ °C}$

(this is the initial temp for the bar as it enters the B container)

2 different mediums (2 phases):

Container A: $T = T_m + Ce^{kt}$, $T = 0 + Ce^{kt}$

Container B: $T = T_m + Ce^{kt}$, $T = 100 + Ce^{kt}$

(different C, k)

(phase 2)
Container B

restart to over again

min t	°C T
0	81
1	91

$T = 100 + Ce^{kt}$

$81 = 100 + Ce^{0}$

$C = 81 - 100 = -19$

$T = 100 - 19e^{kt}$

$91 = 100 - 19e^{k(1)}$

$-19e^{k} = -9$, $e^{k} = \frac{-9}{-19}$

$k = \ln\left(\frac{9}{19}\right) = -0.7472144018$

$T = 100 - 19e^{-0.7472144018t}$

$99.9 = 100 - 19e^{-0.7472144018t}$

$-19e^{-0.7472144018t} = -0.1$

$-0.7472144018t = \ln\left(\frac{-0.1}{-19}\right)$

$t = \frac{\ln\left(\frac{.1}{19}\right)}{-0.7472144018} = 7.022\text{ min}$

time from start of process is $7.022 + 2$ min in container A

$= \boxed{9.022\text{ min}}$

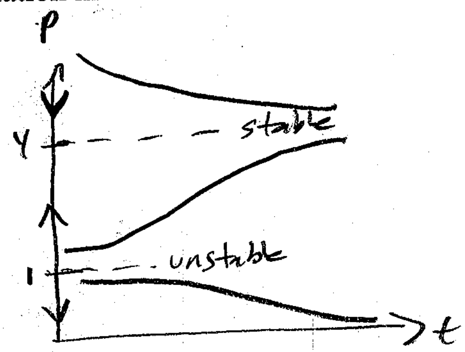
3.2

#1. If a constant number h of fish are harvested from a fishery per unit time, than a model for the population $P(t)$ of the fishery at time t is given by:

$$\frac{dP}{dt} = P(a-bP)-h, \quad P(0)=P_0$$

where a, b, h, and P_0 are positive constants. Suppose $a=5$, $b=1$, and $h=4$.

(a) Since the DE is autonomous, use the phase portrait concept to sketch representative solution curves corresponding to the cases $P_0>4$, $1<P_0<4$, and $0<P_0<1$. Determine the long-term behavior of the population in each case.



(test $P=5$)

$$\frac{dP}{dt} = 5(5-5)-4 < 0$$

(test $P=2$)

$$\frac{dP}{dt} = 2(5-2)-4 > 0$$

(test $P=0.5$)

$$\frac{dP}{dt} = 0.5(5-0.5)-4 < 0$$

for $P_0>4$, population decreases to 4

for $1<P_0<4$, population increases to 4

for $P_0<1$, population decreases to 0

(b) Solve the differential equation initial-value problem. Then verify the results of your phase portrait in part (a) by graphing the solution with an initial condition taken from each of the three intervals given.

$$\frac{dP}{dt} = P(5-P)-4 = -P^2+5P-4$$

separable $\int \frac{1}{P^2-5P+4}\,dP = \int -dt$

partial fraction for left integration:

$$P^2-5P+4 = (P-4)(P-1)$$

$$\frac{1}{(P-4)(P-1)} = \frac{A}{P-4} + \frac{B}{P-1} = \frac{A(P-1)+B(P-4)}{(P-4)(P-1)}$$

$$1 = AP - A + BP - 4B = (A+B)P + (-A-4B)$$

$$\begin{cases} A+B=0 \\ -A-4B=1 \end{cases} \quad A=\tfrac{1}{3},\ B=-\tfrac{1}{3}$$

$$\tfrac{1}{3}\int \frac{1}{P-4}\,dP - \tfrac{1}{3}\int \frac{1}{P-1}\,dP = \int -dt$$

$$\tfrac{1}{3}\ln|P-4| - \tfrac{1}{3}\ln|P-1| = -t + C_1$$

$$\ln|P-4| - \ln|P-1| = -3t + C_2$$

$$\ln\left|\frac{P-4}{P-1}\right| = -3t + C_2$$

$$\frac{P-4}{P-1} = e^{-3t+C_2} = e^{-3t}e^{C_2} = Ce^{-3t}$$

$$P(0)=P_0, \quad \frac{P_0-4}{P_0-1} = Ce^0, \quad C = \frac{P_0-4}{P_0-1}$$

$$P-4 = (P-1)Ce^{-3t} = PCe^{-3t} - Ce^{-3t}$$

$$P - PCe^{-3t} = 4 - Ce^{-3t}$$

continued...

(c) Use the information in parts (a) and (b) to determine whether the fishery population becomes extinct in finite time. If so, find that time.

any $P_0<1$ leads to extinction when numerator $=0$

$$4 - \left(\frac{P_0-4}{P_0-1}\right)e^{-3t} \stackrel{?}{=} 0$$

$$\left(\frac{P_0-4}{P_0-1}\right)e^{-3t} = 4$$

$$e^{-3t} = 4\left(\frac{P_0-1}{P_0-4}\right)$$

$$-3t = \ln\left(4\left(\frac{P_0-1}{P_0-4}\right)\right)$$

$$\boxed{t = \frac{\ln\left(4\left(\frac{P_0-1}{P_0-4}\right)\right)}{-3}}$$

3.2 #16 continued:

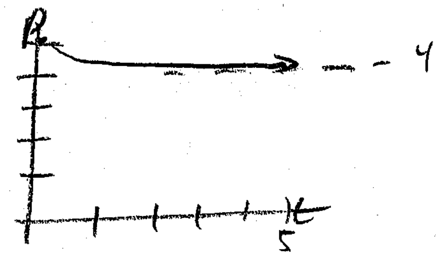
$$P(1-Ce^{-3t}) = 4-Ce^{-3t}$$

$$P = \frac{4-Ce^{-3t}}{1-Ce^{-3t}}$$ from above, $C = \frac{P_0-4}{P_0-1}$

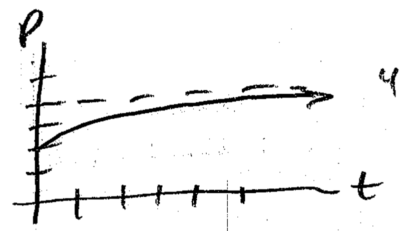
so $$\boxed{P(t) = \frac{4-\left(\frac{P_0-4}{P_0-1}\right)e^{-3t}}{1-\left(\frac{P_0-4}{P_0-1}\right)e^{-3t}}}$$

trying the 3 test values:

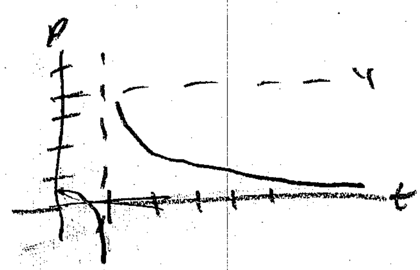
$P_0 = 5$: $$P(t) = \frac{4-\frac{5-4}{5-1}e^{-3t}}{1-\frac{5-4}{5-1}e^{-3t}} = \frac{4-\frac{1}{4}e^{-3t}}{1-\frac{1}{4}e^{-3t}}$$



$P_0 = 2$: $$P(t) = \frac{4-\frac{2-4}{2-1}e^{-3t}}{1-\frac{2-4}{2-1}e^{-3t}} = \frac{4+2e^{-3t}}{1+2e^{-3t}}$$



$P_0 = 0.5$: $$P(t) = \frac{4-\frac{0.5-4}{0.5-1}e^{-3t}}{1-\frac{0.5-4}{0.5-1}e^{-3t}} = \frac{4-7e^{-3t}}{1-7e^{-3t}}$$



vertical asymptote at $1-7e^{-3t}=0$

$7e^{-3t}=1$

$e^{-3t}=\frac{1}{7}$

$-3t=\ln(1/7)$, $t=\frac{\ln(1/7)}{-3} = 0.6486$