DiffEq - Ch 4 - Required Practice

Name: Key

4.1

#1. The given family of functions is the general solution of the differential equation on the indicated interval. Find a member of the family that is a solution of the initial-value problem.

$y = C_1e^x + C_2e^{-x}$, $(-\infty, \infty)$;

$y'' - y = 0$, $y(0) = 0$, $y'(0) = 1$

$y = C_1e^x + C_2e^{-x}$ $\quad$ $y' = C_1e^x - C_2e^{-x}$

$0 = C_1e^0 + C_2e^0$ $\quad$ $1 = C_1e^0 - C_2e^0$

$C_1 - C_2 = 1$

$$\begin{cases} C_1 + C_2 = 0 \\ C_1 - C_2 = 1 \end{cases}$$

$2C_1 = 1$

$C_1 = \frac{1}{2}, \quad C_2 = -\frac{1}{2}$

$$\boxed{y = \tfrac{1}{2}e^x - \tfrac{1}{2}e^{-x}}$$

#2. The given family of functions is the general solution of the differential equation on the indicated interval. Find a member of the family that is a solution of the initial-value problem.

$y = C_1x + C_2x\ln x$, $(0, \infty)$;

$x^2y'' - xy' + y = 0$, $y(1) = 3$, $y'(1) = -1$

$y = C_1x + C_2x\ln x$ $\quad$ $y' = C_1 + C_2x\cdot\frac{1}{x} + \ln x\, C_2$

$y' = C_1 + C_2 + C_2\ln x$

$3 = C_1(1) + C_2(1)\underbrace{\ln(1)}_{0}$ $\quad$ $-1 = C_1 + C_2 + C_2\underbrace{\ln(1)}_{0}$

$C_1 = 3$ $\quad$ $C_2 = -1 - C_1$

System $\begin{cases} C_1 = 3 \\ C_2 = -1 - C_1 \end{cases}$ $\quad$ $C_1 = 3$, $C_2 = -1 - 3 = -4$

$$\boxed{y = 3x - 4x\ln x}$$

#3. Determine whether the given set of functions is linearly independent on the interval $(-\infty, \infty)$.

$f_1(x) = x$, $f_2(x) = x^2$, $f_3(x) = 4x - 3x^2$

$$W = \begin{vmatrix} x & x^2 & 4x - 3x^2 \\ 1 & 2x & 4 - 6x \\ 0 & 2 & -6 \end{vmatrix}$$

$$= x(-12x - (8 - 12x)) - x^2(-6 - 0) + (4x - 3x^2)(2 - 0)$$

$$= -12x^2 - 8x + 12x^2 + 6x^2 + 8x - 6x^2$$

$= 0$

__not__ linearly independent

#4. Determine whether the given set of functions is linearly independent on the interval $(-\infty, \infty)$.

$f_1(x) = 0$, $f_2(x) = x$, $f_3(x) = e^x$

$$W = \begin{vmatrix} 0 & x & e^x \\ 0 & 1 & e^x \\ 0 & 0 & e^x \end{vmatrix}$$

use this column

$= 0(\sim) - 0(\sim) + 0(\sim) = 0$

__not__ linearly independent

#5. Determine whether the given set of functions is linearly independent on the interval $(-\infty,\infty)$.

$f_1(x)=1+x,\quad f_2(x)=x,\quad f_3(x)=x^2$

$$W=\begin{vmatrix} 1+x & x & x^2 \\ 1 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix}$$

$= (1+x)(2-0)-x(2-0)+x^2(0-0)$

$= 2+2x-2x$

$= 2 \neq 0$

these solutions are linearly independent

#6. Verify that the given functions form a fundamental set of solutions of the differential equation on the indicated interval. Then form the general solution.

$y''-y'-12y=0;\quad e^{-3x}, e^{4x}, (-\infty,\infty)$

verify solution of DE;

$y=e^{-3x}$ $\quad y''-y'-12y=0$

$y'=-3e^{-3x}$ $\quad 9e^{-3x}-(-3e^{-3x})-12(e^{-3x})=0$

$0=0$ ✓ verified

$y''=9e^{-3x}$

$y=e^{4x}$ $\quad y''-y'-12y=0$

$y'=4e^{4x}$ $\quad 16e^{4x}-4e^{4x}-12e^{4x}=0$

$0=0$ ✓ verified

$y''=16e^{4x}$

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verify solutions are linearly independent

$$W=\begin{vmatrix} e^{-3x} & e^{4x} \\ -3e^{-3x} & 4e^{4x} \end{vmatrix}$$

$= 4e^{4x}e^{-3x}-(-3e^{-3x}e^{4x})$

$= 4e^{x}+3e^{x}=7e^{x}\neq 0$

Solutions are linearly independ

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general solution:

$$\boxed{y=C_1e^{-3x}+C_2e^{4x}}$$

#7. Verify that the given functions form a fundamental set of solutions of the differential equation on the indicated interval. Then form the general solution.

$y''-2y'+5y=0; \quad e^x\cos 2x, e^x\sin 2x, \ (-\infty,\infty)$

verify solutions to DE:

$y=e^x\cos 2x$

$y'=e^x(-2\sin 2x)+\cos(2x)e^x=-2e^x\sin 2x+e^x\cos 2x$

$y''=(-2e^x)(2\cos 2x)+(\sin 2x)(-2e^x)$

$\quad +(e^x)(-2\sin 2x)+(\cos 2x)e^x=$

$=-4e^x\cos 2x-2e^x\sin 2x-2e^x\sin 2x+e^x\cos 2x$

$=-3e^x\cos 2x-4e^x\sin 2x$

DE: $y''-2y'+5y=0$

$(-3e^x\cos 2x-4e^x\sin 2x)-2(-2e^x\sin 2x+e^x\cos 2x)$

$\quad +5(e^x\cos 2x)\overset{?}{=}0, \quad 0=0$ ✓

$y=e^x\sin 2x$

$y'=e^x(2\cos 2x)+(\sin 2x)e^x=2e^x\cos 2x+e^x\sin 2x$

$y''=(2e^x)(-2\sin 2x)+(\cos 2x)(2e^x)$

$\quad +(e^x)(2\cos 2x)+(\sin 2x)(e^x)$

$=-4e^x\sin 2x+2e^x\cos 2x+2e^x\cos 2x+e^x\sin 2x$

$=-3e^x\sin 2x+4e^x\cos 2x$

DE: $y''-2y'+5y=0$

$(-3e^x\sin 2x+4e^x\cos 2x)-2(2e^x\cos 2x+e^x\sin 2x)$

$\quad +5(e^x\sin 2x)\overset{?}{=}0, \quad 0=0$ ✓

verify linearly independent

$$W=\begin{vmatrix} e^x\cos 2x & e^x\sin 2x \\ e^x(-2\sin 2x)+\cos 2x e^x & e^x(2\cos 2x)+\sin 2x e^x \end{vmatrix}$$

$=e^x\cos 2x(e^x(2\cos 2x)+\sin 2x e^x)$

$-e^x\sin 2x(e^x(-2\sin 2x)+\cos 2x e^x)$

$=2e^{2x}\cos^2 2x+e^{2x}\cos 2x\sin 2x$

$+2e^{2x}\sin^2 2x-e^{2x}\sin 2x\cos 2x$

$=4e^{2x}(\cos^2 2x+\sin^2 2x)=4e^{2x}\neq 0$

are linearly independent

general solution:

$$\boxed{y=c_1e^x\cos 2x+c_2e^x\sin 2x}$$

#8. Verify that the given functions form a fundamental set of solutions of the differential equation on the indicated interval. Then form the general solution.

$x^2y''-6xy'+12y=0; \quad x^3, x^4, \ (0,\infty)$

verify solutions to DE:

$y=x^3 \quad x^2y''-6xy'+12y=0$

$y'=3x^2 \quad x^2(6x)-6x(3x^2)+12(x^3)=0$

$y''=6x \quad 6x^3-18x^3+12x^3=0$

$0=0$ ✓

$y=x^4 \quad x^2y''-6xy'+12y=0$

$y'=4x^3 \quad x^2(12x^2)-6x(4x^3)+12(x^4)=0$

$y''=12x^2 \quad 12x^4-24x^4+12x^4=0$

$0=0$ ✓

verify linearly independent...

$$W=\begin{vmatrix} x^3 & x^4 \\ 3x^2 & 4x^3 \end{vmatrix}$$

$=4x^6-3x^6=x^6\neq 0$ over $(0,\infty)$

so are linearly independent

general solution is...

$$\boxed{y=c_1x^3+c_2x^4}$$

4.2

#1. The indicated function is a solution of the given differential equation (no need to verify). Use reduction of order to find a second solution.

$y'' - 4y' + 4y = 0; \quad y_1 = e^{2x}$

$y_2 = u e^{2x}$

$y' = u(2e^{2x}) + e^{2x}u'$

$y'' = u(4e^{2x}) + (2e^{2x})u' + e^{2x}u'' + u'(2e^{2x})$

$= e^{2x}u'' + 4e^{2x}u' + 4e^{2x}u$

$y'' - 4y' + 4y = 0$

$[e^{2x}u'' + 4e^{2x}u' + 4e^{2x}u] - 4[2e^{2x}u + e^{2x}u']$

$+4[e^{2x}u] = 0$

$(e^{2x})u'' + (4e^{2x} - 4e^{2x})u' + (4e^{2x} - 8e^{2x} + 4e^{2x})u = 0$

$e^{2x}u'' = 0$

$u'' = 0$ (let $w = u'$)

$w' = 0$

$w = \int 0\,dx = c_1$

$u' = c_1$

$u = \int c_1\,dx = c_1x + c_2$

Simplest if $c_1 = 1$, $c_2 = 0$

$u = x$

So $y_2 = ue^{2x}$

$\boxed{y_2 = xe^{2x}}$

#2. The indicated function is a solution of the given differential equation (no need to verify). Use reduction of order to find a second solution.

$y'' + 16y = 0; \quad y_1 = \cos(4x)$

$y_2 = u\cos(4x)$

$y' = u(-4\sin(4x)) + \cos(4x)u'$

$y'' = u(-16\cos(4x)) + (-4\sin(4x))u' + \cos(4x)u''$

$+u'(-4\sin(4x)$

$= \cos(4x)u'' - 8\sin(4x)u' - 16\cos(4x)u$

$y'' + 16y = 0$

$[\cos(4x)u'' - 8\sin(4x)u' - 16\cos(4x)u]$

$+16[u\cos(4x)] = 0$

$(\cos(4x))u'' - (8\sin(4x))u' = 0$

let $w = u'$

$\cos(4x)\,w' - 8\sin(4x)w = 0$

$w' - 8\tan(4x)w = 0$ linear with $P(x) = -8\tan(4x))$

$\int -8\tan(4x)dx \quad n = 4x, \frac{dn}{dx} = 4, dx = \frac{1}{4}d$

$-8(\frac{1}{4})\int \tan n\,dn$

$-2[\ln|\sec n|] = -2\ln|\sec(4x)|$

$I.F. = e^{\int(-8\tan(4x))dx} = e^{-2\ln(\sec(4x))}$

$= e^{\ln(\sec(4x)^{-2})} = \sec(4x)^{-2}$

$= \cos^2(4x)$

So $\cos^2(4x)w = \int 0\,dx$

$\cos^2(4x)w = c_1$

$u' = w = c_1\sec^2(4x) \quad m = 4x, \frac{dm}{dx} = 4$

So $u = \int c_1\sec^2(4x)dx = \quad dx = \frac{1}{4}d$

$\frac{1}{4}\int c_1\sec^2(m)dm = \frac{1}{4}c_1\tan(m) + C$

$= c_3\tan(4x) + c_2$ simplest is $c_3 = 1$, $c_2 =$

$u = \tan(4x)$

So $y_2 = u\cos 4x = \tan 4x\cos 4x = \frac{\sin 4x}{\cos 4x}\cos 4x$

$\boxed{y_2 = \sin 4x}$

#3. The indicated function is a solution of the given differential equation (no need to verify). Use reduction of order to find a second solution.

$9y''-12y'+4y=0; \quad y_1=e^{\left(\frac{2}{3}x\right)}$

$y_2=\underline{u}\,\underline{e^{\frac{2}{3}x}}$

$y'=\underline{u}\,\underline{\tfrac{2}{3}e^{\frac{2}{3}x}}+\underline{e^{2/3x}}\,\underline{u'}$

$y''=u\left(\tfrac{4}{9}e^{2/3x}\right)+\left(\tfrac{2}{3}e^{2/3x}\right)u'+\left(e^{2/3x}\right)u''$

$+u'\left(\tfrac{2}{3}e^{2/3x}\right)$

$9y''-12y'+4y=0$

$9\left[\tfrac{4}{9}e^{2/3x}u+\tfrac{4}{3}e^{2/3x}u'+e^{2/3x}u''\right]$

$-12\left[\tfrac{2}{3}e^{2/3x}u+e^{2/3x}u'\right]+4e^{2/3x}u=0$

$\underbrace{\left(9e^{2/3x}\right)}_{\text{never zero}}u''=0$, so $u''=0$

$w=u'$, $w=\int 0\,dx=C_1$

$u'=C_1$

$u=\int C_1\,dx=C_1x+C_2$

simplest: $C_1=1,\ C_2=0$

$u=x$

$$\boxed{y_2=xe^{\frac{2}{3}x}}$$

#4. The indicated function is a solution of the given differential equation (no need to verify). Use reduction of order to find a second solution.

$x^2y''-7xy'+16y=0; \quad y_1=x^4$

$y_2=\underline{u}\,\underline{x^4}$

$y'=\underline{u}\,\underline{(4x^3)}+\underline{x^4}\,\underline{u'}$

$y''=u(12x^2)+4x^3u'+x^4u''+u'(4x^3)$

$=12x^2u+8x^3u'+x^4u''$

into DE ...

$x^2y''-7xy'+16y=0$

$x^2\left[12x^2u+8x^3u'+x^4u''\right]-7x\left[4x^3u+x^4u'\right]$

$+16\left(x^4u\right)=0$

$x^6u''+\left(8x^5-7x^5\right)u'+(12\underbrace{x^4-28x^4+16x^4)}_{0}u=$

$x^6u''+x^5u'=0$ Let $w=u'$

$\frac{x^6w'}{x^6}+\frac{x^5w}{x^6}=\frac{0}{x^6}$

$w'+\frac{1}{x}w=0$ linear, w/ $P(x)=\frac{1}{x}$

$I.F.=e^{\int\frac{1}{x}dx}=e^{\ln x}=x$

$xw=\int 0x\,dx=\int 0\,dx=C_1$

$w=\frac{C_1}{x}=u'$

so $u=\int\frac{C_1}{x}dx=C_1\ln|x|+C_2$

simplest: $C_1=1,\ C_2=0$

$u=\ln|x|$

$y_2=ux^4=(\ln|x|)x^4$

$$\boxed{y_2=x^4\ln|x|}$$

#5. The indicated function is a solution of the given differential equation (no need to verify). Use reduction of order to find a second solution.

$xy'' + y' = 0; \quad y_1 = \ln x$

$y_2 = u \ln x$

$y' = u\frac{1}{x} + \ln x\, u' = ux^{-1} + \ln x\, u'$

$y'' = u(-x^{-2}) + \frac{1}{x}u' + \ln x\, u'' + u'\frac{1}{x}$

$= -\frac{1}{x^2}u + \frac{2}{x}u' + \ln x\, u''$

into DE: $xy'' + y' = 0$

$x[-\frac{1}{x^2}u + \frac{2}{x}u' + \ln x\, u''] + [\frac{1}{x}u + \ln x\, u'] = 0$

$(x\ln x)u'' + (2+\ln x)u' + (\underbrace{-\frac{1}{x} + \frac{1}{x}}_{\Rightarrow}) = 0$ let $w = u'$

$x\ln x\, w' + (2+\ln x)w = 0$ linear, w/ $P(x) = \frac{2+\ln x}{x\ln x} = \frac{2}{x\ln x} + \frac{1}{x}$ can integrate, but I.F. is messy, switch to formula..

$w' + (\frac{2+\ln x}{x\ln x})w = 0$

$y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{(y_1(x))^2} dx$

$xy'' + y' = 0$

$y'' + \frac{1}{x}y' = 0$

here, $P(x) = \frac{1}{x}$

$y_2 = \ln x \int \frac{e^{-\int \frac{1}{x}dx}}{(\ln x)^2} dx$

$= \ln x \int \frac{e^{-\ln|x|}}{(\ln x)^2} dx = \ln x \int \frac{e^{\ln(|x|^{-1})}}{(\ln x)^2} dx = \ln x \int \frac{1}{|x|(\ln x)^2} dx$

$w = \ln x$ 'sub' w/ w

$\frac{dw}{dx} = \frac{1}{x}$

$dw = \frac{1}{x}dx$

$y_2 = \ln x \int u^{-2} du$

$= \ln x(-\frac{1}{u} + c)$

$= -\ln x \frac{1}{\ln x} + C$

$= -1 - \ln x C$

$\boxed{y_2 = -1 - \ln x}$

not simplest

$y_2 = 1$

4.3

#1. Find the general solution of the differential equation.

$4y'' + y' = 0$

$4m^2 + 1m = 0$

$m(4m+1) = 0$

$m = 0, \; m = -\frac{1}{4}$

$y = C_1 e^{0x} + C_2 e^{-1/4x}$

$$\boxed{y = C_1 + C_2 e^{-\frac{1}{4}x}}$$

#2. Find the general solution of the differential equation.

$y'' + 8y' + 16y = 0$

$m^2 + 8m + 16 = 0$

$(m+4)(m+4) = 0$

$m = -4$ (repeated)

$$\boxed{y = C_1 e^{-4x} + C_2 x e^{-4x}}$$

#3. Find the general solution of the differential equation.

$12y'' - 5y' - 2y = 0$

M	A
-24	-5
$(8)(-3)$	
$(-8)(3)$	-5 ✓

$12m^2 - 5m - 2 = 0$

$$\frac{(12m-8)}{4}\frac{(12m+3)}{3}$$

$(3m-2)(4m+1) = 0$

$m = 2/3, \; m = -1/4$

$$\boxed{y = C_1 e^{2/3x} + C_2 e^{-1/4x}}$$

#4. Find the general solution of the differential equation.

$y'' + 9y = 0$

$m^2 + 9 = 0$

$m^2 = -9$

$m = \pm\sqrt{-9} = \pm\sqrt{9}\sqrt{-1} = \pm 3i = \overset{\alpha}{0} \pm \overset{\beta}{3}i$

$y = C_1 e^{0x}\cos(3x) + C_2 e^{0x}\sin(3x)$

$$\boxed{y = C_1 \cos(3x) + C_2 \sin(3x)}$$

#5. Find the general solution of the differential equation.

$y''-4y'+5y=0$

$m^2-4m+5=0$

$$m=\frac{4\pm\sqrt{16-4(1)(5)}}{2(1)}=\frac{4\pm\sqrt{-4}}{2}$$

$$=\frac{4\pm\sqrt{4}\sqrt{-1}}{2}=\frac{4\pm 2i}{2}=2\pm i$$

$$\boxed{y=c_1e^{2x}\cos(x)+c_2e^{2x}\sin(x)}$$

#6. Find the general solution of the differential equation.

$3y''+2y'+y=0$

$3m^2+2m+1=0$

$$m=\frac{-2\pm\sqrt{4-4(3)(1)}}{2(3)}=\frac{-2\pm\sqrt{-8}}{6}$$

$$=\frac{-2\pm\sqrt{4}\sqrt{2}\sqrt{-1}}{6}=\frac{-2\pm 2\sqrt{2}\,i}{6}$$

$$=-\frac{1}{3}\pm\frac{\sqrt{2}}{3}i$$

$$\boxed{y=c_1e^{\left(-\frac{1}{3}x\right)}\cos\left(\frac{\sqrt{2}}{3}x\right)+c_2e^{\left(-\frac{1}{3}x\right)}\sin\left(\frac{\sqrt{2}}{3}x\right)}$$

#7. Find the general solution of the differential equation.

$y'''-5y''+3y'+9y=0$

guess a root: 1?

$m^3-5m^2+3m+9=0$

try 1:

$$\begin{array}{r|rrrr} 1 & 1 & -5 & 3 & 9 \\ & & 1 & -4 & -1 \\ \hline & 1 & -4 & -1 & 8 \ne 0 \end{array}$$

try -1:

$$\begin{array}{r|rrrr} -1 & 1 & -5 & 3 & 9 \\ & & -1 & 6 & -9 \\ \hline & 1 & -6 & 9 & 0 \; ✓ \end{array}$$

m^2-6m+9

$(m+1)(m^2-6m+9)$

$(m+1)(m-3)(m-3)=0$

$m=-1$

$m=3$ (repeated)

so

$$\boxed{y=c_1e^{-x}+c_2e^{3x}+c_3xe^{3x}}$$

#8. Find the general solution of the differential equation.

$$y''' + 3y'' + 3y' + y = 0$$

$$m^3+3m^2+3m+1=0$$

$$\text{try } 1 \; \begin{array}{|cccc} 1 & 3 & 3 & 1 \\ & 1 & 4 & 7 \\ \hline 1 & 4 & 7 & \boxed{8} \neq 0 \end{array}$$

$$\text{try } -1 \; \begin{array}{|cccc} 1 & 3 & 3 & 1 \\ & -1 & -2 & -1 \\ \hline 1 & 2 & 1 & \boxed{0} \checkmark \end{array}$$

$$(m+1)(m^2+2m+1)=0$$

$$(m+1)(m+1)(m+1)=0$$

$m=-1$ multiplicity 3

$$\boxed{y=C_1e^{-x}+C_2xe^{-x}+C_3x^2e^{-x}}$$

#9. Solve the initial-value problem.

$$\frac{d^2y}{dt^2}-4\frac{dy}{dt}-5y=0, \quad y(1)=0, \; y'(1)=2$$

$$m^2-4m-5=0$$

$$(m+1)(m-5)=0$$

$$m=-1, \; m=5$$

$$y=C_1e^{-x}+C_2e^{5x} \qquad y'=-C_1e^{-x}+5C_2e^{5x}$$

$$y(1)=0 \qquad\qquad y'(1)=2$$

$$0=C_1e^{-(1)}+C_2e^{5(1)} \qquad 2=-C_1e^{-(1)}+5C_2e^{5(1}$$

system:

$$+\begin{cases} e^{-1}C_1+e^5C_2=0 \\ -e^{-1}C_1+5e^5C_2=2 \end{cases}$$

$$6e^5C_2=2$$

$$C_2=\frac{2}{6e^5}=\frac{1}{3}e^{-5}$$

$$e^{-1}C_1+e^5C_2=0$$

$$e^{-1}C_1+e^5\left(\frac{1}{3}e^{-5}\right)=0$$

$$e^{-1}C_1+\frac{1}{3}e^0=0$$

$$e^{-1}C_1+\frac{1}{3}=0$$

$$(e^1)e^{-1}C_1=-\frac{1}{3}(e^1)$$

$$C_1=-\frac{1}{3}e^1$$

so $\boxed{y=\left(-\frac{1}{3}e\right)e^{-x}+\left(\frac{1}{3}e^{-5}\right)e^{5x}}$

or

$$y=-\frac{1}{3}e^1e^{-x}+\frac{1}{3}e^{-5}e^{5x}$$

$$\boxed{y=-\frac{1}{3}e^{(1-x)}+\frac{1}{3}e^{(-5+5x)}}$$

#10. Solve the initial-value problem.

$y'' + y' + 2y = 0, \quad y(0) = 0, \quad y'(0) = 0$

$m^2 + m + 2 = 0$

$m = \frac{-1 \pm \sqrt{1-(4)(1)(2)}}{2(1)} = \frac{-1 \pm \sqrt{-7}}{2} = \frac{-1 \pm \sqrt{7}\sqrt{-1}}{2} = \frac{-1 \pm \sqrt{7}\, i}{2} = -\frac{1}{2} \pm \frac{\sqrt{7}}{2} i$

$y = \underline{c_1 e^{-1/2x}} \underline{\cos\left(\frac{\sqrt{7}}{2}x\right)} + \underline{c_2 e^{-1/2x}} \underline{\sin\left(\frac{\sqrt{7}}{2}x\right)}$

$y(0) = 0$

$0 = c_1 e^0 \cos(0) + c_2 e^0 \sin 0$

$0 = c_1 + 0 \rightarrow c_1 = 0$

$y = \underline{c_2 e^{-1/2x}} \underline{\sin\left(\frac{\sqrt{7}}{2}x\right)}$

$y' = \left(c_2 e^{-1/2x}\right)\left(\frac{\sqrt{7}}{2}\cos\left(\frac{\sqrt{7}}{2}x\right)\right) + \left(\sin\left(\frac{\sqrt{7}}{2}x\right)\right)\left(-\frac{1}{2}c_2 e^{-1/2x}\right)$

$y'(0) = 0$

$0 = c_2 e^0 \frac{\sqrt{7}}{2} \cos 0 + \sin 0 \left(-\frac{1}{2}\right) c_2 e^0$

$0 = \frac{\sqrt{7}}{2} c_2 + 0$, so c_2 also $= 0$

then $y = (0) e^{-1/2x} \cos\left(\frac{\sqrt{7}}{2}x\right) + (0) e^{-1/2x} \sin\left(\frac{\sqrt{7}}{2}x\right)$

$\boxed{y = 0}$ (the x-axis is the solution curve)

#11. Solve the initial-value problem.

$y''' + 12y'' + 36y' = 0, \quad y(0) = 0, \quad y'(0) = 1, \quad y''(0) = -7$

$m^3 + 12m^2 + 36m = 0$

$m(m^2 + 12m + 36) = 0$

$m(m+6)(m+6) = 0$

$m = 0, \quad m = -6$ (repeated) so $y = C_1 e^{0x} + C_2 e^{-6x} + C_3 x e^{-6x}$

$y = C_1 + C_2 e^{-6x} + C_3 x e^{-6x}$

$y(0) = 0$

$0 = C_1 + C_2 e^0 + C_3(0)e^0 \rightarrow \underline{C_1 + C_2 = 0}$

$y' = -6C_2 e^{-6x} + C_3 x(-6e^{-6x}) + e^{-6x}(C_3) = -6C_2 e^{-6x} - 6C_3 x e^{-6x} + C_3 e^{-6x}$

if $y'(0) = 1$:

$1 = -6C_2 e^0 - 6C_3(0)e^0 + C_3 e^0$

$1 = -6C_2 + C_3 \rightarrow \underline{-6C_2 + C_3 = 1}$

$y'' = 36C_2 e^{-6x} + (-6C_3 x)(-6e^{-6x}) + (e^{-6x})(-6C_3) - 6C_3 e^{-6x}$

if $y''(0) = -7$

$-7 = 36C_2 e^0 + 36C_3(0)e^0 - 6C_3 e^0 - 6C_3 e^0$

$-7 = 36C_2 - 12C_3 \rightarrow \underline{36C_2 - 12C_3 = -7}$

System: $\begin{cases} C_1 + C_2 = 0 \\ -6C_2 + C_3 = 1 \\ 36C_2 - 12C_3 = -7 \end{cases}$ $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -6 & 1 & 1 \\ 0 & 36 & -12 & -7 \end{array}\right]$ rref $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5/36 \\ 0 & 1 & 0 & -5/36 \\ 0 & 0 & 1 & 1/6 \end{array}\right]$

$C_1 = 5/36, \quad C_2 = -5/36, \quad C_3 = 1/6$

So $\boxed{y = \left(\frac{5}{36}\right) + \left(-\frac{5}{36}\right)e^{-6x} + \left(\frac{1}{6}\right)xe^{-6x}}$

4.4

#1. Solve the differential equation using the method of undermined coefficients.

$y''+3y'+2y=6$

y_c $m^2+3m+2=0$

$(m+2)(m+1)=0$

$m=-2,\ m=-1$

$y_c=C_1e^{-2x}+C_2e^{-x}$

y_p

$y_p=A$ (no absorption)

$y'=0$

$y''=0$

into DE: $y''+3y'+2y=6$

$[0]+3[0]+2A=6$

$A=\frac{6}{2}=3$

$y_p=3$

$y=y_c+y_p$

$$\boxed{y=C_1e^{-2x}+C_2e^{-x}+3}$$

#2. Solve the differential equation using the method of undermined coefficients.

$y''-10y'+25y=30x+3$

y_c: $y''-10y'+25y=0$

$m^2-10m+25=0$

$(m-5)(m-5)=0$

$m=5$ repeated

$y_c=C_1e^{5x}+C_2xe^{5x}$

y_p: from table for $30x+3$

$y_p=Ax+B$ (no absorption)

$y'=A$

$y''=0$

into DE: $y''-10y'+25y=30x+3$

$[0]-10[A]+25[Ax+B]=30x+3$

$(25A)x+(-10A+25B)=(30)x+(3)$

System: $25A=30 \rightarrow A=\frac{30}{25}=\frac{6}{5}$

$\{-10A+25B=3 \leftarrow$

$-10\left(\frac{6}{5}\right)+25B=3$

$-12+25B=3$

$25B=15$

$B=\frac{15}{25}=\frac{3}{5}$

$y_p=\frac{6}{5}x+\frac{3}{5}$

general solution:

$y=y_c+y_p$

$$\boxed{y=C_1e^{5x}+C_2xe^{5x}+\frac{6}{5}x+\frac{3}{5}}$$

& update posted

#3. Solve the differential equation using the method of undermined coefficients.

$$\frac{1}{4}y'' + y' + y = x^2 - 2x$$

$\underline{y_c}$: $\frac{1}{4}y''+y'+y=0$

$y''+4y'+4y=0$

$m^2+4m+4=0$

$(m+2)(m+2)=0$

$m=-2$ repeated

$y_c = C_1e^{-2x}+C_2xe^{-2x}$

$\underline{y_p}$: from table for x^2-2x

$y_p = Ax^2+Bx+C$ (no absorption)

$y' = 2Ax+B$

$y''=2A$

into DE:

$\frac{1}{4}y''+y'+y=x^2-2x$

$\frac{1}{4}[2A]+[2Ax+B]+[Ax^2+Bx+C]=x^2-2x$

$(A)x^2+(2A+B)x+(\frac{1}{2}A+B+C)=(1)x^2+(-2)x+(0)$

system:

$$\begin{cases} A=1 \\ 2A+B=-2 \\ \frac{1}{2}A+B+C=0 \end{cases}$$

$\rightarrow A=1$

$\leftarrow 2(1)+B=-2$

$B=-4$

$\frac{1}{2}(1)+(-4)+C=0$

$C=4-\frac{1}{2}=\frac{7}{2}$

$y_p = x^2-4x+\frac{7}{2}$

general solution:

$y=y_c+y_p$

$$\boxed{y=C_1e^{-2x}+C_2xe^{-2x}+x^2-4x+\frac{7}{2}}$$

unans M posted soln

✓ correct

#4. Solve the differential equation using the method of undermined coefficients.

$$y'' + 3y = -48x^2e^{3x}$$

$\underline{y_c}$: $y''+3y=0$

$m^2+3=0$, $m^2=-3$, $m=\pm\sqrt{-3}$

$m=0\pm\sqrt{3}i$

$y_c = C_1\cos(\sqrt{3}x)+C_2\sin(\sqrt{3}x)$

$\underline{y_p}$: from table

$y_p = \underline{(Ax^2+Bx+C)}\underline{e}^{3x}$ (no absorption)

$y' = \underline{(Ax^2+Bx+C)}(3\underline{e}^{3x})+\underline{e}^{3x}\underline{(2Ax+B)}$

$y''=(Ax^2+Bx+C)(9e^{3x})+(3e^{3x})(2Ax+B)$
$+(e^{3x})(2A)+(2Ax+B)(3e^{3x})$

into DE... $y''+3y=-48x^2e^{3x}$

$[(\underline{Ax^2}+\underline{Bx}+\underline{C})(9e^{3x})+6e^{3x}(\underline{2Ax}+\underline{B})+\underline{2A}e^{3x}]$
$+3[(\underline{Ax^2}+\underline{Bx}+\underline{C})e^{3x}] = -48x^2e^{3x}$

$(9A+3A)x^2e^{3x}+(9B+12A+3B)xe^{3x}$
$+(9C+6B+2A+3C)e^{3x}=-48x^2e^{3x}$

system:

$$\begin{cases} 12A=-48 \\ 12A+12B=0 \\ 2A+6B+12C=0 \end{cases}$$

$\rightarrow A=\frac{-48}{12}=-4$

$12(-4)+12B=0$

$B=4$

$2(-4)+6(4)+12C=0$

$12C=-16$, $C=\frac{-16}{12}=\frac{-4}{3}$

$y_p = (-4x^2+4x-\frac{4}{3})e^{3x}$

general solution:

$y=y_c+y_p$

$$\boxed{y=C_1\cos(\sqrt{3}x)+C_2\sin(\sqrt{3}x)-4x^2e^{3x}+4xe^{3x}-\frac{4}{3}e^{3}}$$

#5. Solve the differential equation using the method of undermined coefficients.

$y'' - 16y = 2e^{4x}$

$\underline{y_c}$: $y''-16y=0$

$m^2-16=0$

$(m+4)(m-4)=0$

$m=-4,\ m=4$

$y_c = C_1e^{-4x}+C_2e^{4x}$

$\underline{y_p}$: table for $2e^{4x}$

$y_p = Ae^{4x}$ (<u>absorbed</u> (same form as y_c term C_2e^{4x})

So $y_p = \underline{Axe^{4x}}$

$y' = Ax(4e^{4x})+e^{4x}(A) = \underline{4Axe^{4x}}+Ae^{4x}$

$y'' = 4Ax(4e^{4x})+e^{4x}(4A)+4Ae^{4x} = 16Axe^{4x}+8Ae^{4x}$

into DE... $y''-16y=2e^{4x}$

$[16Axe^{4x}+8Ae^{4x}]-16[Axe^{4x}]=2e^{4x}$

$\underset{0}{(16A \cancel{-16A})}\,xe^{4x}+(8A)e^{4x}=(2)e^{4x}$

$8A=2,\quad A=\frac{2}{8}=\frac{1}{4}$

$y_p=\frac{1}{4}xe^{4x}$

general solution:

$y=y_c+y_p$

$$\boxed{y=C_1e^{-4x}+C_2e^{4x}+\frac{1}{4}xe^{4x}}$$

#6. Solve the differential equation using the method of undermined coefficients.

$y'' + y = 2x\sin x$

y_c: $y'' + y = 0$

$m^2 + 1 = 0$

$m^2 = -1$

$m = \pm\sqrt{-1} = 0 \pm i$

$y_c = C_1 e^{0x}\cos x + C_2 e^{0x}\sin x$

$y_c = C_1\cos x + C_2\sin x$

y_p: from table for $2x\sin x$

$y_p = (Ax^2+Bx+C)\sin x + (Dx^2+Ex+F)\cos x$ but $C\sin x$ & $F\cos x$ would be absorbed

so we will just omit those. Instead:

$y_p = (Ax^2+Bx)\sin x + (Cx^2+Dx)\cos x$ (no absorption)

$y_p = Ax^2\sin x + Bx\sin x + Cx^2\cos x + Dx\cos x$

$y' = Ax^2\cos x + 2Ax\sin x + Bx\cos x + B\sin x - Cx^2\sin x + 2Cx\cos x - Dx\sin x + D\cos x$

$y'' = -Ax^2\sin x + 2Ax\cos x + 2Ax\cos x + 2A\sin x - Bx\sin x + B\cos x + B\cos x$

$y'' = -Ax^2\sin x + 2Ax\cos x + 2Ax\cos x + 2A\sin x - Bx\sin x + B\cos x + B\cos x - Dx\cos x - D\sin x - D\sin x$

$\quad -Cx^2\cos x - 2Cx\sin x - 2Cx\sin x + 2C\cos x - Dx\cos x - D\sin x - D\sin x$

into DE... $y'' + y = 2x\sin x$

$[-Ax^2\sin x - Cx^2\cos x + (4A-D)x\cos x + (-B-4C)x\sin x + (2A-2D)\sin x + (2B+2C)\cos x] = 2x\sin x$

$+[Ax^2\sin x + Bx\sin x + Cx^2\cos x + Dx\cos x]$

$(-\cancel{A}+\cancel{A})x^2\sin x + (-\cancel{C}+\cancel{C})x^2\cos x + (4A-D+D)x\cos x + (-B-4C+B)x\sin x + (2A-2D)\sin x + (2B+2C)\cos x = (2)x\sin x$

$0 \qquad\qquad 0$

system:
$$\begin{cases} 4A = 0 \\ -4C = 2 \\ 2A - 2D = 0 \\ 2B + 2C = 0 \end{cases} \quad \left[\begin{array}{cccc|c} 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & 2 \\ 2 & 0 & 0 & -2 & 0 \\ 0 & 2 & 2 & 0 & 0 \end{array}\right] \text{rref} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1/2 \\ 0 & 0 & 1 & 0 & -1/2 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \begin{matrix} =A \\ =B \\ =C \\ =D \end{matrix}$$

$y_p = (0x^2 + \frac{1}{2}x)\sin x + (-\frac{1}{2}x^2 + 0x)\cos x$

$y_p = \frac{1}{2}x\sin x - \frac{1}{2}x^2\cos x$

general solution:

$y = y_c + y_p$

$$\boxed{y = C_1\cos x + C_2\sin x + \frac{1}{2}x\sin x - \frac{1}{2}x^2\cos x}$$

#7. Solve the initial-value problem.

$5y'' + y' = -6x, \quad y(0) = 0, \quad y'(0) = -10$

$\underline{y_c}$: $5y''+y'=0$ $\qquad y_c = C_1e^{0x}+C_2e^{-1/5x}$

$5m^2+m=0$ $\qquad y_c = C_1+C_2e^{-1/5x}$

$m(5m+1)=0$

$m=0, m=-1/5$

$\underline{y_p}$: from table for $-6x$: $y_p = Ax+\textcircled{B}$ $\quad$ B absorbed (C_1)

try $y_p = Ax^2+Bx$

$y' = 2Ax+B$

$y'' = 2A$

into full DE ... $5y''+y'=-6x$ $\qquad$ system: $\begin{cases} 2A=-6 & A=-3 \\ 10A+B=0 & 10(-3)+B=0 \Rightarrow B=30 \end{cases}$

$5[2A]+[2Ax+B]=-6x$

$(2A)x+(10A+B)=(-6)x+(0)$ $\qquad y_p = -3x^2+30x$

general solution: $y=y_c+y_p$

$y = C_1+C_2e^{-1/5x}-3x^2+30x$

particular solution: use $y(0)=0, y'(0)=-10$

$y = C_1+C_2e^{-1/5x}-3x^2+30x$ $\qquad y' = -\frac{1}{5}C_2e^{-1/5x}-6x+30$

$0 = C_1+C_2e^0-0+0$ $\qquad -10 = -\frac{1}{5}C_2e^0-0+30$

$C_1+C_2=0$ $\qquad -\frac{1}{5}C_2 = -40$

System: $\begin{cases} C_1+C_2=0 \\ \frac{1}{5}C_2=40 \end{cases}$ $\qquad C_2 = 40(5) = 200$

$C_1 = -200$

particular solution: $\boxed{y = -200+200e^{-1/5x}-3x^2+30x}$

4.6 (we skip 4.5)

#1. Solve the differential equation using the method of variation of parameters.
$y'' + y = \sec x$.

y_c: $y''+y=0$ $\quad m=\pm\sqrt{-1}=0\pm i$

$m^2+1=0$ $\quad y_c = C_1\cos x + C_2 \sin x$

y_p: $y_p = u_1 \cos x + u_2 \sin x$

$$u_1' = \frac{\begin{vmatrix} 0 & \sin x \\ \sec x & \cos x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \frac{0-\sin x \sec x}{\cos^2 x+\sin^2 x} = \frac{-\tan x}{1}$$

$$u_1 = -\int \tan x\,dx = -\int \frac{\sin x}{\cos x}dx \qquad \begin{matrix} u=\cos x \\ du=-\sin x\,dx \end{matrix}$$

$$= \int \frac{1}{u}du = \ln|u| = \ln|\cos x|$$

$$u_2' = \frac{\begin{vmatrix} \cos x & 0 \\ -\sin x & \sec x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \frac{\cos x \sec x}{1} = \frac{1}{1} = 1$$

$$u_2 = \int 1\,dx = x$$

$y_p = \ln|\cos x|\cos x + x\sin x$

general Solution:

$$\boxed{y = C_1\cos x + C_2 \sin x + \ln|\cos x|\cos x + x\sin x}$$

#2. Solve the differential equation using the method of variation of parameters.
$y'' + y = \sin x$

y_c: $y''+y=0$ $\quad m=\pm\sqrt{-1}=0\pm i$

$m^2+1=0$ $\quad y_c = C_1\cos x + C_2 \sin x$

y_p: $y_p = u_1 \cos x + u_2 \sin x$

$$u_1' = \frac{\begin{vmatrix} 0 & \sin x \\ \sin x & \cos x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \frac{0-\sin^2 x}{\cos^2 x+\sin^2 x} = -\sin^2 x$$

(trig identity)

$$u_1 = -\int \sin^2 x\,dx = -\int\left(\frac{1}{2}-\frac{1}{2}\cos(2x)\right)dx$$

$$= -\frac{1}{2}x + \frac{1}{4}\sin(2x)$$

$$u_2' = \frac{\begin{vmatrix} \cos x & 0 \\ -\sin x & \sin x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \frac{\cos x \sin x}{1} = \cos x \sin x$$

$$u_2 = \int \cos x \sin x\,dx \qquad \begin{matrix} u=\sin x \\ du=\cos x\,dx \end{matrix}$$

$$= \int u\,du = \frac{1}{2}u^2 = \frac{1}{2}\sin^2 x$$

$y_p = \left(\frac{1}{2}x - \frac{1}{4}\sin(2x)\right)\cos x + \frac{1}{2}\sin^2 x \sin x$

general Solution:

$$\boxed{y = C_1\cos x + C_2\sin x - \frac{1}{2}x\cos x + \frac{1}{4}\sin(2x)\cos x + \frac{1}{2}\sin^3 x}$$

#3. Solve the differential equation using the method of variation of parameters.

$$y'' - 9y = \frac{9x}{e^{3x}}$$

$\underline{y_c}$: $y'' - 9y = 0$

$m^2 - 9 = 0$

$m^2 = 9, \; m = 3, \; m = -3$ $\qquad y_c = C_1 e^{3x} + C_2 e^{-3x}$

$\underline{y_p}$: $y_p = u_1 e^{3x} + u_2 e^{-3x}$

$$u_1' = \frac{\begin{vmatrix} 0 & e^{-3x} \\ \frac{9x}{e^{3x}} & -3e^{-3x} \end{vmatrix}}{\begin{vmatrix} e^{3x} & e^{-3x} \\ 3e^{3x} & -3e^{-3x} \end{vmatrix}} = \frac{0 - \frac{9x}{e^{3x}} e^{-3x}}{-3e^{3x}e^{-3x} - 3e^{3x}e^{-3x}} = \frac{-9xe^{-3x}e^{-3x}}{-3e^{0} - 3e^{0}} = \frac{-9xe^{-6x}}{-6} = \frac{3}{2}xe^{-6x}$$

by parts:

$u = x \qquad dv = e^{-6x}dx \qquad uv - \int v\,du$

$\frac{du}{dx} = 1 \qquad \int dv = \int e^{-6x}dx \qquad -\frac{1}{6}xe^{-6x} - \int\left(-\frac{1}{6}e^{-6x}\right)dx$

$du = dx \qquad v = -\frac{1}{6}e^{-6x}$

$-\frac{1}{6}xe^{-6x} + \frac{1}{6}\int e^{-6x}dx = -\frac{1}{6}xe^{-6x} - \frac{1}{36}e^{-6x}$

$$u_1 = \frac{3}{2}\int xe^{-6x}dx$$

$$u_1 = \frac{3}{2}\left[-\frac{1}{6}xe^{-6x} - \frac{1}{36}e^{-6x}\right] = -\frac{1}{4}xe^{-6x} - \frac{1}{24}e^{-6x}$$

$$u_2' = \frac{\begin{vmatrix} e^{3x} & 0 \\ 3e^{3x} & \frac{9x}{e^{3x}} \end{vmatrix}}{\begin{vmatrix} e^{3x} & e^{-3x} \\ 3e^{3x} & -3e^{-3x} \end{vmatrix}} = \frac{e^{3x}\frac{9x}{e^{3x}} - 0}{-6} = -\frac{3}{2}x$$

$$u_2 = -\frac{3}{2}\int x\,dx = -\frac{3}{2}\left(\frac{1}{2}x^2\right) = -\frac{3}{4}x^2$$

$$y_p = \left(-\frac{1}{4}xe^{-6x} - \frac{1}{24}e^{-6x}\right)e^{3x} + \left(-\frac{3}{4}x^2\right)e^{-3x} = -\frac{1}{4}xe^{-6x}e^{3x} - \frac{1}{24}e^{-6x}e^{3x} - \frac{3}{4}x^2e^{-3x}$$

$$y_p = -\frac{1}{4}xe^{-3x} - \frac{1}{24}e^{-3x} - \frac{3}{4}x^2e^{-3x}$$

general solution:

$$y = C_1e^{3x} + \underline{C_2e^{-3x}} - \frac{1}{4}xe^{-3x} - \underline{\frac{1}{24}e^{-3x}} - \frac{3}{4}x^2e^{-3x}$$

combining is OKAY: $C_2 - \frac{1}{24}$ = new constant C_3

$$\boxed{y = C_1e^{3x} + C_3e^{-3x} - \frac{1}{4}xe^{-3x} - \frac{3}{4}x^2e^{-3x}}$$

#4. Solve the differential equation using the method of variation of parameters.

$$y'' + 3y' + 2y = \frac{1}{1+e^x}$$

$\underline{y_c}$: $y'' + 3y' + 2y = 0$ $\quad m = -1,\ m = -2$

$m^2 + 3m + 2 = 0$ $\quad y_c = C_1 e^{-x} + C_2 e^{-2x}$

$(m+1)(m+2) = 0$

$\underline{y_p}$: $y_p = u_1 e^{-x} + u_2 e^{-2x}$

$$u_1' = \frac{\begin{vmatrix} 0 & e^{-2x} \\ \frac{1}{1+e^x} & -2e^{-2x} \end{vmatrix}}{\begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix}} = \frac{0 - \frac{e^{-2x}}{1+e^x}}{-2e^{-x}e^{-2x} + e^{-x}e^{-2x}} = \frac{-\frac{e^{-2x}}{1+e^x}}{-e^{-3x}} = \frac{e^{-2x}}{e^{-3x}(1+e^x)} = \frac{e^x}{1+e^x}$$

$$u_1 = \int \frac{e^x}{1+e^x}\,dx \quad \begin{matrix} u = 1+e^x \\ du = e^x dx \end{matrix} \quad u_1 = \int \frac{1}{u}\,du = \ln|u| = \ln|1+e^x|$$

$$u_2' = \frac{\begin{vmatrix} e^{-x} & 0 \\ -e^{-x} & \frac{1}{1+e^x} \end{vmatrix}}{\begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix}} = \frac{e^{-x}\frac{1}{1+e^x} - 0}{-e^{-3x}} = \frac{\left(\frac{e^{-x}}{1+e^x}\right)}{-e^{-3x}} = \frac{e^{-x}}{-e^{-3x}(1+e^x)} = -\frac{e^{2x}}{1+e^x} = \frac{e^x}{1+e^x} + \underset{\text{"something else"}}{A}$$

try to match this form!

$$\frac{-e^{2x}}{1+e^x} = \frac{e^x}{1+e^x} + \frac{A(1+e^x)}{1+e^x};$$ numerators:

$$A(1+e^x) = -e^{2x} - e^x = -e^x e^x - e^x$$

$$A = \frac{-e^x(e^x+1)}{1+e^x} = -e^x$$

$$u_2 = \int \frac{-e^{2x}}{1+e^x}\,dx = \int \frac{e^x}{1+e^x}\,dx - \int e^x\,dx$$

$$= \ln|1+e^x| - e^x$$

$$y_p = \ln|1+e^x|e^{-x} + (\ln|1+e^x| - e^x)e^{-2x} = e^{-x}\ln|1+e^x| + e^{-2x}\ln|1+e^x| - e^{-x}$$

general solution:

$$y = C_1 e^{-x} + C_2 e^{-2x} + e^{-x}\ln|1+e^x| + e^{-2x}\ln|1+e^x| - e^{-x}$$

OKAY to combine these; $C_1 - 1 =$ new constant C_3

$$\boxed{y = C_3 e^{-x} + C_2 e^{-2x} + e^{-x}\ln|1+e^x| + e^{-2x}\ln|1+e^x|}$$

#5. Solve the initial value problem using variation of parameters.

$4y'' - y = xe^{\left(\frac{1}{2}x\right)}$ $\rightarrow$ $y'' - \frac{1}{4}y = \frac{xe^{1/2x}}{4}\Big\}$ R+S

$\underline{y_c}$: $4y'' - y = 0$ $\qquad m = \pm\frac{1}{2}$

$4m^2 - 1 = 0$ $\qquad y_c = C_1e^{1/2x} + C_2e^{-1/2x}$

$m^2 = \frac{1}{4}$

$\underline{y_p}$: $y_p = u_1e^{1/2x} + u_2e^{-1/2x}$

$$u_1' = \frac{\begin{vmatrix} 0 & e^{-1/2x} \\ \frac{xe^{1/2x}}{4} & -\frac{1}{2}e^{-1/2x} \end{vmatrix}}{\begin{vmatrix} e^{1/2x} & e^{-1/2x} \\ \frac{1}{2}e^{1/2x} & -\frac{1}{2}e^{-1/2x} \end{vmatrix}} = \frac{0 - \frac{1}{4}xe^{1/2x}e^{-1/2x}}{-\frac{1}{2}e^{1/2x}e^{-1/2x} - \frac{1}{2}e^{1/2x}e^{-1/2x}} = \frac{-\frac{1}{4}xe^0}{-e^0} = \frac{1}{4}x$$

$$u_1 = \frac{1}{4}\int x\,dx = \frac{1}{8}x^2$$

$$u_2' = \frac{\begin{vmatrix} e^{1/2x} & 0 \\ \frac{1}{2}e^{1/2x} & \frac{xe^{1/2x}}{4} \end{vmatrix}}{\begin{vmatrix} e^{1/2x} & e^{-1/2x} \\ \frac{1}{2}e^{1/2x} & -\frac{1}{2}e^{-1/2x} \end{vmatrix}} = \frac{\frac{1}{4}xe^{1/2x}e^{1/2x} - 0}{-1} = -\frac{1}{4}xe^x$$

by parts; $u = x$ $\quad dv = e^x dx$

$\frac{du}{dx} = 1$ $\quad \int dv = \int e^x dx$

$du = dx$ $\quad v = e^x$

$$u_2 = -\frac{1}{4}\underbrace{\int xe^x dx}_{uv - \int v du}$$

$$= -\frac{1}{4}\left[xe^x - \int e^x dx\right] = -\frac{1}{4}\left[xe^x - e^x\right] = -\frac{1}{4}xe^x + \frac{1}{4}e^x$$

$$y_p = \left(\frac{1}{8}x^2\right)e^{1/2x} + \left(-\frac{1}{4}xe^x + \frac{1}{4}e^x\right)e^{-1/2x} = \frac{1}{8}x^2e^{1/2x} - \frac{1}{4}xe^{1/2x} + \frac{1}{4}e^{1/2x}$$

general solution:

$$y = \underline{C_1e^{1/2x}} + C_2e^{-1/2x} + \frac{1}{8}x^2e^{1/2x} - \frac{1}{4}xe^{1/2x} + \underline{\frac{1}{4}e^{1/2x}}$$

these can be combined $\quad C_1 + 1/4 = $ new constant C_3

$$\boxed{y = C_3e^{1/2x} + C_2e^{-1/2x} + \frac{1}{8}x^2e^{1/2x} - \frac{1}{4}xe^{1/2x}}$$

4.7

#1. Solve the differential equation.

$x^2y''-2y=0$

$am^2+(b-a)m+c=0$

$1m^2+(0-1)m-2=0$

$m^2-m-2=0$

$(m+1)(m-2)=0$

$m=-1,\ m=2$

$$\boxed{y=C_1x^{-1}+C_2x^2}$$

#2. Solve the differential equation.

$x^2y''+5xy'+3y=0$

$am^2+(b-a)m+c=0$

$1m^2+(5-1)m+3=0$

$m^2+4m+3=0$

$(m+1)(m+3)=0$

$m=-1,\ m=-3$

$$\boxed{y=C_1x^{-1}+C_2x^{-3}}$$

#3. Solve the differential equation.

$x^2y''-3xy'-2y=0$

$am^2+(b-a)m+c=0$

$1m^2+(-3-1)m+(-2)=0$

$m^2-4m-2=0$

$$m=\frac{4\pm\sqrt{16+4(1)(2)}}{2(1)}=\frac{4\pm\sqrt{24}}{2}=\frac{4\pm\sqrt{4}\sqrt{6}}{2}$$

$$m=\frac{4\pm2\sqrt{6}}{2}=2\pm\sqrt{6}$$

$$\boxed{y=C_1x^{(2+\sqrt{6})}+C_2x^{(2-\sqrt{6})}}$$

#4. Solve the differential equation by variation of parameters.

$xy'' - 4y' = x^4$

$\underline{y_c}$: $x(xy'' - 4y') = 0(x)$

$x^2y'' - 4xy' = 0$ Cauchy-Euler form

$am^2 + (b-a)m + c = 0$ $\quad m(m-5) = 0$

$1m^2 + (-4-1)m + 0 = 0$ $\quad m = 0,\ m = 5$

$m^2 - 5m = 0$ $\nearrow$

$y_c = C_1x^0 + C_2x^5 = C_1 + C_2x^5$

$\underline{y_p}$: $\left(\text{RHS}: \frac{x^4}{x} = x^3\right)$

$y_p = u_1(1) + u_2x^5$

$$u_1' = \frac{\begin{vmatrix} 0 & x^5 \\ x^3 & 5x^4 \end{vmatrix}}{\begin{vmatrix} 1 & x^5 \\ 0 & 5x^4 \end{vmatrix}} = \frac{0 - x^5x^3}{5x^4 - 0} = \frac{-x^8}{5x^4} = -\frac{1}{5}x^4$$

$u_1 = -\frac{1}{5}\int x^4 dx = -\frac{1}{5}\left(\frac{1}{5}x^5\right) = -\frac{1}{25}x^5$

$$u_2' = \frac{\begin{vmatrix} 1 & 0 \\ 0 & x^3 \end{vmatrix}}{\begin{vmatrix} 1 & x^5 \\ 0 & 5x^4 \end{vmatrix}} = \frac{x^3}{5x^4} = \frac{1}{5}\frac{1}{x}$$

$u_2 = \frac{1}{5}\int \frac{1}{x}dx = \frac{1}{5}\ln|x|$.

$y_p = \left(-\frac{1}{25}x^5\right)(1) + \left(\frac{1}{5}\ln|x|\right)x^5$

general solution

$y = C_1 + \underline{C_2x^5 - \frac{1}{25}x^5} + \frac{1}{5}x^5\ln|x|$

$\boxed{y = C_1 + C_3x^5 + \frac{1}{5}x^5\ln|x|}$

#5. Use the substitution $x = e^t$ to transform the given Cauchy-Euler equation to a differential equation with constant coefficients, then solve.

$x^2y'' + 9xy' - 20y = 0$

$x = e^t,\ t = \ln x$

$y' = \frac{1}{x}\frac{dy}{dt},\ y'' = \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$

into DE ...

$x^2\left[\frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)\right] + 9x\left[\frac{1}{x}\frac{dy}{dt}\right] - 20y = 0$

$\frac{d^2y}{dt^2} - \frac{dy}{dt} + 9\frac{dy}{dt} - 20y = 0$

$\frac{d^2y}{dt^2} + 8\frac{dy}{dt} - 20y = 0$

$m^2 + 8m - 20 = 0$

$(m+10)(m-2) = 0$

$m = -10,\ m = 2$

$y = C_1e^{-10t} + C_2e^{2t}$

resubstitute:

$y = C_1e^{-10(\ln x)} + C_2e^{2\ln x}$

$y = C_1e^{\ln(x^{-10})} + C_2e^{\ln(x^2)}$

$\boxed{y = C_1x^{-10} + C_2x^2}$

#6. Use the substitution $x = e^t$ to transform the given Cauchy-Euler equation to a differential equation with constant coefficients, then solve.

$x^2y'' + 10xy' + 8y = x^2$

$x = e^t,\ t = \ln x,\ y' = \frac{1}{x}\frac{dy}{dt},\ y'' = \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$

into DE...

$x^2\left[\frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)\right] + 10x\left[\frac{1}{x}\frac{dy}{dt}\right] + 8y = (e^t)^2$

$\frac{d^2y}{dt^2} - \frac{dy}{dt} + 10\frac{dy}{dt} + 8y = e^{2t}$

$\frac{d^2y}{dt^2} + 9\frac{dy}{dt} + 8y = e^{2t}$

$\underline{y_c}$: $\frac{d^2y}{dt^2} + 9\frac{dy}{dt} + 8y = 0$ $\quad m = -1,\ m = -8$

$m^2 + 9m + 8 = 0$ $\quad y_c = C_1e^{-t} + C_2e^{-8t}$

$(m+1)(m+8) = 0$ $\quad$ resubstitute: $y_c = C_1e^{-\ln x} + C_2e^{-8\ln x} = C_1e^{\ln(x^{-1})} + C_2\ln(x^{-8})$

$y_c = C_1x^{-1} + C_2x^{-8}$

for y_p, let's do table method for x^2

$y_p = Ax^2 + Bx + C$ (no absorption)

$y' = 2Ax + B$

$y'' = 2A$

into full DE... $x^2y'' + 10xy' + 8y = x^2$

$x^2[2A] + 10x[2Ax+B] + 8[Ax^2 + Bx + C] = x^2$

$(2A + 20A + 8A)x^2 + (10B + 8B)x + (8C) = (1)x^2 + (0)x + (0)$

System: $\begin{cases} 30A = 1 \\ 18B = 0 \\ 8C = 0 \end{cases}$ $\quad A = \frac{1}{30},\ B = 0,\ C = 0$ $\quad$ so $y_p = \frac{1}{30}x^2$

general solution:

$$\boxed{y = C_1x^{-1} + C_2x^{-8} + \frac{1}{30}x^2}$$

DiffEq Ch4 Test Review

#1. Given one solution to the differential equation (no need to verify), use reduction of order to find a second solution.

$y'' - 25y = 0; \quad y_1 = e^{5x}$

$y_2 = \underline{u}\,\underline{e^{5x}}$

$y' = \underline{u}(\underline{5e^{5x}}) + \underline{e^{5x}}\,\underline{u'}$

$y'' = u(25e^{5x}) + 5e^{5x}u' + e^{5x}u'' + u'(5e^{5x})$

$[25e^{5x}u + 10e^{5x}u' + e^{5x}u''] - 25[e^{5x}u] = 0$

$(e^{5x})u'' + (10e^{5x})u' + \underset{0}{(25e^{5x}\cancel{-25e^{5x}})}u = 0$

let $w = u'$

$\frac{e^{5x}w'}{e^{5x}} + \frac{10e^{5x}w}{e^{5x}} = \frac{0}{e^{5x}}$

$w' + 10w = 0$ linear, w/ $P(x) = 10$

$I.F. = e^{\int 10dx} = e^{10x}$

$\frac{e^{10x}w}{e^{10x}} = \int 0e^{10x}dx = \int 0dx = \frac{c_1}{e^{10x}}$

$w = c_1e^{-10x}$

$u' = c_1e^{-10x}$

$u = c_1\int e^{-10x}dx = c_1\left(-\frac{1}{10}e^{-10x}\right) + c_2$

simplest form if $-\frac{c_1}{10} = 1,\ c_2 = 0$

$u = e^{-10x}$

then $y_2 = ue^{5x} = e^{-10x}e^{5x}$

$\boxed{y_2 = e^{-5x}}$

#2. Given one solution to the differential equation (no need to verify), use reduction of order to find a second solution.

$9y'' - 12y' + 4y = 0; \quad y_1 = e^{\left(\frac{2}{3}x\right)}$

$y_2 = \underline{u}\,\underline{e^{2/3x}}$

$y' = \underline{u}(\underline{\frac{2}{3}e^{2/3x}}) + \underline{e^{2/3x}}\,\underline{u'}$

$y'' = u(\frac{4}{9}e^{2/3x}) + \frac{2}{3}e^{2/3x}u' + e^{2/3x}u'' + u'(\frac{2}{3}e^{2/3x}$

$9[\frac{4}{9}e^{2/3x}u + \frac{4}{3}e^{2/3x}u' + e^{2/3x}u'']$

$-12[\frac{2}{3}e^{2/3x}u + e^{2/3x}u'] + 4[e^{2/3x}u] = 0$

$(9e^{2/3x})u'' + \underset{0}{(12e^{2/3x}\cancel{-12e^{2/3x}})}u' + \underset{0}{(4e^{2/3x}\cancel{-8e^{2/3x}+4e^{2/3x}})}$

Let $w = u'$

$\frac{9e^{2/3x}w'}{9e^{2/3x}} = \frac{0}{9e^{2/3x}}$

$w' = 0$

$w = \int 0dx = c_1$

$u' = c_1$

$u = \int c_1dx = c_1x + c_2$

simplest is $c_1 = 1,\ c_2 = 0$

$u = x$

so $y_2 = ue^{2/3x}$

$\boxed{y_2 = xe^{2/3x}}$

#3. Use a Wronskian to determine if the following set of functions is linearly independent on the interval $(0,\infty)$

$f_1(x)=e^{4x}$, $f_2(x)=e^{2x}$

$$\begin{vmatrix} e^{4x} & e^{2x} \\ 4e^{4x} & 2e^{2x} \end{vmatrix}$$

$$2e^{2x}e^{4x}-4e^{4x}e^{2x}$$

$$-2e^{6x}\neq 0$$

So these solutions

<u>are</u> linearly independent.

#4. Use a Wronskian to determine if the following set of functions is linearly independent on the interval $(0,\infty)$

$f_1(x)=x$, $f_2(x)=x-1$, $f_3(x)=x+3$

$$\begin{vmatrix} \overset{+}{x} & \overset{-}{x-1} & \overset{+}{x+3} \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{vmatrix}$$

$$x(0-0)-(x-1)(0-0)+(x+3)(0-0)$$

$$0-0-0$$

$$=0$$

So these solutions

<u>are not</u> linearly independent.

#5. Solve the differential equation.

$y''-2y'-2y=0$

$$m^2-2m-2=0$$

$$m=\frac{2\pm\sqrt{4+4(1)2}}{2(1)}=\frac{2\pm\sqrt{12}}{2}=\frac{2\pm\sqrt{4}\sqrt{3}}{2}$$

$$m=\frac{2\pm2\sqrt{3}}{2}=1\pm\sqrt{3}$$

$$y=C_1e^{(1+\sqrt{3})x}+C_2e^{(1-\sqrt{3})x}$$

#6. Solve the differential equation.

$2y''+2y'+3y=0$

$$2m^2+2m+3=0$$

$$m=\frac{-2\pm\sqrt{4-4(2)(3)}}{2(2)}=\frac{-2\pm\sqrt{-20}}{4}$$

$$m=\frac{-2\pm\sqrt{4}\sqrt{5}\sqrt{-1}}{4}=\frac{-2\pm2\sqrt{5}i}{4}$$

$$m=-\frac{1}{2}\pm\frac{\sqrt{5}}{2}i$$

$$\boxed{y=C_1e^{-1/2x}\cos\left(\frac{\sqrt{5}}{2}x\right)+C_2e^{-1/2x}\sin\left(\frac{\sqrt{5}}{2}x\right)}$$

#7. Solve the differential equation.

$y''' - 5y'' + 3y' + 9y = 0$

$m^3 - 5m^2 + 3m + 9 = 0$

$$\text{try } -1 \;\Big|\; \begin{array}{rrrr} 1 & -5 & 3 & 9 \\ & -1 & 6 & -9 \\ \hline 1 & -6 & 9 & \underline{/0} \end{array}$$

$(m+1)(m^2 - 6m + 9) = 0$

$(m+1)(m-3)(m-3) = 0$

$m = -1$, $m = 3$ repeated

$\boxed{y = C_1 e^{-x} + C_2 e^{3x} + C_3 x e^{3x}}$

#8. Solve the differential equation by method of undetermined coefficients (table method).

$y'' + 4y = 3\sin(2x)$

$\underline{y_c}$: $y'' + 4y = 0$ $\quad m^2 = -4$

$m^2 + 4 = 0$ $\quad m = \pm\sqrt{-4} = 0 \pm 2i$

$y_c = C_1 \cos(2x) + C_2 \sin(2x)$

$\underline{y_p}$: table for $3\sin(2x)$

$y_p = A\cos(2x) + B\sin(2x)$ (absorbed)

so $y_p = \underline{Ax}\,\underline{\cos(2x)} + \underline{Bx}\,\underline{\sin(2x)}$

$y' = \underline{Ax}(\underline{-2\sin(2x)}) + \cos(2x)A + \underline{Bx}(\underline{2\cos(2x)}) + \sin(2x)B$

$y'' = Ax(-4\cos(2x)) + (-2\sin(2x))A - 2A\sin(2x) + Bx(-4\sin(2x)) + (2\cos(2x))B + 2B\cos(2x)$

into full DE.. $y'' + 4y = 3\sin(2x)$

$[-4Ax\cos(2x) - 4A\sin(2x) - 4B\,x\sin(2x) + 4B\cos(2x)] + 4[Ax\cos(2x) + Bx\sin(2x)] = 3\sin(2x)$

$(\underbrace{-4A\cancel{+4A}}_{0})x\cos(2x) + (\underbrace{-4B\cancel{+4B}}_{0})x\sin(2x) + (-4A)\sin(2x) + (4B)\cos(2x) = 3\sin(2x)$

System: $\begin{cases} -4A = 3 \\ 4B = 0 \end{cases}$ $\quad A = -\frac{3}{4}, \quad B = 0$

$y_p = -\frac{3}{4}x\cos(2x) + (0)x\sin(2x)$

$y_p = -\frac{3}{4}x\cos(2x)$

general solution: $\boxed{y = C_1\cos(2x) + C_2\sin(2x) - \frac{3}{4}x\cos(2x)}$

#9. Solve the differential equation by method of undetermined coefficients (table method).

$$y''+8y'+16y=2x^2-3,\quad y(0)=\frac{247}{64},\quad y'(0)=\frac{-153}{8}$$

$\underline{y_c}$: $y''+8y'+16y=0$ $m=-4$ repeated

$m^2+8m+16=0$ $y_c=C_1e^{-4x}+C_2xe^{-4x}$

$(m+4)(m+4)=0$

$\underline{y_p}$: table for $2x^2-3$: $y_p=Ax^2+Bx+C$ (no absorption)

$y'=2Ax+B$

$y''=2A$

into full DE: $y''+8y'+16y=2x^2-3$

$[2A]+8[2Ax+B]+16[Ax^2+Bx+C]=2x^2-3$

$(16A)x^2+(16A+16B)x+(2A+8B+16C)=(2)x^2+(0)x+(-3)$

system: $\begin{cases}16A=2\\16A+16B=0\\2A+8B+16C=-3\end{cases}$

$A=\frac{2}{16}=\frac{1}{8}$

$16(\frac{1}{8})+16B=0$, $B=-\frac{1}{8}$

$2(\frac{1}{8})+8(-\frac{1}{8})+16C=-3$

$\frac{1}{4}-1+16C=-3$, $16C=-\frac{9}{4}$, $C=-\frac{9}{64}$

$y_p=\frac{1}{8}x^2-\frac{1}{8}x-\frac{9}{64}$

general solution: $y=C_1e^{-4x}+C_2xe^{-4x}+\frac{1}{8}x^2-\frac{1}{8}x-\frac{9}{64}$

use $y(0)=\frac{247}{64}$ $\frac{247}{64}=C_1e^0+C_2(0)e^0+\frac{1}{8}(0)^2-\frac{1}{8}(0)-\frac{9}{64}$

$\frac{247}{64}=C_1-\frac{9}{64}$, $C_1=4$

$y=4e^{-4x}+\underline{C_2x}\,\underline{e}^{-4x}+\frac{1}{8}x^2-\frac{1}{8}x-\frac{9}{64}$

$y'=-16e^{-4x}+C_2x(-4e^{-4x})+e^{-4x}C_2+\frac{1}{4}x-\frac{1}{8}$

use $y'(0)=\frac{-153}{8}$ $\frac{-153}{8}=-16e^0-4C_2(0)e^0+C_2e^0+\frac{1}{4}(0)-\frac{1}{8}$

$\frac{-153}{8}=-16+C_2-\frac{1}{8}$, $C_2=-3$

$$\boxed{y=4e^{-4x}-3xe^{-4x}+\frac{1}{8}x^2-\frac{1}{8}x-\frac{9}{64}}$$

#10. Solve the differential equation by method of undetermined coefficients (table method).

$y'' - 16y = 2e^{4x}$

$\underline{y_c}$: $y'' - 16y = 0$ $\quad m = 4,\ m = -4$

$m^2 - 16 = 0$ $\quad y_c = C_1 e^{4x} + C_2 e^{-4x}$

$(m-4)(m+4) = 0$

$\underline{y_p}$: table for $2e^{4x}$:

$y_p = Ae^{4x}$ (absorbed)

So $y_p = \underline{Axe}^{4x}$

$y' = \underline{Ax}(\underline{4e^{4x}}) + e^{4x}A$

$y'' = Ax(16e^{4x}) + 4e^{4x}A + 4Ae^{4x}$

into full DE: $y'' - 16y = 2e^{4x}$

$[16Axe^{4x} + 8Ae^{4x}] - 16[Axe^{4x}] = 2e^{4x}$

$(\underbrace{16A - \cancel{16A}}_{0})xe^{4x} + (8A)e^{4x} = 2e^{4x}$, $\quad 8A = 2, \quad A = \frac{2}{8} = \frac{1}{4}$

$y_p = \frac{1}{4}xe^{4x}$

general solution: $\boxed{y = C_1 e^{4x} + C_2 e^{-4x} + \frac{1}{4}xe^{4x}}$

#11. Solve the differential equation by variation of parameters (Wronskian method).

$$y'' - 9y = \frac{9x}{e^{3x}}$$

$\underline{y_c}$: $y'' - 9y = 0$ $\quad m = 3, m = -3$

$m^2 - 9 = 0$ $\quad y_c = C_1e^{3x} + C_2e^{-3x}$

$(m-3)(m+3) = 0$

$\underline{y_p}$: $y_p = u_1e^{3x} + u_2e^{-3x}$

$$u_1' = \frac{\begin{vmatrix} 0 & e^{-3x} \\ \frac{9}{e^{3x}} & -3e^{-3x} \end{vmatrix}}{\begin{vmatrix} e^{3x} & e^{-3x} \\ 3e^{3x} & -3e^{-3x} \end{vmatrix}} = \frac{0 - \frac{9x}{e^{3x}}e^{-3x}}{-3e^{3x}e^{-3x} - 3e^{3x}e^{-3x}} = \frac{-9xe^{-3x}e^{-3x}}{-3e^0 - 3e^0} = \frac{-9xe^{-6x}}{-6} = \frac{3}{2}xe^{-6x}$$

$u_1 = \frac{3}{2}\int xe^{-6x}\,dx$

by parts:

$u = x \quad dv = e^{-6x}dx$

$\frac{du}{dx} = 1 \quad \int dv = \int e^{-6x}dx$

$du = dx \quad v = -\frac{1}{6}e^{-6x}$

$uv - \int v\,du$

$-\frac{1}{6}xe^{-6x} + \int(-\frac{1}{6}e^{-6x})dx$

$-\frac{1}{6}xe^{-6x} + \frac{1}{6}\int e^{-6x}dx = -\frac{1}{6}xe^{-6x} - \frac{1}{36}e^{-6x}$

$u_1 = \frac{3}{2}[-\frac{1}{6}xe^{-6x} - \frac{1}{36}e^{-6x}] = -\frac{1}{4}xe^{-6x} - \frac{1}{24}e^{-6x}$

$$u_2' = \frac{\begin{vmatrix} e^{3x} & 0 \\ 3e^{3x} & \frac{9}{e^{3x}} \end{vmatrix}}{\begin{vmatrix} e^{3x} & e^{-3x} \\ 3e^{3x} & -3e^{-3x} \end{vmatrix}} = \frac{e^{3x}\frac{9x}{e^{3x}}}{-6} = -\frac{3}{2}x$$

$u_2 = -\frac{3}{2}\int x\,dx = -\frac{3}{2}(\frac{1}{2}x^2) = -\frac{3}{4}x^2$

$y_p = (-\frac{1}{4}xe^{-6x} - \frac{1}{24}e^{-6x})e^{3x} + (-\frac{3}{4}x^2)e^{-3x} = -\frac{1}{4}xe^{-6x}e^{3x} - \frac{1}{24}e^{-6x}e^{3x} - \frac{3}{4}x^2e^{-3x}$

$y_p = -\frac{1}{4}xe^{-3x} - \frac{1}{24}e^{-3x} - \frac{3}{4}x^2e^{-3x}$

general solution:

$y = C_1e^{3x} + \underbrace{C_2e^{-3x}} - \frac{1}{4}xe^{-3x} \underbrace{- \frac{1}{24}e^{-3x}} - \frac{3}{4}x^2e^{-3x}$

$$\boxed{y = C_1e^{3x} + C_3e^{-3x} - \frac{1}{4}xe^{-3x} - \frac{3}{4}x^2e^{-3x}}$$

#12. Solve the differential equation by variation of parameters (Wronskian method).

$y'' + y = \sec^3 x$

y_c: $y''+y=0$ $\quad m=\pm\sqrt{-1}=0\pm i$ $\quad y_c=C_1\cos x+C_2\sin x$

$m^2+1=0$

y_p: $y_p=u_1\cos x+u_2\sin x$

$$u_1'=\frac{\begin{vmatrix} 0 & \sin x \\ \sec^3 x & \cos x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}}=\frac{0-\sin x\sec^3 x}{\cos^2 x+\sin^2 x}=\frac{-\frac{\sin x}{\cos^3 x}}{1}=-\frac{\sin x}{\cos^3 x}$$

$u_1=\int\frac{-\sin x}{\cos^3 x}dx$ $\quad u=\cos x$, $\frac{du}{dx}=-\sin x$, $du=-\sin x\,dx$ $\quad u_1=\int u^{-3}du=\frac{u^{-2}}{-2}=\frac{-1}{2\cos^2 x}$

$$u_2'=\frac{\begin{vmatrix} \cos x & 0 \\ -\sin x & \sec^3 x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}}=\frac{\cos x\sec^3 x-0}{1}=\frac{\cos x}{\cos^3 x}=\frac{1}{\cos^2 x}=\sec^2 x$$

$u_2=\int\sec^2 x\,dx=\tan x$

$y_p=\left(\frac{-1}{2\cos^2 x}\right)\cos x+(\tan x)\sin x=\frac{-1}{2\cos x}+\tan x\sin x$

general solution: $\boxed{y=C_1\cos x+C_2\sin x-\frac{1}{2\cos x}+\tan x\sin x}$

#13. Solve this Cauchy-Euler form differential equation using the Cauchy-Euler procedure for the homogeneous solution. Then use the variation of parameters (Wronskian method) to find the particular solution. Finally, join the solutions to produce the general solution.

$2x^2y'' + 5xy' + y = x^2 - x$

y_c: $2x^2y''+5xy'+y=0$ $\quad (2m+1)(m+1)=0 \quad y_c = c_1 x^{-1/2} + c_2 x^{-1}$

$am^2+(b-a)m+c=0 \quad m=-1/2 \quad m=-1$

$2m^2+(5-2)m+1=0$

$2m^2+3m+1=0$

y_p: $y_p = u_1 x^{-1/2} + u_2 x^{-1} \qquad$ RHS: $\frac{x^2-x}{2x^2}$

$$u_1' = \frac{\begin{vmatrix} 0 & x^{-1} \\ \frac{x^2-x}{2x^2} & -x^{-2} \end{vmatrix}}{\begin{vmatrix} x^{-1/2} & x^{-1} \\ -\frac{1}{2}x^{-3/2} & -x^{-2} \end{vmatrix}} = \frac{0 - x^{-1}\frac{x^2-x}{2x^2}}{-x^{-1/2}x^{-2}+\frac{1}{2}x^{-3/2}x^{-1}} = \frac{-\left(\frac{x-1}{2x^2}\right)}{-x^{-5/2}+\frac{1}{2}x^{-5/2}} = \frac{-(x-1)}{2x^2(-\frac{1}{2}x^{-5/2})} = \frac{x-1}{x^{-1/2}} = x^{3/2} - x^{1/2}$$

$$u_1 = \int (x^{3/2} - x^{1/2})dx = \frac{2}{5}x^{5/2} - \frac{2}{3}x^{3/2}$$

$$u_2' = \frac{\begin{vmatrix} x^{-1/2} & 0 \\ -\frac{1}{2}x^{-3/2} & \frac{x^2-x}{2x^2} \end{vmatrix}}{\begin{vmatrix} x^{-1/2} & x^{-1} \\ -\frac{1}{2}x^{-3/2} & -x^{-2} \end{vmatrix}} = \frac{\frac{x^{-1/2}(x^2-x)}{2x^2} - 0}{-\frac{1}{2}x^{-5/2}} = -(x^2-x) = -x^2+x$$

$$u_2 = \int (-x^2+x)dx = -\frac{1}{3}x^3 + \frac{1}{2}x^2$$

$$y_p = \left(\frac{2}{5}x^{5/2} - \frac{2}{3}x^{3/2}\right)x^{-1/2} + \left(-\frac{1}{3}x^3 + \frac{1}{2}x^2\right)x^{-1} = \frac{2}{5}x^2 - \frac{2}{3}x - \frac{1}{3}x^2 + \frac{1}{2}x$$

$$y_p = \frac{1}{15}x^2 - \frac{1}{6}x$$

so $\boxed{y = c_1 x^{-1/2} + c_2 x^{-1} + \frac{1}{15}x^2 - \frac{1}{6}x}$

#14. Solve this Cauchy-Euler form differential equation using the Cauchy-Euler procedure for the homogeneous solution. Then use the variation of parameters (Wronskian method) to find the particular solution. Finally, join the solutions to produce the general solution.

$xy'' - 4y' = x^4$

y_c: $xy''-4y'=0$ (multiply by x)

$x^2y''-4xy'=0$ (now, Cauchy-Euler)

$am^2+(b-a)m+c=0$

$1\cdot m^2+(-4-1)m+0=0$

$m^2-5m=0 \nearrow$

$m(m-5)=0$

$m=0,\ m=5$

$y_c = C_1x^0 + C_2x^5$

$y_c = C_1 + C_2x^5$

y_p: $xy''-4y'=x^4$

$y''-\frac{4}{x}y' = x^3$

$y_p = u_1 + u_2x^5$

$$u_1' = \frac{\begin{vmatrix}0 & x^5\\ x^3 & 5x^4\end{vmatrix}}{\begin{vmatrix}1 & x^5\\ 0 & 5x^4\end{vmatrix}} = \frac{0-x^8}{5x^4-0} = \frac{-x^8}{5x^4} = -\frac{1}{5}x^4$$

$$u_1 = -\frac{1}{5}\int x^4\,dx = -\frac{1}{5}\frac{1}{5}x^5 = -\frac{1}{25}x^5$$

$$u_2' = \frac{\begin{vmatrix}1 & 0\\ 0 & x^3\end{vmatrix}}{\begin{vmatrix}1 & x^5\\ 0 & 5x^4\end{vmatrix}} = \frac{x^3-0}{5x^4} = \frac{1}{5}\frac{1}{x}$$

$$u_2 = \frac{1}{5}\int \frac{1}{x}\,dx = \frac{1}{5}\ln|x|$$

$$y_p = -\frac{1}{25}x^5(1) + \left(\frac{1}{5}\ln|x|\right)x^5 = -\frac{1}{25}x^5 + \frac{1}{5}x^5\ln|x|$$

so $y = C_1 + \underbrace{C_2x^5 - \frac{1}{25}x^5} + \frac{1}{5}x^5\ln|x|$

$$\boxed{y = C_1 + C_3x^5 + \frac{1}{5}x^5\ln|x|}$$