DiffEq - Ch 5 - Required Practice

Name: full solutions

#1. A mass weighing 4 pounds is attached to a spring whose spring constant is 16 lb/ft. What is the period of simple harmonic motion?

$T=\frac{1}{f}$, $w=2\pi f$, $w^2=\frac{k}{m}$, weight $= mg$

$mg = 4$ lbs

$m = \frac{4}{32} = \frac{1}{8}$ slug, $w^2 = \frac{k}{m} = \frac{16}{(\frac{1}{8})} = 128$

$w = \sqrt{128} = 2\pi f$

$f = \frac{\sqrt{128}}{2\pi}$ $T = \frac{1}{f}$

$\boxed{T = \frac{2\pi}{\sqrt{128}} \approx 0.555 \text{ sec}}$

#2. A 20-kilogram mass is attached to a spring. If the frequency of simple harmonic motion is $\frac{2}{\pi}$ cycles/s:

a) What is the spring constant k?

b) What is the frequency of simple harmonic motion if the original mass is replaced with an 80-kilogram mass?

(a) $f = \frac{2}{\pi}$, $w = 2\pi f = 2\pi\left(\frac{2}{\pi}\right) = 4$

$w^2 = \frac{k}{m}$

$(4)^2 = \frac{k}{20}$, $\boxed{k = 20(4)^2 = 320 \text{ N/m}}$

(b) (Same k)

$w^2 = \frac{k}{m} = \frac{320}{80} = 4$

$w = \sqrt{4} = 2 = 2\pi f$

$f = \frac{2}{2\pi} = \frac{1}{\pi}$

$\boxed{f = \frac{1}{\pi} = 0.318 \text{ cycles/s}}$

#3. A mass weighing 24 pounds, attached to the end of a spring, stretches it 4 inches. Initially, the mass is released from rest from a point 3 inches above the equilibrium position. Find the equation of motion.

$\frac{3}{12}$ ft

$= \frac{4}{12}$ ft

$mg = ks$

$24 = k\left(\frac{4}{12}\right) = \frac{k}{3}$, $k = 3(24) = 72$

$mg = ks$

$m(32) = 72\left(\frac{4}{12}\right)$, $m = \frac{3}{4}$ slug

$mx'' + \beta x' + kx = 0$ ($\beta = 0$ no damping)

$\frac{3}{4}x'' + 72x = 0$

$x'' + 96x = 0$

$m^2 + 96 = 0$, $m^2 = -96$, $m = \pm\sqrt{-96}$

$m = 0 \pm \sqrt{96}\, i$

$x(t) = C_1 e^{0t}\cos(\sqrt{96}t) + C_2 e^{0t}\sin(\sqrt{96}t)$

$x = C_1\cos(\sqrt{96}t) + C_2\sin(\sqrt{96}t)$

now, $x(0) = \frac{-3}{12}$ (must be in ft)

$x(0) = -\frac{1}{4}$

$-\frac{1}{4} = C_1\cos(0) + C_2\sin(0)$, $C_1 = -\frac{1}{4}$

$x = -\frac{1}{4}\cos(\sqrt{96}t) + C_2\sin(\sqrt{96}t)$

also, "from rest" $x'(0) = 0$

$x'(t) = -\frac{1}{4}\left(-\sqrt{96}\sin(\sqrt{96}t)\right) + C_2\left(\sqrt{96}\cos(\sqrt{96}t)\right)$

$x' = \frac{\sqrt{96}}{4}\sin(\sqrt{96}t) + \sqrt{96}C_2\cos(\sqrt{96}t)$

$0 = \frac{\sqrt{96}}{4}\sin(0) + \sqrt{96}\,C_2\cos(0)$

$\sqrt{96}\,C_2 = 0$, $C_2 = 0$

so $\boxed{x(t) = -\frac{1}{4}\cos(\sqrt{96}t)}$

#4. A mass weighing 8 pounds is attached to a spring. When set in motion, the spring/mass system exhibits simple harmonic motion. Determine the equation of motion if the spring constant is $1\ ^{lb}/_{ft}$ and the mass is initially released from a point 6 inches below the equilibrium position with a downward velocity of $\frac{3}{2}\ ^{ft}/_{s}$.

"positive"

$\frac{1}{2}$ ft

$mx'' + \beta x' + kx = 0$ $\qquad mg = 8$ lbs $\qquad k = 1 \qquad \beta = 0$ (no damping)

$m = \frac{8}{32} = \frac{1}{4}$ slug

$\frac{1}{4}x'' + (1)x = 0$

$x'' + 4x = 0$

$m^2 + 4 = 0, \quad m^2 = -4, \quad m = 0 \pm \sqrt{4}\, i$ $\quad \alpha \pm \beta$

$m = 0 \pm 2i$

$x(t) = C_1 e^{0t}\cos(2t) + C_2 e^{0t}\sin(2t)$

$x = C_1\cos(2t) + C_2\sin(2t)$

$x(0) = \frac{1}{2}$ (must be in ft, downward = +)

$\frac{1}{2} = C_1\cos(0) + C_2\sin(0) \rightarrow C_1 = \frac{1}{2}$

$x' = -2C_1\sin(2t) + 2C_2\cos(2t)$

$x'(0) = 3/2$ (+ "downward velocity")

$\frac{3}{2} = -2(\frac{1}{2})\sin(0) + 2C_2\cos(0) \longrightarrow 2C_2 = \frac{3}{2} \rightarrow C_2 = \frac{3}{4}$

So $\boxed{x(t) = \frac{1}{2}\cos(2t) + \frac{3}{4}\sin(2t)}$

#5. A mass weighing 4 pounds is attached to a spring whose constant is 2 lb/ft. The medium offers a damping force that is numerically equal to the instantaneous velocity. The mass is initially released from a point 1 foot above the equilibrium position with a downward velocity of 8 ft/s.

a) Determine the time at which the mass passes through the equilibrium position.
b) Find the time at which the mass attains its extreme displacement from the equilibrium position.
c) What is the position of the mass at this instant?

(a) $mg = 4\,lbs$ $\quad k=2$

$m = \frac{4}{32} = \frac{1}{8}$ slug $\quad \beta = 1$

$mx'' + \beta x' + kx = 0$

$\frac{1}{8}x'' + (1)x' + 2x = 0$

$x'' + 8x' + 16x = 0$

$m^2 + 8m + 16 = 0$

$(m+4)(m+4) = 0 \quad m = -4$ repeated

$x(t) = C_1 e^{-4t} + C_2 t e^{-4t}$

$x(0) = -1$

$-1 = C_1 e^0 + C_2(0)e^0 \rightarrow \underline{C_1 = -1}$

$x'(t) = -4C_1 e^{-4t} + -4C_2 t e^{-4t} + C_2 e^{-4t}$

$x'(0) = 8$

$8 = -4(-1)e^0 - 4C_2(0)e^0 + C_2 e^0$

$8 = 4 + C_2 \rightarrow \underline{C_2 = 4}$

$x(t) = -e^{-4t} + 4te^{-4t}$

time when at equilibrium: $X = 0$

$-e^{-4t} + 4te^{-4t} = 0$

$e^{-4t}(-1 + 4t) = 0$

$-1 + 4t = 0$

$\boxed{t = \frac{1}{4}\text{ sec}}$

(b) extreme displacement at local max of $x(t)$: when $x'(t) = 0$

$x'(t) = -4C_1 e^{-4t} - 4C_2 t e^{-4t} + C_2 e^{-4t}$

$x' = -4(-1)e^{-4t} - 4(4)te^{-4t} + (4)e^{-4t}$

$x' = 8e^{-4t} - 16te^{-4t} = 0$

$8e^{-4t}(1 - 2t) = 0$

$1 - 2t = 0$

$2t = 1$

$\boxed{t = \frac{1}{2}\text{ sec}}$

(c) $x(\frac{1}{2}) = -e^{-4(\frac{1}{2})} + 4(\frac{1}{2})e^{-4(\frac{1}{2})}$

$= -e^{-2} + 2e^{-2}$

$= e^{-2} = \boxed{\frac{1}{e^2} \approx 0.1353\text{ ft}}$

(below equilibrium)

#6. A 4-foot spring measures 8 feet long after a mass weighing 8 pounds is attached to it. The medium through which the mass moves offers a damping force numerically equal to $\sqrt{2}$ times the instantaneous velocity.

a) Find the equation of motion if the mass is initially released from the equilibrium position with a downward velocity of $5\ ^{ft}/_{s}$.

b) Find the time at which the mass attains its extreme displacement from the equilibrium position.

c) What is the position of the mass at this instant?

(a) $mg = ks$ $\quad mg = 8$

$8 = k(4)$ $\quad m = \frac{8}{32} = \frac{1}{4}$ slugs

$k = \frac{8}{4} = 2$ lbs/ft $\quad \beta = \sqrt{2}$

$mx'' + \beta x' + kx = 0$

$\frac{1}{4}x'' + \sqrt{2}x' + 2x = 0$

$x'' + 4\sqrt{2}x' + 8x = 0$

$m^2 + 4\sqrt{2}m + 8 = 0$

$m = \frac{-4\sqrt{2} \pm \sqrt{16(2) - 4(1)(8)}}{2(1)} = \frac{-4\sqrt{2} \pm 0}{2}$

$m = -2\sqrt{2} = -\sqrt{8}$ repeated.

$x(t) = c_1 e^{-\sqrt{8}t} + c_2 t e^{-\sqrt{8}t}$

$x(0) = 0$

$0 = c_1 e^0 + c_2(0)e^0 \rightarrow c_1 = 0$

$x'(t) = -\sqrt{8}c_1 e^{-\sqrt{8}t} + c_2 t(-\sqrt{8}e^{-\sqrt{8}t}) + c_2 e^{-\sqrt{8}t}$

$x'(0) = 5$

$5 = -\sqrt{8}(0)e^0 + c_2(0)(-\sqrt{8})e^0 + c_2 e^0 \rightarrow c_2 = 5$

$\boxed{x(t) = 5te^{-\sqrt{8}t}}$

(b) extreme displacement at max/min x, when $x' = 0$

$x'(t) = 5t(-\sqrt{8}e^{-\sqrt{8}t}) + e^{-\sqrt{8}t}(5) = 0$

$= 5e^{-\sqrt{8}t}(-\sqrt{8}t + 1) = 0$

$-\sqrt{8}t + 1 = 0$

$\boxed{t = \frac{-1}{-\sqrt{8}} = \frac{1}{\sqrt{8}} \approx 0.35355 \text{ sec}}$

(c) $x\left(\frac{1}{\sqrt{8}}\right) = 5\left(\frac{1}{\sqrt{8}}\right)e^{-\sqrt{8}\frac{1}{\sqrt{8}}} = \boxed{\frac{5}{\sqrt{8}}e^{-1} \approx 0.6503 \text{ ft}}$

#7. A 1-kilogram mass is attached to a spring whose constant is 16 N/m, and the entire system is then submerged in a liquid that imparts a damping force numerically equal to 10 times the instantaneous velocity.

a) Find the equation of motion if the mass is initially released from rest from a point 1 meter below the equilibrium position.

b) Find the equation of motion if the mass is initially released from a point 1 meter below the equilibrium position with an upward velocity of 12 m/s.

$m=1,\ k=16,\ \beta=10$

$mx''+\beta x'+kx=0$

$x''+10x'+16x=0$

$m^2+10m+16=0$

$(m+2)(m+8)=0$

$m=-2,\ m=-8$

$x(t)=C_1e^{-2t}+C_2e^{-8t}$

$x'(t)=-2C_1e^{-2t}-8C_2e^{-8t}$

(a) $x(0)=1 \quad x'(0)=0$

$1=C_1e^0+C_2e^0 \rightarrow C_1+C_2=1$

$0=-2C_1e^0-8C_2e^0 \rightarrow -2C_1-8C_2=0$

$\begin{cases} C_1+C_2=1 \\ 2C_1+8C_2=0 \end{cases}$

$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 8 & 0 \end{array}\right] \text{rref} \left[\begin{array}{cc|c} 1 & 0 & 4/3 \\ 0 & 1 & -1/3 \end{array}\right] \begin{array}{l} = C_1 \\ = C_2 \end{array}$

$\boxed{x(t)=\frac{4}{3}e^{-2t}-\frac{1}{3}e^{-8t}}$

(b) $x(0)=1 \quad x'(0)=-12$

$1=C_1e^0+C_2e^0 \rightarrow C_1+C_2=1$

$-12=-2C_1e^0-8C_2e^0 \rightarrow -2C_1-8C_2=-12$

$\begin{cases} C_1+C_2=1 \\ -2C_1-8C_2=-12 \end{cases}$

$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ -2 & -8 & -12 \end{array}\right] \text{rref} \left[\begin{array}{cc|c} 1 & 0 & -2/3 \\ 0 & 1 & 5/3 \end{array}\right] \begin{array}{l} = C_1 \\ = C_2 \end{array}$

$\boxed{x(t)=-2/3\,e^{-2t}+5/3\,e^{-8t}}$

#8. A force of 2 pounds stretches a spring 1 foot. A mass weighing 3.2 pounds is attached to the spring, and the system is then immersed in a medium that offer a damping force that is numerically equal to 0.4 times the instantaneous velocity.

a) Find the equation of motion if the mass is initially released from rest from a point 1 foot above the equilibrium position.

b) Use the fact that…

$A\sin(\omega t+\phi)=C_1\cos(\omega t)+C_2\sin(\omega t)$

where $A=\sqrt{C_1^2+C_2^2}$ *and* $\tan\phi=\dfrac{C_1}{C_2}$

…to express the equation of motion as a sum of two terms with the same frequency without phase shift.

c) Find the first time at which the mass passes through the equilibrium position heading upward.

(a) $mg=ks$ $\quad mg=3.2$

$2=k(1)$ $\quad m=\frac{3.2}{32}=0.1$ slug

$k=2$ lb/ft $\quad \beta=0.4$

$mx''+\beta x'+kx=0$

$0.1x''+0.4x'+2x=0$

$x''+4x'+20x=0$

$m=\dfrac{-4\pm\sqrt{16-4(1)(20)}}{2(1)}=\dfrac{-4\pm 8i}{2}=-2\pm 4i$

$x(t)=C_1e^{-2t}\cos(4t)+C_2e^{-2t}\sin(4t)$

$x(0)=-1$ ("above" = negative)

$-1=C_1e^0\cos(0)+C_2e^0\sin 0\rightarrow C_1=-1$

$x'(t)=C_1e^{-2t}(-4\sin(4t))+\cos(4t)(-2C_1e^{-2t})$
$\quad +C_2e^{-2t}(4\cos(4t))+\sin(4t)(-2C_2e^{-2t})$

$x'(0)=0$ ("from rest" = 0)

$0=(-1)e^0(-4)\sin(0)+\cos(0)(-2)(-1)e^0$
$\quad +C_2e^0(4)\cos(0)+\sin(0)(-2)C_2e^0$

$0=0+2+4C_2+0\rightarrow 4C_2=-2\rightarrow C_2=-\frac12$

$$\boxed{x(t)=-e^{-2t}\cos(4t)-\tfrac12 e^{-2t}\sin(4t)}$$

(b) $C_1\cos(wt)+C_2\sin(wt)=A\sin(wt+\phi)$

$x(t)=e^{-2t}((-1)\cos(4t)-\frac12\sin(4t))$

$A=\sqrt{C_1^2+C_2^2}=\sqrt{(-1)^2+(-\frac12)^2}=\sqrt{1+\frac14}=\sqrt{\frac54}=\frac{\sqrt5}{2}$

$\tan\phi=\frac{C_1}{C_2}=\frac{-1}{-\frac12}=2$ $\quad \phi=\arctan(2)$

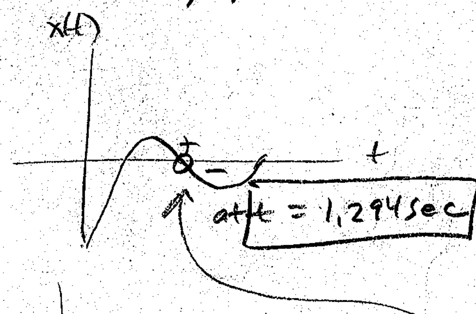
by calculator is 1.107…

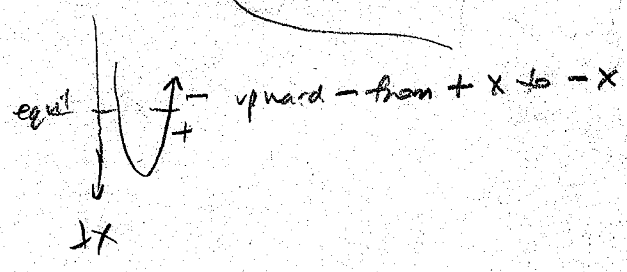
but C_1 & C_2 are <0 so in quad. III

$\phi=1.107+\pi=4.2487$

so $\boxed{x(t)=e^{-2t}\left[\frac{\sqrt5}{2}\sin(4t+4.2487)\right]}$

(c) calculator graph of $x(t)$:





#9. A mass weighing 16 pounds stretches a spring $\frac{8}{3}$ feet. The mass is initially released from rest from a point 2 feet below the equilibrium position, and the subsequent motion takes placed in a medium that offers a damping force that is numerically equal to $\frac{1}{2}$ the instantaneous velocity. Find the equation of motion if the mass is driven by an external force equal to $f(t)=10\cos(3t)$.

$mg = ks$ $\quad$ $mg = 16$

$16 = k\left(\frac{8}{3}\right)$ $\quad$ $m = \frac{16}{32} = \frac{1}{2}$ slug

$k = \frac{16(3)}{8} = 6$ $\quad$ $\beta = \frac{1}{2}$

$mx'' + \beta x' + kx = f(t)$

$\frac{1}{2}x'' + \frac{1}{2}x' + 6x = 10\cos(3t)$

$x'' + x' + 12x = 20\cos(3t)$

① find x_c: $x'' + x' + 12x = 0$

$m^2 + m + 12 = 0$

$m = \frac{-1 \pm \sqrt{1^2 - 4(12)}}{2} = \frac{-1 \pm \sqrt{47}\,i}{2}$

$m = -\frac{1}{2} \pm \frac{\sqrt{47}}{2} i$

$\alpha \pm \beta$

$x_c = c_1 e^{-1/2t}\cos\left(\frac{\sqrt{47}}{2}t\right) + c_2 e^{-1/2t}\sin\left(\frac{\sqrt{47}}{2}t\right)$

② find x_p

$x_p = A\cos 3t + B\sin 3t$ (no absorption)

$x' = -3A\sin 3t + 3B\cos 3t$

$x'' = -9A\cos 3t - 9B\sin 3t$

$x'' + x' + 12x = 20\cos(3t)$

$[-9A\cos 3t - 9B\sin 3t] + [-3A\sin 3t + 3B\cos 3t]$

$+12[A\cos 3t + B\sin 3t] = 20\cos 3t$

$(-9A + 3B + 12A)\cos 3t + (-9B - 3A + 12B)\sin 3t = 20\cos 3t$

$(3A + 3B)\cos 3t + (-3A + 3B)\sin 3t = (20)\cos 3t + (0)\sin 3t$

System is $\begin{cases} 3A + 3B = 20 \\ -3A + 3B = 0 \end{cases}$ $\quad$ $-3A + 3\left(\frac{10}{3}\right) = 0$

$6B = 20$ $\quad$ $-3A + 10 = 0$

$B = \frac{20}{6} = \frac{10}{3}$ $\quad$ $A = \frac{10}{3}$

$x_p = \frac{10}{3}\cos 3t + \frac{10}{3}\sin 3t$

So $x = x_c + x_p$

$x = c_1 e^{-1/2t}\cos\left(\frac{\sqrt{47}}{2}t\right) + c_2 e^{-1/2t}\sin\left(\frac{\sqrt{47}}{2}t\right)$

$+ \frac{10}{3}\cos 3t + \frac{10}{3}\sin 3t$

③ use initial conditions:

$x(0) = 2$:

$2 = c_1 e^0 \cos 0 + c_2 e^0 \sin 0 + \frac{10}{3}\cos 0 + \frac{10}{3}\sin 0$

$2 = c_1 + \frac{10}{3} \rightarrow c_1 = 2 - \frac{10}{3} = \frac{-4}{3}$

$x'(0) = 0$ "from rest"

$x' = c_1 e^{-1/2t}\left(-\frac{\sqrt{47}}{2}\sin\left(\frac{\sqrt{47}}{2}t\right)\right) + \cos\left(\frac{\sqrt{47}}{2}t\right)\left(-\frac{1}{2}c_1 e^{-1/2t}\right)$

$+ c_2 e^{-1/2t}\left(\frac{\sqrt{47}}{2}\cos\left(\frac{\sqrt{47}}{2}t\right)\right) + \sin\left(\frac{\sqrt{47}}{2}t\right)\left(-\frac{1}{2}c_2 e^{-1/2t}\right)$

$+ \frac{10}{3}(-3\sin 3t) + \frac{10}{3}(3\cos 3t)$

$0 = c_1 e^0\left(-\frac{\sqrt{47}}{2}\right)\sin 0 + \cos 0\left(-\frac{1}{2}\right)c_1 e^0$

$+ c_2 e^0\left(\frac{\sqrt{47}}{2}\right)\cos 0 + \sin 0\left(-\frac{1}{2}\right)c_2 e^0 - 10\sin 0 + 10\cos 0$

$0 = -\frac{1}{2}c_1 + \frac{\sqrt{47}}{2}c_2 + 10$

$0 = -\frac{1}{2}\left(-\frac{4}{3}\right) + \frac{\sqrt{47}}{2}c_2 + 10$

$\frac{\sqrt{47}}{2}c_2 = -\frac{2}{3} - 10 = -\frac{32}{3} \rightarrow c_2 = \frac{-32}{3}\cdot\frac{2}{\sqrt{47}} = \frac{-64}{3\sqrt{47}}$

④ final answer:

$$\boxed{x(t) = -\frac{4}{3}e^{-1/2t}\cos\left(\frac{\sqrt{47}}{2}t\right) - \frac{64}{3\sqrt{47}}e^{-1/2t}\sin\left(\frac{\sqrt{47}}{2}t\right) + \frac{10}{3}\cos(3t) + \frac{10}{3}\sin(3t)}$$

#10. For the given LRC series circuit…

$L = \frac{5}{3}\, H, \quad R = 10\,\Omega, \quad C = \frac{1}{30}\, F,$

$E(t) = 300\, V, \quad q(0) = 0\ Coulombs, \quad i(0) = 0\ A$

a) Find the charge on the capacitor as a function of time.

b) Find the maximum charge on the capacitor.

$$Lq'' + Rq' + \frac{1}{C}q = E$$

$$\frac{5}{3}q'' + 10q' + 30q = 300$$

$$q'' + 6q' + 18q = 180$$

a) ① q_c: $q'' + 6q' + 18q = 0$

$$m^2 + 6m + 18 = 0$$

$$m = \frac{-6 \pm \sqrt{36 - 4(18)}}{2} = \frac{-6 \pm 6i}{2} = -3 \pm 3i$$

$\alpha \pm \beta$

$$\underline{q_c(t) = C_1 e^{-3t}\cos(3t) + C_2 e^{-3t}\sin(3t)}$$

② q_p: (table)

$q_p = A$ (no absorption)

$q' = 0$

$q'' = 0$

$q'' + 6q' + 18q = 180$

$[0] + 6[0] + 18[A] = 180$

$A = \frac{180}{18} = 10$

$\underline{q_p = 10}$

so

$q(t) = q_c + q_p$

$$\underline{q(t) = C_1 e^{-3t}\cos(3t) + C_2 e^{-3t}\sin(3t) + 10}$$

③ use initial conditions:

$\underline{q(0) = 0}$

$0 = C_1 e^0 \cos 0 + C_2 e^0 \sin 0 + 10$

$0 = C_1 + 10 \rightarrow \underline{C_1 = -10}$

$\underline{i(0) = q'(0) = 0}$

$q'(t) = C_1 e^{-3t}(-3\sin(3t)) + \cos(3t)(-3C_1 e^{-3t})$
$+ C_2 e^{-3t}(3\cos(3t)) + \sin(3t)(-3C_2 e^{-3t}) + 0$

$0 = C_1 e^0(-3)\sin 0 - 3C_1 e^0 \cos 0 + 3C_2 e^0 \cos 0 - 3C_2 e^0 \sin 0$

$0 = -3C_1 + 3C_2$

$0 = -3(-10) + 3C_2 \rightarrow C_2 = \frac{-30}{3} = -10$

$$\boxed{q(t) = -10e^{-3t}\cos(3t) - 10e^{-3t}\sin(3t) + 10}$$

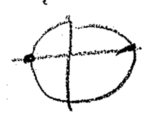
b) max charge occurs when $q'(t) = 0$

$q'(t) = -10e^{-3t}(-3\sin(3t)) + \cos(3t)(30e^{-3t})$
$+ (-10e^{-3t})(3\cos(3t)) + \sin(3t)(30e^{-3t}) = 0$

$30e^{-3t}\sin 3t + \cancel{30e^{-3t}\cos 3t} - \cancel{30e^{-3t}\cos 3t} + 30e^{-3t}\sin 3t = 0$

$60e^{-3t}\sin(3t) = 0$ $\sin(3t) = 0$ $\theta = 3t$

never zero $\sin\theta = 0$



$\theta = 0, \pi$

$3t = 0, \pi$

$t = 0, \pi/3$

$q(0) = 0$ so must be $\boxed{t = \pi/3 \text{ sec}}$

max charge is:

$q(\pi/3) = -10e^{-3(\frac{\pi}{3})}\cos(3\tfrac{\pi}{3}) - 10e^{-3\pi/3}\sin(3\pi/3) + 10$

$= -10e^{-\pi}(\underbrace{\cos\pi}_{-1} - \underbrace{\sin\pi}_{0}) + 10$

$$\boxed{= 10e^{-\pi} + 10 \approx 10.432 \text{ Coulombs}}$$

#11. For the given LRC series circuit…
$L = 1\,H$, $R = 2\,\Omega$, $C = 0.25\,F$,
$E(t) = 50\cos(t)\ V$

a) Find the steady-state charge on the capacitor.
b) Find the steady-state current in the circuit.

$Lq'' + Rq' + \frac{1}{C}q = E$

$q'' + 2q' + 4q = 50\cos t$

a) ① q_c: $q'' + 2q' + 4q = 0$

$m^2 + 2m + 4 = 0$

$m = \frac{-2 \pm \sqrt{4 - 4(4)}}{2} = \frac{-2 \pm \sqrt{12}\,i}{2}$

$m = -1 \pm \sqrt{3}\,i$

$q_c(t) = C_1 e^{-t}\cos(\sqrt{3}t) + C_2 e^{-t}\sin(\sqrt{3}t)$

② q_p: (table) $q_p = A\cos t + B\sin t$
(no absorption)

$q' = -A\sin t + B\cos t$
$q'' = -A\cos t - B\sin t$
$q'' + 2q' + 4q = 50\cos t$

$[-A\cos t - B\sin t] + 2[-A\sin t + B\cos t]$
$+ 4[A\cos t + B\sin t] = 50\cos t$

$(-A + 2B + 4A)\cos t + (-B - 2A + 4B)\sin t = (50)\cos t$

$(3A + 2B)\cos t + (-2A + 3B)\sin t = (50)\cos t$

system: $\begin{cases} 3A + 2B = 50 \\ -2A + 3B = 0 \end{cases}$

$\left[\begin{array}{cc|c} 3 & 2 & 50 \\ -2 & 3 & 0 \end{array}\right] \text{rref} \left[\begin{array}{cc|c} 1 & 0 & \frac{150}{13} \\ 0 & 1 & \frac{100}{13} \end{array}\right] \begin{array}{l} = A \\ = B \end{array}$

$q_p = \frac{150}{13}\cos t + \frac{100}{3}\sin t$

so

$q_c(t) = q_c + q_p$

$q_c(t) = C_1 e^{-t}\cos(\sqrt{3}t) + C_2 e^{-t}\sin(\sqrt{3}t)$
$+ \frac{150}{13}\cos t + \frac{100}{3}\sin t$

③

$q_c(t) = \underbrace{C_1 e^{-t}\cos(\sqrt{3}t) + C_2 e^{-t}\sin(\sqrt{3}t)}_{\text{transient terms}} + \underbrace{\frac{150}{13}\cos t + \frac{100}{3}\sin t}_{\text{steady-state terms}}$

Steady-state charge:

$\boxed{q_{ss}(t) = \frac{150}{13}\cos t + \frac{100}{13}\sin t}$

b) current, $i = \frac{dq}{dt}$

$\boxed{i_{ss}(t) = -\frac{150}{13}\sin t + \frac{100}{13}\cos t}$