**DiffEq - Ch 8 - Required Practice** Name: full solutions

## 8.1

#1. Write the linear system in matrix form:

$$\frac{dx}{dt} = 3x - 5y$$

$$\frac{dy}{dt} = 4x + 8y$$

$$\vec{X}' = \begin{bmatrix} 3 & -5 \\ 4 & 8 \end{bmatrix} \vec{X}$$

$$\left(\vec{X} = \begin{pmatrix} x \\ y \end{pmatrix}\right)$$

#3. Write the linear system in matrix form:

$$\frac{dx}{dt} = x - y + z + t - 1$$

$$\frac{dy}{dt} = 2x + y - z - 3t^2$$

$$\frac{dz}{dt} = x + y + z + t^2 - t + 2$$

$$\boxed{\vec{X}' = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \vec{X} + \vec{F} \quad \vec{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \vec{F} = \begin{bmatrix} t-1 \\ -3t^2 \\ t^2 - t + 2 \end{bmatrix}}$$

#2. Write the linear system in matrix form:

$$\frac{dx}{dt} = -3x + 4y - 9z$$

$$\frac{dy}{dt} = 6x - y$$

$$\frac{dz}{dt} = 10x + 4y + 3z$$

$$\boxed{\vec{X}' = \begin{bmatrix} -3 & 4 & -9 \\ 6 & -1 & 0 \\ 10 & 4 & 3 \end{bmatrix} \vec{X}}$$

$$\left(\vec{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)$$

#4. Write the given system without the use of matrices:

$$\vec{X}' = \begin{bmatrix} 4 & 2 \\ -1 & 3 \end{bmatrix} \vec{X} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t$$

$$\boxed{\begin{aligned} \frac{dx}{dt} &= 4x + 2y + e^t \\ \frac{dy}{dt} &= -x + 3y - e^t \end{aligned}}$$

#5. Write the given system without the use of matrices:

$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -1 & 2 \\ 3 & -4 & 1 \\ -2 & 5 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} e^{-t} - \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} t$$

$$\boxed{\begin{aligned} \frac{dx}{dt} &= x - y + 2z + e^{-t} - 3t \\ \frac{dy}{dt} &= 3x - 4y + z + 2e^{-t} + t \\ \frac{dz}{dt} &= -2x + 5y + 6z + 2e^{-t} - t \end{aligned}}$$

#6. Verify that the vector $\vec{X}$ is a solution of the given system:

$$\frac{dx}{dt} = 3x - 4y \qquad \vec{X} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-5t}$$

$$\frac{dy}{dt} = 4x - 7y \qquad x = e^{-5t} \quad y = 2e^{-5t}$$

$$\frac{dx}{dt} = -5e^{-5t} \quad \frac{dy}{dt} = -10e^{-5t}$$

$$\frac{dx}{dt} \stackrel{?}{=} 3x - 4y$$

$$-5e^{-5t} \stackrel{?}{=} 3(e^{-5t}) - 4(2e^{-5t})$$

$$-5e^{-5t} \stackrel{?}{=} (3-8)e^{-5t}$$

$$-5e^{-5t} = -5e^{-5t} \checkmark$$

$$\frac{dy}{dt} \stackrel{?}{=} 4x - 7y$$

$$-10e^{-5t} \stackrel{?}{=} 4(e^{-5t}) - 7(2e^{-5t})$$

$$-10e^{-5t} \stackrel{?}{=} (4-14)e^{-5t}$$

$$-10e^{-5t} = -10e^{-5t} \checkmark$$

#7. Verify that the vector $\vec{X}$ is a solution of the given system:

$$\vec{X}' = \begin{bmatrix} -1 & \frac{1}{4} \\ 1 & -1 \end{bmatrix}\vec{X}; \qquad \vec{X} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{-\frac{3}{2}t}$$

$$x = -e^{-3/2t} \quad y = 2e^{-3/2t}$$

$$\frac{dx}{dt} = \frac{3}{2}e^{-3/2t} \quad \frac{dy}{dt} = -3e^{-3/2t}$$

$$\frac{dx}{dt} \stackrel{?}{=} -x + \frac{1}{4}y$$

$$\frac{3}{2}e^{-3/2t} \stackrel{?}{=} -(-e^{-3/2t}) + \frac{1}{4}(2e^{-3/2t})$$

$$\frac{3}{2}e^{-3/2t} \stackrel{?}{=} (1 + \tfrac{1}{2})e^{-3/2t}$$

$$\frac{3}{2}e^{-3/2t} = \frac{3}{2}e^{-3/2t} \checkmark$$

$$\frac{dy}{dt} \stackrel{?}{=} x - y$$

$$-3e^{-3/2t} \stackrel{?}{=} (-e^{-3/2t}) - (2e^{-3/2t})$$

$$-3e^{-3/2t} \stackrel{?}{=} (-1-2)e^{-3/2t}$$

$$-3e^{-3/2t} = -3e^{-3/2t} \checkmark$$

#8. Verify that the vector $\vec{X}$ is a solution of the given system:

$$\vec{X}' = \begin{bmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix}\vec{X}; \quad \vec{X} = \begin{bmatrix} 1 \\ 6 \\ -13 \end{bmatrix}$$

$x=1 \quad y=6 \quad z=-13$

$\frac{dx}{dt}=0 \quad \frac{dy}{dt}=0 \quad \frac{dz}{dt}=0$

$\frac{dx}{dt} \overset{?}{=} x+2y+z$

$0 \overset{?}{=} (1)+2(6)+(-13) = 0$ ✓

$\frac{dy}{dt} \overset{?}{=} 6x-y$

$0 \overset{?}{=} 6(1)-(6) = 0$ ✓

$\frac{dz}{dt} \overset{?}{=} -x-2y-z$

$0 = -(1)-2(6)-(-13) = 0$ ✓

#9. The given vectors are solutions of a system $\vec{X}' = \vec{A}\vec{X}$. Determine whether the vectors form a fundamental set on the interval $(-\infty, \infty)$:

$$\vec{X}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t}, \quad \vec{X}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-6t}$$

$$W = \begin{vmatrix} e^{-2t} & e^{-6t} \\ e^{-2t} & -e^{-6t} \end{vmatrix}$$

$$= -e^{-2t}e^{-6t} - e^{-2t}e^{-6t}$$

$$= -e^{-8t} - e^{-8t} = -2e^{-8t} \neq 0 \quad (-\infty,\infty)$$

so these <u>do</u> form a fundamental set

#10. Verify that the vector $\vec{X}_p$ is a particular solution of the given system:

$$\vec{X}' = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}\vec{X} + \begin{bmatrix} 2t-7 \\ -4t-18 \end{bmatrix}; \quad \vec{X}_p = \begin{bmatrix} 2 \\ -1 \end{bmatrix} t + \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

$x = 2t+5 \quad y = -t+1$

$\frac{dx}{dt} = 2 \quad \frac{dy}{dt} = -1$

$\frac{dx}{dt} \overset{?}{=} x+4y+2t-7$

$2 \overset{?}{=} (2t+5)+4(-t+1)+2t-7$

$2 = 2$ ✓

$\frac{dy}{dt} \overset{?}{=} 3x+2y-4t-18$

$-1 \overset{?}{=} 3(2t+5)+2(-t+1)-4t-18$

$-1 = -1$ ✓

## 8.2 day 1

#1. Find the general solution of the system:

$$\frac{dx}{dt} = x + 2y$$

$$\frac{dy}{dt} = 4x + 3y$$

$\vec{X}' = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\vec{X}$

$\det(\vec{A} - \lambda \vec{I}) = 0$

$\begin{vmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix} = 0$

$(1-\lambda)(3-\lambda) - 8 = 0$

$3 - 4\lambda + \lambda^2 - 8 = 0$

$\lambda^2 - 4\lambda - 5 = 0$

$(\lambda - 5)(\lambda + 1) = 0$

$\underline{\lambda = 5} \quad \underline{\lambda = -1}$

$(\vec{A} - \lambda \vec{I})\vec{K} = 0$

$\underline{\lambda = 5}$

$\begin{bmatrix} 1-5 & 2 \\ 4 & 3-5 \end{bmatrix}\begin{pmatrix} K_1 \\ K_2 \end{pmatrix} = 0$

$\left[\begin{array}{cc|c} -4 & 2 & 0 \\ 4 & -2 & 0 \end{array}\right]$

rref

$\left[\begin{array}{cc|c} 1 & -1/2 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$K_1 - \frac{1}{2}K_2 = 0$

$K_1 = \frac{1}{2}K_2$

$(\frac{1}{2}K_2, K_2) \to (1, 2)$

$\lambda = 5: \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\underline{\lambda = -1}$

$\begin{bmatrix} 1+1 & 2 \\ 4 & 3-1 \end{bmatrix}\begin{bmatrix} K_1 \\ K_2 \end{bmatrix} = 0$

$\left[\begin{array}{cc|c} 2 & 2 & 0 \\ 4 & 4 & 0 \end{array}\right]$

rref

$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$K_1 + K_2 = 0$

$K_1 = -K_2$

$(-K_2, K_2) \to (-1, 1)$

$\lambda = -1: \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$\boxed{\vec{X} = c_1\begin{bmatrix} 1 \\ 2 \end{bmatrix}e^{5t} + c_2\begin{bmatrix} -1 \\ 1 \end{bmatrix}e^{-t}}$$

#2. Find the general solution of the system:

$$\frac{dx}{dt} = -4x + 2y$$

$$\frac{dy}{dt} = -\frac{5}{2}x + 2y$$

$\vec{X}' = \begin{bmatrix} -4 & 2 \\ -5/2 & 2 \end{bmatrix}\vec{X}$

$\begin{vmatrix} -4-\lambda & 2 \\ -5/2 & 2-\lambda \end{vmatrix} = 0$

$(-4-\lambda)(2-\lambda) + 5 = 0$

$-8 + 2\lambda + \lambda^2 + 5 = 0$

$\lambda^2 + 2\lambda - 3 = 0$

$(\lambda + 3)(\lambda - 1) = 0$

$\underline{\lambda = -3} \quad \underline{\lambda = 1}$

$\underline{\lambda = -3}$

$\begin{bmatrix} -4+3 & 2 \\ -5/2 & 2+3 \end{bmatrix}\begin{bmatrix} K_1 \\ K_2 \end{bmatrix} = \vec{0}$

$\left[\begin{array}{cc|c} -1 & 2 & 0 \\ -5/2 & 5 & 0 \end{array}\right]$ rref $\left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$K_1 - 2K_2 = 0, \quad K_1 = 2K_2, \quad (2K_2, K_2) \; (2, 1)$

$\lambda = -3 \; \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\underline{\lambda = 1}$

$\begin{bmatrix} -4-1 & 2 \\ -5/2 & 2-1 \end{bmatrix}\begin{bmatrix} K_1 \\ K_2 \end{bmatrix} = \vec{0}$

$\left[\begin{array}{cc|c} -5 & 2 & 0 \\ -5/2 & 1 & 0 \end{array}\right]$ rref $\left[\begin{array}{cc|c} 1 & -2/5 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$K_1 - \frac{2}{5}K_2 = 0, \quad K_1 = \frac{2}{5}K_2, \quad (\frac{2}{5}K_2, K_2) \; (2, 5)$

$\lambda = 1 \; \begin{bmatrix} 2 \\ 5 \end{bmatrix}$

$$\boxed{X = c_1\begin{bmatrix} 2 \\ 1 \end{bmatrix}e^{-3t} + c_2\begin{bmatrix} 2 \\ 5 \end{bmatrix}e^{t}}$$

#3. Find the general solution of the system:

$$\vec{X}' = \begin{bmatrix} 10 & -5 \\ 8 & -12 \end{bmatrix} \vec{X}$$

$$\begin{vmatrix} 10-\lambda & -5 \\ 8 & -12-\lambda \end{vmatrix} = 0$$

$(10-\lambda)(-12-\lambda) + 40 = 0$

$-120 + 2\lambda + \lambda^2 + 40 = 0$

$\lambda^2 + 2\lambda - 80 = 0$

$(\lambda + 10)(\lambda - 8) = 0$

$\lambda = -10 \quad \lambda = 8$

$\lambda = -10$

$$\begin{bmatrix} 10+10 & -5 \\ 8 & -12+10 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \vec{0}$$

$$\left[\begin{array}{cc|c} 20 & -5 & 0 \\ 8 & -2 & 0 \end{array}\right] \text{rref} \left[\begin{array}{cc|c} 1 & -1/4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$k_1 - \frac{1}{4}k_2 = 0, \quad k_1 = \frac{1}{4}k_2, \quad (\frac{1}{4}k_2, k_2), \quad (1, 4)$

$\lambda = -10 \quad \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

$\lambda = 8$

$$\begin{bmatrix} 10-8 & -5 \\ 8 & -12-8 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \vec{0}$$

$$\left[\begin{array}{cc|c} 2 & -5 & 0 \\ 8 & -20 & 0 \end{array}\right] \text{rref} \left[\begin{array}{cc|c} 1 & -5/2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$k_1 - \frac{5}{2}k_2 = 0, \quad k_1 = \frac{5}{2}k_2, \quad (\frac{5}{2}k_2, k_2) \quad (5, 2)$

$\lambda = 8 \quad \begin{bmatrix} 5 \\ 2 \end{bmatrix}$

$$\boxed{\vec{X} = c_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} e^{-10t} + c_2 \begin{bmatrix} 5 \\ 2 \end{bmatrix} e^{8t}}$$

#4. Find the general solution of the system:

$$\frac{dx}{dt} = x + y - z \qquad \vec{X}' = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 0 \\ 0 & 1 & -1 \end{bmatrix} \vec{X}$$

$$\frac{dy}{dt} = 2y$$

$$\frac{dz}{dt} = y - z \qquad \begin{vmatrix} 1-\lambda & 1 & -1 \\ 0 & 2-\lambda & 0 \\ 0 & 1 & -1-\lambda \end{vmatrix} = 0$$

$(1-\lambda)[(2-\lambda)(-1-\lambda)-0] - 0(\sim) + 0(-) = 0$

$(1-\lambda)(2-\lambda)(-1-\lambda) = 0$

$\underline{\lambda=1}$ $\underline{\lambda=2}$ $\underline{\lambda=-1}$

$\underline{\lambda=1}$

$\begin{bmatrix} 1-1 & 1 & -1 \\ 0 & 2-1 & 0 \\ 0 & 0 & -1-1 \end{bmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \vec{0}$

$\left[\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right]$

rref

$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$k_2 = 0,\ k_3 = 0$

$(k_1, 0, 0) \to (1, 0, 0)$

$\lambda = 1 \quad \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\underline{\lambda=2}$

$\begin{bmatrix} 1-2 & 1 & -1 \\ 0 & 2-2 & 0 \\ 0 & 1 & -1-2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \vec{0}$

$\left[\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -3 & 0 \end{array}\right]$

rref

$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$k_1 - 2k_3 = 0$

$k_2 - 3k_3 = 0$

$k_1 = 2k_3$

$k_2 = 3k_3$

$(2k_3, 3k_3, k_3)$

$(2, 3, 1)$

$\lambda = 2 \quad \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$

$\underline{\lambda=-1}$

$\begin{bmatrix} 1+1 & 1 & -1 \\ 0 & 2+1 & 0 \\ 0 & 1 & -1+1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \vec{0}$

$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$

rref

$\left[\begin{array}{ccc|c} 1 & 0 & -1/2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$k_1 - \frac{1}{2}k_3 = 0 \to k_1 = \frac{1}{2}k_3$

$k_2 = 0$

$(\frac{1}{2}k_3, 0, k_3)$

$(1, 0, 2)$

$\lambda = -1 \quad \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

$$\boxed{\vec{X} = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} e^{2t} + c_3 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} e^{-t}}$$

#5. Find the general solution of the system:

$$\vec{X}' = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 3 & -1 \end{bmatrix} \vec{X}$$

$$\begin{vmatrix} -1-\lambda & 1 & 0 \\ 1 & 2-\lambda & 1 \\ 0 & 3 & -1-\lambda \end{vmatrix} = 0$$

$$(-1-\lambda)[(2-\lambda)(-1-\lambda)-3] - 1[1(-1-\lambda)-0] + 0[\sim]$$

$$(-1-\lambda)(2-\lambda)(-1-\lambda) - 3(-1-\lambda) - (-1-\lambda) = 0$$

$$(-1-\lambda)[(2-\lambda)(-1-\lambda)-3-1] = 0$$

$$(-1-\lambda)(-2-\lambda+\lambda^2-4] = 0$$

$$(-1-\lambda)(\lambda^2-\lambda-6] = 0$$

$$(-1-\lambda)(\lambda-3)(\lambda+2) = 0$$

$$\lambda = -1 \quad \lambda = 3 \quad \lambda = -2$$

$\lambda = -1$

$$\begin{bmatrix} -1+1 & 1 & 0 \\ 1 & 2+1 & 1 \\ 0 & 3 & -1+1 \end{bmatrix}\begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix} = \vec{0}$$

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 3 & 0 & 0 \end{array}\right]$$

rref

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$K_1 + K_3 = 0 \rightarrow K_1 = -K_3$

$K_2 = 0$

$(-K_3, 0, K_3)$

$(-1, 0, 1)$

$$\lambda = -1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$\lambda = 3$

$$\begin{bmatrix} -1-3 & 1 & 0 \\ 1 & 2-3 & 1 \\ 0 & 3 & -1-3 \end{bmatrix}\begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix} = \vec{0}$$

$$\left[\begin{array}{ccc|c} -4 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 3 & -4 & 0 \end{array}\right]$$

rref

$$\left[\begin{array}{ccc|c} 1 & 0 & -1/3 & 0 \\ 0 & 1 & -4/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$K_1 - 1/3 K_3 = 0, \ K_1 = 1/3 K_3$

$K_2 - 4/3 K_3 = 0, \ K_2 = 4/3 K_3$

$(1/3 K_3, 4/3 K_3, K_3)$

$(1, 4, 3)$

$$\lambda = 3 \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$$

$\lambda = -2$

$$\begin{bmatrix} -1+2 & 1 & 0 \\ 1 & 2+2 & 1 \\ 0 & 3 & -1+2 \end{bmatrix}\begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix} = \vec{0}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 3 & 1 & 0 \end{array}\right]$$

rref

$$\left[\begin{array}{ccc|c} 1 & 0 & -1/3 & 0 \\ 0 & 1 & 1/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$K_1 - 1/3 K_3 = 0 \rightarrow K_1 = 1/3 K_3$

$K_2 + 1/3 K_3 = 0 \rightarrow K_2 = -1/3 K_3$

$(1/3 K_3, -1/3 K_3, K_3)$

$(1, -1, 3)$

$$\lambda = -2 \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$$

$$\boxed{\vec{X} = c_1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} e^{3t} + c_3 \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} e^{-2t}}$$

#6. Solve the initial-value problem:

$$\vec{X}' = \begin{bmatrix} \frac{1}{2} & 0 \\ 1 & -\frac{1}{2} \end{bmatrix} \vec{X}, \quad \vec{X}(0) = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$\begin{vmatrix} \frac{1}{2}-\lambda & 0 \\ 1 & -\frac{1}{2}-\lambda \end{vmatrix} = 0$$

$$\left(\tfrac{1}{2}-\lambda\right)\left(-\tfrac{1}{2}-\lambda\right) - 0 = 0$$

$$\underline{\lambda = 1/2 \quad \lambda = -1/2}$$

$\underline{\underline{\lambda = 1/2}}$

$$\begin{bmatrix} 1/2 - 1/2 & 0 \\ 1 & -1/2 - 1/2 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \vec{0}$$

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & -1 & 0 \end{array}\right]$$

rref

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$k_1 - k_2 = 0 \rightarrow k_1 = k_2$

$(k_2, k_2)$

$(1, 1)$

$\lambda = 1/2 \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\underline{\underline{\lambda = -1/2}}$

$$\begin{bmatrix} 1/2 + 1/2 & 0 \\ 1 & -1/2 + 1/2 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \vec{0}$$

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right]$$

rref

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$k_1 = 0$

$(0, k_2)$

$(0, 1)$

$\lambda = -1/2 \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$\underline{\vec{X} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{1/2 t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-1/2 t}}$$

now $\vec{X}(0) = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

$$\begin{bmatrix} 3 \\ 5 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{0} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{0}$$

$$\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_1 \end{bmatrix} + \begin{bmatrix} 0 \\ c_2 \end{bmatrix}$$

$c_1 = 3$

$c_1 + c_2 = 3 + c_2 = 5, \quad c_2 = 2$

$$\boxed{\vec{X} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{1/2 t} + 2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-1/2 t}}$$

## 8.2 day 2

#1. Find the general solution of the system:

$$\frac{dx}{dt} = 3x - y$$

$$\frac{dy}{dt} = 9x - 3y$$

$\vec{X}' = \begin{bmatrix} 3 & -1 \\ 9 & -3 \end{bmatrix}\vec{X}$

$\begin{vmatrix} 3-\lambda & -1 \\ 9 & -3-\lambda \end{vmatrix} = 0$

$(3-\lambda)(-3-\lambda) + 9 = 0$

$\lambda^2 - 9 + 9 = 0$

$\lambda^2 = 0$

$\lambda = 0$ (repeated)

$\lambda = 0$

$\begin{bmatrix} 3-0 & -1 \\ 9 & -3-0 \end{bmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \vec{0}$

$\left[\begin{array}{cc|c} 3 & -1 & 0 \\ 9 & -3 & 0 \end{array}\right]$

rref

$\left[\begin{array}{cc|c} 1 & -1/3 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$k_1 - \frac{1}{3}k_2 = 0, \quad k_1 = \frac{1}{3}k_2$

$(\frac{1}{3}k_2, k_2) \quad (1, 3)$

$\lambda = 0 \quad \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

repeated

$\left[\begin{array}{cc|c} 3 & -1 & 1 \\ 9 & -3 & 3 \end{array}\right]$

rref

$\left[\begin{array}{cc|c} 1 & -1/3 & 1/3 \\ 0 & 0 & 0 \end{array}\right]$

$k_1 - \frac{1}{3}k_2 = \frac{1}{3} \to k_1 = \frac{1}{3}k_2 + \frac{1}{3}$

$(\frac{1}{3}k_2 + \frac{1}{3}, k_2)$

$(1, 2) \quad \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\vec{X} = c_1\begin{bmatrix} 1 \\ 3 \end{bmatrix}e^{0t} + c_2\left(\begin{bmatrix} 1 \\ 3 \end{bmatrix}te^{0t} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}e^{0t}\right)$

$$\boxed{\vec{X} = c_1\begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2\left(\begin{bmatrix} 1 \\ 3 \end{bmatrix}t + \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)}$$

#2. Find the general solution of the system:

$$\vec{X}' = \begin{bmatrix} -1 & 3 \\ -3 & 5 \end{bmatrix}\vec{X}$$

$\begin{vmatrix} -1-\lambda & 3 \\ -3 & 5-\lambda \end{vmatrix} = 0$

$(-1-\lambda)(5-\lambda) + 9 = 0$

$\lambda^2 - 4\lambda + 4 = 0$

$(\lambda-2)(\lambda-2) = 0$

$\lambda = 2$ (repeated)

$\lambda = 2$

$\begin{bmatrix} -1-2 & 3 \\ -3 & 5-2 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \vec{0}$

$\left[\begin{array}{cc|c} -3 & 3 & 0 \\ -3 & 3 & 0 \end{array}\right]$

rref

$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $k_1 - k_2 = 0 \to k_1 = k_2$

$(k_2, k_2) \quad (1,1) \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ for $\lambda = 2$

repeat

$\left[\begin{array}{cc|c} -3 & 3 & 1 \\ -3 & 3 & 1 \end{array}\right]$

rref

$\left[\begin{array}{cc|c} 1 & -1 & -1/3 \\ 0 & 0 & 0 \end{array}\right]$ $k_1 - k_2 = -\frac{1}{3}$

$k_1 = k_2 - \frac{1}{3}$

$(k_2 - \frac{1}{3}, k_2)$

$(-\frac{1}{3}, 0) \quad \begin{pmatrix} -1/3 \\ 0 \end{pmatrix}$

$$\boxed{\vec{X} = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{2t} + c_2\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}te^{2t} + \begin{bmatrix} -1/3 \\ 0 \end{bmatrix}e^{2t}\right)}$$

#3. Find the general solution of the system:

$$\frac{dx}{dt} = 3x - y - z$$

$$\frac{dy}{dt} = x + y - z$$

$$\frac{dz}{dt} = x - y + z$$

$$\vec{X}' = \begin{bmatrix} 3 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \vec{X}$$

$$\begin{array}{ccc} + & - & + \end{array}$$

$$\begin{vmatrix} 3-\lambda & -1 & -1 \\ 1 & 1-\lambda & -1 \\ 1 & -1 & 1-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)[(1-\lambda)^2 - 1] - (-1)[(1-\lambda)+1] + (-1)[-1-(1-\lambda)] = 0 \Rightarrow$$

$$(3-\lambda)(1-\lambda)^2 - (3-\lambda) + (1-\lambda) + 1 + 1 + (1-\lambda) = 0$$

$$(3-\lambda)(1-2\lambda+\lambda^2) - 3 + \lambda + 1 - \lambda + 2 + 1 - \lambda = 0$$

$$3 - 6\lambda + 3\lambda^2 - \lambda + 2\lambda^2 - \lambda^3 - 3 + \lambda + 1 - \lambda + 3 - \lambda = 0$$

$$-\lambda^3 + 5\lambda^2 - 8\lambda + 4 = 0$$

$$-(\lambda^3 - 5\lambda^2 + 8\lambda - 4) = 0$$

try 2
$$\begin{array}{c|cccc} & 1 & -5 & 8 & -4 \\ & & 2 & -6 & 4 \\ \hline & 1 & -3 & 2 & 0 \end{array}$$

$$(\lambda-2)(\lambda^2 - 3\lambda + 2)$$

$$(\lambda-2)(\lambda-2)(\lambda-1) = 0$$

$\lambda = 1$ & $\lambda = 2$ (repeated)

$\lambda = 1$

$$\left[\begin{array}{ccc|c} 3-1 & -1 & -1 & 0 \\ 1 & 1-1 & -1 & 0 \\ 1 & -1 & 1-1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \end{array}\right]$$

rref

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$k_1 - k_3 = 0 \to k_1 = k_3$

$k_2 - k_3 = 0 \to k_2 = k_3$

$(k_3, k_3, k_3)$ $(1,1,1)$

$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ for $\lambda = 1$

$\lambda = 2$

$$\left[\begin{array}{ccc|c} 3-2 & -1 & -1 & 0 \\ 1 & 1-2 & -1 & 0 \\ 1 & -1 & 1-2 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 1 & -1 & -1 & 0 \\ 1 & -1 & -1 & 0 \end{array}\right]$$

rref

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$k_1 - k_2 - k_3 = 0$

$k_1 = k_2 + k_3$

$(k_2 + k_3, k_2, k_3)$

choose: $k_2 = 0, k_3 = 1$

$(1, 0, 1)$ $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

also choose: $k_2 = 1, k_3 = 0$

$(1, 1, 0)$ $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

both for $\lambda = 2$

$$\boxed{\vec{X} = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{2t} + c_3 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{2t}}$$

#4. Find the general solution of the system:

$$\vec{X}' = \begin{bmatrix} 5 & -4 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 5 \end{bmatrix} \vec{X} \qquad \begin{vmatrix} 5-\lambda & -4 & 0 \\ 1 & 0-\lambda & 2 \\ 0 & 2 & 5-\lambda \end{vmatrix} = 0$$

$$(5-\lambda)[-\lambda(5-\lambda)-4] - (-4)[5-\lambda-0] + 0[\;\;]$$

$$-\lambda(5-\lambda)^2 - 4(5-\lambda) + 4(5-\lambda) = 0$$

$$-\lambda(5-\lambda)^2 = 0 \qquad \underline{\lambda = 0},\ \underline{\lambda = 5} \text{ (repeated)}$$

$\lambda = 0$

$$\left[\begin{array}{ccc|c} 5 & -4 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 5 & 0 \end{array}\right]$$

rref

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 5/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$k_1 + 2k_3 = 0 \rightarrow k_1 = -2k_3$

$k_2 + 5/2 k_3 = 0 \rightarrow k_2 = -\frac{5}{2} k_3$

$(-2k_3, -\frac{5}{2}k_3, k_3)$

$(-4, -5, 2)$

$$\begin{bmatrix} -4 \\ -5 \\ 2 \end{bmatrix}$$

for $\lambda = 0$

$\lambda = 5$

$$\left[\begin{array}{ccc|c} 5-5 & -4 & 0 & 0 \\ 1 & 0-5 & 2 & 0 \\ 0 & 2 & 5-5 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 0 & -4 & 0 & 0 \\ 1 & -5 & 2 & 0 \\ 0 & 2 & 0 & 0 \end{array}\right]$$

rref

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$k_1 + 2k_3 = 0 \rightarrow k_1 = -2k_3$

$k_2 = 0$

$(-2k_3, 0, k_3)$

$(-2, 0, 1)$

$$\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

for $\lambda = 5$

repeat $\lambda = 5$

$$\left[\begin{array}{ccc|c} 0 & -4 & 0 & -2 \\ 1 & -5 & 2 & 0 \\ 0 & 2 & 0 & 1 \end{array}\right]$$

rref

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 5/2 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$k_1 + 2k_3 = 5/2 \rightarrow k_1 = 5/2 - 2k_3$

$k_2 = 1/2$

$(5/2 - 2k_3, \frac{1}{2}, k_3)$

$(\frac{5}{2}, \frac{1}{2}, 0)$

$$\begin{bmatrix} 5/2 \\ 1/2 \\ 0 \end{bmatrix}$$

repeat for $\lambda = 5$

$$\vec{X} = c_1 \begin{bmatrix} -4 \\ -5 \\ 2 \end{bmatrix} \cancel{e^{0t}} + c_2 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} e^{5t} + c_3\left(\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} te^{5t} + \begin{bmatrix} 5/2 \\ 1/2 \\ 0 \end{bmatrix} e^{5t}\right)$$

$$\boxed{\vec{X} = c_1 \begin{bmatrix} -4 \\ -5 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} e^{5t} + c_3\left(\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} te^{5t} + \begin{bmatrix} 5/2 \\ 1/2 \\ 0 \end{bmatrix} e^{5t}\right)}$$

#5. Find the general solution of the system:

$$\vec{X}' = \begin{bmatrix}1&0&0\\2&2&-1\\0&1&0\end{bmatrix}\vec{X} \qquad \begin{vmatrix}1-\lambda&0&0\\2&2-\lambda&-1\\0&1&0-\lambda\end{vmatrix}=0$$

$$(1-\lambda)[-\lambda(2-\lambda)+1]-0(\sim)+0(\sim)=0$$
$$(1-\lambda)[\lambda^2-2\lambda+1]=0$$
$$(1-\lambda)(\lambda-1)(\lambda-1)=0 \qquad \underline{\lambda=1} \text{ multiplicity } 3$$

$\underline{\lambda=1}$

$$\left[\begin{array}{ccc|c}1-1&0&0&0\\2&2-1&-1&0\\0&1&0-1&0\end{array}\right]$$
$$\left[\begin{array}{ccc|c}0&0&0&0\\2&1&-1&0\\0&1&-1&0\end{array}\right]$$
rref
$$\left[\begin{array}{ccc|c}1&0&0&0\\0&1&-1&0\\0&0&0&0\end{array}\right]$$
$K_1=0$
$K_2-K_3=0 \rightarrow K_2=K_3$
$(0, K_3, K_3)$
$(0, 1, 1)$
$$\begin{bmatrix}0\\1\\1\end{bmatrix}$$
for $\lambda=1$

$\underline{1^{st} \text{ repeat}}$

$$\left[\begin{array}{ccc|c}0&0&0&0\\2&1&-1&1\\0&1&-1&1\end{array}\right]$$
rref
$$\left[\begin{array}{ccc|c}1&0&0&0\\0&1&-1&1\\0&0&0&0\end{array}\right]$$
$K_1=0$
$K_2-K_3=1 \rightarrow K_2=1+K_3$
$(0, 1+K_3, K_3)$
$(0, 1, 0)$
$$\begin{bmatrix}0\\1\\0\end{bmatrix}$$
$1^{st}$ repeat
$\lambda=1$

$\underline{2^{nd} \text{ repeat}}$

$$\left[\begin{array}{ccc|c}0&0&0&0\\2&1&-1&1\\0&1&-1&0\end{array}\right]$$
rref
$$\left[\begin{array}{ccc|c}1&0&0&1/2\\0&1&-1&0\\0&0&0&0\end{array}\right]$$
$K_1=1/2$
$K_2-K_3=0 \rightarrow K_2=K_3$
$(\frac{1}{2}, K_3, K_3)$
$(\frac{1}{2}, 0, 0)$
$$\begin{bmatrix}1/2\\0\\0\end{bmatrix}$$
$2^{nd}$ repeat
$\lambda=1$

$$\boxed{\vec{X}=c_1\begin{bmatrix}0\\1\\1\end{bmatrix}e^t+c_2\left(\begin{bmatrix}0\\1\\1\end{bmatrix}te^t+\begin{bmatrix}0\\1\\0\end{bmatrix}e^t\right)+c_3\left(\begin{bmatrix}0\\1\\1\end{bmatrix}\frac{t^2}{2}e^t+\begin{bmatrix}0\\1\\0\end{bmatrix}te^t+\begin{bmatrix}1/2\\0\\0\end{bmatrix}e^t\right)}$$

#6. Solve the initial-value problem:

$$\vec{X}' = \begin{bmatrix} 2 & 4 \\ -1 & 6 \end{bmatrix} \vec{X}, \qquad \vec{X}(0) = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$$

$$\begin{vmatrix} 2-\lambda & 4 \\ -1 & 6-\lambda \end{vmatrix} = 0 \qquad (2-\lambda)(6-\lambda)+4=0$$

$$\lambda^2 - 8\lambda + 16 = 0$$

$$(\lambda-4)(\lambda-4) = 0 \qquad \underline{\lambda=4} \text{ repeated}$$

$\underline{\lambda=4}$

$$\left[\begin{array}{cc|c} 2-4 & 4 & 0 \\ -1 & 6-4 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} -2 & 4 & 0 \\ -1 & 2 & 0 \end{array}\right]$$

rref

$$\left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$k_1 - 2k_2 = 0 \rightarrow k_1 = 2k_2$$

$$(2k_2, k_2)$$

$$(2,1) \quad \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

for $\lambda=4$

$\underline{\text{repeat } \lambda=4}$

$$\left[\begin{array}{cc|c} -2 & 4 & 2 \\ -1 & 2 & 1 \end{array}\right]$$

rref

$$\left[\begin{array}{cc|c} 1 & -2 & -1 \\ 0 & 0 & 0 \end{array}\right]$$

$$k_1 - 2k_2 = -1$$

$$\rightarrow k_1 = 2k_2 - 1$$

$$(2k_2-1, k_2)$$

$$(-1, 0)$$

$$\begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

for repeat $\lambda=4$

$$\vec{X} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{4t} + c_2\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} te^{4t} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} e^{4t}\right) \qquad \text{now } \vec{X}(0) = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$$

$$x = 2c_1 e^{4t} + 2c_2 te^{4t} - c_2 e^{4t}$$

$$-1 = 2c_1 e^0 + 2c_2(0)e^0 - c_2 e^0$$

$$-1 = 2c_1 - c_2$$

$$-1 = 2(6) - c_2$$

$$c_2 = 12 + 1 = \underline{13}$$

$$y = c_1 e^{4t} + c_2 te^{4t}$$

$$6 = c_1 e^0 + c_2(0)e^0$$

$$\leftarrow \quad \underline{6 = c_1}$$

$$\boxed{\vec{X} = 6\begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{4t} + 13\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} te^{4t} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} e^{4t}\right)}$$

$$\vec{X} = \begin{bmatrix} 12 \\ 6 \end{bmatrix} e^{4t} + \begin{bmatrix} 26 \\ 13 \end{bmatrix} te^{4t} + \begin{bmatrix} -13 \\ 0 \end{bmatrix} e^{4t}$$

$$\boxed{\vec{X} = \begin{bmatrix} -1 \\ 6 \end{bmatrix} e^{4t} + \begin{bmatrix} 26 \\ 13 \end{bmatrix} te^{4t}}$$

$\leftarrow$ (this will be identical, regardless of which eigenvectors you chose)

**8.2 day 3**

#1. Find the general solution of the system:

$$\frac{dx}{dt} = 6x - y$$

$$\frac{dy}{dt} = 5x + 2y$$

$$\begin{vmatrix} 6-\lambda & -1 \\ 5 & 2-\lambda \end{vmatrix} = 0 \qquad (6-\lambda)(2-\lambda) + 5 = 0$$

$$\lambda^2 - 8\lambda + 17 = 0$$

$$\lambda = \frac{8 \pm \sqrt{64 - 4(17)}}{2}$$

$$\lambda = \frac{8 \pm 2i}{2} = 4 \pm i$$

$\underline{\lambda = 4 + i}$

$$\left[\begin{array}{cc|c} 6-(4+i) & -1 & 0 \\ 5 & 2-(4+i) & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 2-i & -1 & 0 \\ 5 & -2-i & 0 \end{array}\right]$$

$(2-i)k_1 - k_2 = 0$

$k_2 = (2-i)k_1$

$(k_1, (2-i)k_1)$

$(1, 2-i)$ $\begin{bmatrix} 1 \\ 2-i \end{bmatrix}$

$\vec{B_1} = Re\begin{bmatrix} 1 \\ 2-i \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\vec{B_2} = Im\begin{bmatrix} 1 \\ 2-i \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

$\vec{X_1} = \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\cos t - \begin{bmatrix} 0 \\ -1 \end{bmatrix}\sin t\right)e^{4t}$

$\vec{X_2} = \left(\begin{bmatrix} 0 \\ -1 \end{bmatrix}\cos t + \begin{bmatrix} 0 \\ -1 \end{bmatrix}\sin t\right)e^{4t}$

$$\boxed{\vec{X} = C_1\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\cos t - \begin{bmatrix} 0 \\ -1 \end{bmatrix}\sin t\right)e^{4t} + C_2\left(\begin{bmatrix} 0 \\ -1 \end{bmatrix}\cos t + \begin{bmatrix} 1 \\ 2 \end{bmatrix}\sin t\right)e^{4t}}$$

or

— or —

$5k_1 + (-2-i)k_2 = 0$

$k_1 = \frac{(2+i)}{5}k_2$ $\left(\frac{(2+i)}{5}k_2, k_2\right)$

$(2+i, 5)$ $\begin{bmatrix} 2+i \\ 5 \end{bmatrix}$

$\vec{B_1} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ $\vec{B_2} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\vec{X_1} = \left(\begin{bmatrix} 2 \\ 5 \end{bmatrix}\cos t - \begin{bmatrix} 1 \\ 0 \end{bmatrix}\sin t\right)e^{4t}$

$\vec{X_2} = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\cos t + \begin{bmatrix} 2 \\ 5 \end{bmatrix}\sin t\right)e^{4t}$

$$\boxed{\begin{aligned} \vec{X} &= C_1\left(\begin{bmatrix} 2 \\ 5 \end{bmatrix}\cos t - \begin{bmatrix} 1 \\ 0 \end{bmatrix}\sin t\right)e^{4t} + C_2\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\cos t + \begin{bmatrix} 2 \\ 5 \end{bmatrix}\sin t\right)e^{4t} \\ &\text{or} \\ \vec{X} &= C_1\begin{bmatrix} 2\cos t - \sin t \\ 5\cos t \end{bmatrix}e^{4t} + C_2\begin{bmatrix} \cos t + 2\sin t \\ 5\sin t \end{bmatrix}e^{4t} \end{aligned}}$$

#2. Find the general solution of the system:

$$\vec{X}' = \begin{bmatrix} 4 & -5 \\ 5 & -4 \end{bmatrix} \vec{X}$$

$\begin{vmatrix} 4-\lambda & -5 \\ 5 & -4-\lambda \end{vmatrix} = 0$ $(4-\lambda)(-4-\lambda)+25=0$ $\lambda = 0 \pm 3i$

$\lambda^2 + 9 = 0$

$\lambda = 0+3i$

$\left[\begin{array}{cc|c} 4-(0+3i) & -5 & 0 \\ 5 & -4-(0+3i) & 0 \end{array}\right]$

$\left[\begin{array}{cc|c} 4-3i & -5 & 0 \\ 5 & -4-3i & 0 \end{array}\right]$

$(4-3i)k_1 - 5k_2 = 0$

$k_2 = \frac{4-3i}{5} k_1$

$(k_1, \frac{4-3i}{5} k_1)$ $(5, 4-3i)$

$\begin{bmatrix} 5 \\ 4-3i \end{bmatrix}$ $\vec{B_1} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$ $\vec{B_2} = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$

$\vec{X_1} = \left(\begin{bmatrix} 5 \\ 4 \end{bmatrix}\cos 3t - \begin{bmatrix} 0 \\ -3 \end{bmatrix}\sin 3t\right)\cancel{e^{0t}}$

$\vec{X_2} = \left(\begin{bmatrix} 0 \\ -3 \end{bmatrix}\cos 3t + \begin{bmatrix} 5 \\ 4 \end{bmatrix}\sin 3t\right)\cancel{e^{0t}}$

$$\boxed{\begin{aligned} \vec{X} &= C_1\left(\begin{bmatrix} 5 \\ 4 \end{bmatrix}\cos 3t - \begin{bmatrix} 0 \\ -3 \end{bmatrix}\sin 3t\right) + C_2\left(\begin{bmatrix} 0 \\ -3 \end{bmatrix}\cos 3t + \begin{bmatrix} 5 \\ 4 \end{bmatrix}\sin 3t\right) \\ &\text{—or—} \\ \vec{X} &= C_1\begin{bmatrix} 5\cos t \\ 4\cos 3t + 3\sin 3t \end{bmatrix} + C_2\begin{bmatrix} 5\sin t \\ -3\cos 3t + 4\sin 3t \end{bmatrix} \end{aligned}}$$

only this one posted

—or—

$5k_1 - (4+3i)k_2 = 0$

$k_1 = \frac{4+3i}{5} k_2$

$(\frac{4+3i}{5} k_2, k_2)$ $(4+3i, 5)$

$\begin{bmatrix} 4+3i \\ 5 \end{bmatrix}$ $\vec{B_1} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ $\vec{B_2} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$

$\vec{X_1} = \left(\begin{bmatrix} 4 \\ 5 \end{bmatrix}\cos 3t - \begin{bmatrix} 3 \\ 0 \end{bmatrix}\sin 3t\right)\cancel{e^{0t}}$

$\vec{X_2} = \left(\begin{bmatrix} 3 \\ 0 \end{bmatrix}\cos 3t + \begin{bmatrix} 4 \\ 5 \end{bmatrix}\sin 3t\right)\cancel{e^{0t}}$

$\vec{X} = C_1\left(\begin{bmatrix} 4 \\ 5 \end{bmatrix}\cos 3t - \begin{bmatrix} 3 \\ 0 \end{bmatrix}\sin 3t\right) + C_2\left(\begin{bmatrix} 3 \\ 0 \end{bmatrix}\cos 3t + \begin{bmatrix} 4 \\ 5 \end{bmatrix}\sin 3t\right)$

—or—

$$\boxed{\vec{X} = C_1\begin{bmatrix} 4\cos 3t - 3\sin 3t \\ 5\cos 3t \end{bmatrix} + C_2\begin{bmatrix} 3\cos 3t + 4\sin 3t \\ 5\sin 3t \end{bmatrix}}$$

#3. Find the general solution of the system:

$$\frac{dx}{dt} = z \qquad \frac{dy}{dt} = -z \qquad \frac{dz}{dt} = y$$

$$\vec{X}' = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \vec{X} \qquad \begin{vmatrix} 0-\lambda & 0 & 1 \\ 0 & 0-\lambda & -1 \\ 0 & 1 & 0-\lambda \end{vmatrix} = 0$$

$-\lambda(\lambda^2+1) = 0 \quad (\lambda^2 = -1)$

$\lambda = 0 \quad \lambda = 0 \pm i$

$(-\lambda)[\lambda^2+1] - 0(\sim) + 1(0-0) = 0$

$\lambda = 0$

$$\left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

mat

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$K_2 = 0, K_3 = 0$

$(K_1, 0, 0)$ $(1,0,0)$

$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ for $\lambda = 0$

$\vec{X}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{0t}$

$\lambda = 0 + 1i$

$$\left[\begin{array}{ccc|c} 0-(0+i) & 0 & 1 & 0 \\ 0 & 0-(0+i) & -1 & 0 \\ 0 & 1 & 0-(0+i) & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} -i & 0 & 1 & 0 \\ 0 & -i & -1 & 0 \\ 0 & 1 & -i & 0 \end{array}\right]$$

$-iK_1 + K_3 = 0 \rightarrow K_3 = iK_1$

$-iK_2 - K_3 = 0 \rightarrow K_3 = -iK_2$

$K_2 - iK_3 = 0 \rightarrow K_2 = iK_3$

$K_2 = i(iK_1) = i^2 K_1$

$K_2 = -K_1$

$(K_1, -K_1, iK_1)$

$(1, -1, i)$

$\begin{bmatrix} 1 \\ -1 \\ i \end{bmatrix} \quad \vec{B}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \vec{B}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\vec{X}_2 = \left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \cos t - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \sin t\right) e^{0t}$

$\vec{X}_3 = \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cos t + \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \sin t\right) e^{0t}$

$-iK_1 + K_3 \rightarrow \quad \rightarrow iK_1 = K_3$

$-iK_2 - K_3 = 0$

$K_1 = \frac{1}{i} K_3 \frac{i}{i}$

$= \frac{i}{-1} K_3$

$K_3 = -iK_2$

$K_2 = -\frac{1}{i} K_3 \frac{i}{i}$

$K_1 = -iK_3$

$(-iK_3, iK_3, K_3)$

$\vec{X} = C_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{0t} + C_2\left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \cos t - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \sin t\right) e^{0t} + C_3\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cos t + \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \sin t\right) e^{0}$

$$\boxed{\begin{gathered} \vec{X} = C_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + C_2\left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \cos t - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \sin t\right) + C_3\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cos t + \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \sin t\right) \\ \text{— or —} \\ \vec{X} = C_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + C_2 \begin{bmatrix} \cos t \\ -\cos t \\ -\sin t \end{bmatrix} + C_3 \begin{bmatrix} \sin t \\ -\sin t \\ \cos t \end{bmatrix} \end{gathered}}$$

(there are many other correct answers)

#4. Find the general solution of the system:

$$\vec{X}' = \begin{bmatrix} 2 & 5 & 1 \\ -5 & -6 & 4 \\ 0 & 0 & 2 \end{bmatrix} \vec{X}$$

$$\begin{vmatrix} 2-\lambda & 5 & 1 \\ -5 & -6-\lambda & 4 \\ 0 & 0 & 2-\lambda \end{vmatrix}$$

$(2-\lambda)[(-6-\lambda)(2-\lambda)-0] - 5[-5(2-\lambda)-0] + 1[0-0]$

$(2-\lambda)(-6-\lambda)(2-\lambda) + 25(2-\lambda) = 0$

$(2-\lambda)[\lambda^2+4\lambda+13] = 0 \qquad \lambda = \frac{-4 \pm \sqrt{16-4(13)}}{2}$

$\underline{\lambda = 2}$

$\lambda = \frac{-4 \pm 6i}{2} = \underline{-2 \pm 3i}$

$\underline{\lambda = 2}$

$$\left[\begin{array}{ccc|c} 2-2 & 5 & 1 & 0 \\ -5 & -6-2 & 4 & 0 \\ 0 & 0 & 2-2 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 0 & 5 & 1 & 0 \\ -5 & -8 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

rref

$$\left[\begin{array}{ccc|c} 1 & 0 & -28/25 & 0 \\ 0 & 1 & 1/5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$k_1 - \frac{28}{25}k_3 = 0 \rightarrow k_1 = \frac{28}{25}k_3$

$k_2 + \frac{1}{5}k_3 = 0 \rightarrow k_2 = -\frac{1}{5}k_3$

$(\frac{28}{25}k_3, -\frac{1}{5}k_3, k_3)$

$(28, -5, 25) \quad \begin{bmatrix} 28 \\ -5 \\ 25 \end{bmatrix}$

$\underline{\lambda = -2+3i}$

$$\left[\begin{array}{ccc|c} 2-(-2+3i) & 5 & 1 & 0 \\ -5 & -6-(-2+3i) & 4 & 0 \\ 0 & 0 & 2-(-2+3i) & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 4-3i & 5 & 1 & 0 \\ -5 & -4-3i & 4 & 0 \\ 0 & 0 & 4-3i & 0 \end{array}\right]$$

$(4-3i)k_1 + 5k_2 + k_3 = 0$

$-5k_1 + (-4-3i)k_2 + 4k_3 = 0$

$(4-3i)k_3 = 0 \rightarrow \underline{k_3 = 0}$

$(4-3i)k_1 + 5k_2 = 0 \rightarrow k_2 = \frac{-(4-3i)}{5}k_1$

$(k_1, \frac{-4+3i}{5}k_1, 0)$

$(5, -4+3i, 0) \quad \begin{bmatrix} 5 \\ -4+3i \\ 0 \end{bmatrix} \quad \vec{B}_1 = \begin{bmatrix} 5 \\ -4 \\ 0 \end{bmatrix} \quad \vec{B}_2 = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$

$\lambda = 2$

$\vec{X}_1 = \begin{bmatrix} 28 \\ -5 \\ 25 \end{bmatrix} e^{2t}$

$\vec{X}_2 = \left(\begin{bmatrix} 5 \\ -4 \\ 0 \end{bmatrix}\cos 3t - \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}\sin 3t\right)e^{-2t}$

$\vec{X}_3 = \left(\begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}\cos 3t + \begin{bmatrix} 5 \\ -4 \\ 0 \end{bmatrix}\sin 3t\right)e^{-2t}$

$$\boxed{\vec{X} = C_1\begin{bmatrix} 28 \\ -5 \\ 25 \end{bmatrix}e^{2t} + C_2\left(\begin{bmatrix} 5 \\ -4 \\ 0 \end{bmatrix}\cos 3t - \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}\sin 3t\right)e^{-2t} + C_3\left(\begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}\cos 3t + \begin{bmatrix} 5 \\ -4 \\ 0 \end{bmatrix}\sin 3t\right)e^{-2t}}$$

—or—

$$\boxed{\vec{X} = C_1\begin{bmatrix} 28 \\ -5 \\ 25 \end{bmatrix}e^{2t} + C_2\begin{bmatrix} 5\cos 3t \\ -4\cos 3t - 3\sin 3t \\ 0 \end{bmatrix}e^{-2t} + C_3\begin{bmatrix} 5\sin 3t \\ 3\cos 3t - 4\sin 3t \\ 0 \end{bmatrix}e^{-2t}}$$

also:

$$\left(\vec{X} = C_1\begin{bmatrix} 28 \\ -5 \\ 25 \end{bmatrix}e^{2t} + C_2\begin{bmatrix} 4\cos 3t - 3\sin 3t \\ -5\cos 3t \\ 0 \end{bmatrix}e^{-2t} + C_3\begin{bmatrix} 4\sin 3t + 3\cos 3t \\ -5\sin 3t \\ 0 \end{bmatrix}e^{-2t}\right)$$

#5. Solve the initial-value problem:

$\vec{X}' = \begin{bmatrix} 6 & -1 \\ 5 & 4 \end{bmatrix} \vec{X}, \qquad \vec{X}(0) = \begin{bmatrix} -2 \\ 8 \end{bmatrix}$

$\begin{vmatrix} 6-\lambda & -1 \\ 5 & 4-\lambda \end{vmatrix} = 0 \quad (6-\lambda)(4-\lambda)+5=0$

$\lambda^2 - 10\lambda + 29 = 0$

$\lambda = \frac{-10 \pm \sqrt{100-4(29)}}{2} = \frac{10 \pm 4i}{2} = 5 \pm 2i$

$\underline{\lambda = 5+2i}$

$\left[\begin{array}{cc|c} 6-(5+2i) & -1 & 0 \\ 5 & 4-(5+2i) & 0 \end{array}\right]$

$\left[\begin{array}{cc|c} 1-2i & -1 & 0 \\ 5 & -1+2i & 0 \end{array}\right]$

$(1-2i)k_1 - k_2 = 0 \rightarrow k_2 = (1-2i)k_1$

$(k_1, (1-2i)k_1) \quad \begin{bmatrix} 1 \\ 1-2i \end{bmatrix} \quad \vec{B}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \vec{B}_2 = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$

$(1, 1-2i)$

$\vec{X} = C_1\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\cos 2t - \begin{bmatrix} 0 \\ -2 \end{bmatrix}\sin 2t\right)e^{5t} + C_2\left(\begin{bmatrix} 0 \\ -2 \end{bmatrix}\cos 2t + \begin{bmatrix} 1 \\ 1 \end{bmatrix}\sin 2t\right)e^{5t}$

$x = C_1 \cos 2t\, e^{5t} + C_2 \sin 2t\, e^{5t}$

$-2 = C_1(1)e^0 + C_2(0)e^0$

$-2 = C_1$

$y = C_1(\cos 2t + 2\sin 2t)e^{5t} + C_2(-2\cos 2t + \sin 2t)e^{5t}$

$8 = C_1(1+2(0))e^0 + C_2(-2(1)+0)e^0$

$8 = C_1 - 2C_2$

$8 = (-2) - 2C_2$

$10 = -2C_2$

$C_2 = -5$

$$\boxed{\vec{X} = -2\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\cos 2t - \begin{bmatrix} 0 \\ -2 \end{bmatrix}\sin 2t\right)e^{5t} - 5\left(\begin{bmatrix} 0 \\ -2 \end{bmatrix}\cos 2t + \begin{bmatrix} 1 \\ 1 \end{bmatrix}\sin 2t\right)e^{5t}}$$

or

$\vec{X} = \begin{bmatrix} -2\cos 2t - 5\sin 2t \\ -2\cos 2t + 2\sin 2t + 10\cos 2t - 5\sin 2t \end{bmatrix} e^{5t}$

$$\boxed{\vec{X} = \begin{bmatrix} -2\cos 2t - 5\sin 2t \\ 8\cos 2t - 3\sin 2t \end{bmatrix} e^{5t}}$$

## 8.3 day 1

#1. Use the method of undermined coefficients to solve the system:

$$\frac{dx}{dt} = 2x + 3y - 7$$

$$\frac{dy}{dt} = -x - 2y + 5$$

$$\vec{X}' = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}\vec{X} + \begin{bmatrix} -7 \\ 5 \end{bmatrix}$$

$$\begin{vmatrix} 2-\lambda & 3 \\ -1 & -2-\lambda \end{vmatrix} = 0 \qquad (2-\lambda)(-2-\lambda) + 3 = 0$$

$$\lambda^2 - 1 = 0$$

$$\underline{\lambda = 1,\ \lambda = -1}$$

$\underline{\lambda = 1}$

$$\left[\begin{array}{cc|c} 2-1 & 3 & 0 \\ -1 & -2-1 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 3 & 0 \\ -1 & -3 & 0 \end{array}\right]$$

rref

$$\left[\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$k_1 + 3k_2 = 0 \rightarrow k_1 = -3k_2$

$(-3k_2, k_2)\ (-3, 1)\ \begin{bmatrix} -3 \\ 1 \end{bmatrix}$

$\underline{\lambda = -1}$

$$\left[\begin{array}{cc|c} 2+1 & 3 & 0 \\ -1 & -2+1 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 3 & 3 & 0 \\ -1 & -1 & 0 \end{array}\right]$$

rref

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$k_1 + k_2 = 0 \rightarrow k_1 = -k_2$

$(-k_2, k_2)\ (-1, 1)\ \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$\vec{X}_c = c_1\begin{bmatrix} -3 \\ 1 \end{bmatrix}e^{t} + c_2\begin{bmatrix} -1 \\ 1 \end{bmatrix}e^{-t}$$

for $\begin{bmatrix} -7 \\ 5 \end{bmatrix}$, $\vec{X}_p = \begin{bmatrix} A \\ B \end{bmatrix}$ $\quad \vec{X}' = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} + \begin{bmatrix} -7 \\ 5 \end{bmatrix}$$

$2A + 3B - 7 = 0$

$-1A - 2B + 5 = 0$

$$\begin{cases} 2A + 3B = 7 \\ -A - 2B = -5 \end{cases}$$

$$\left[\begin{array}{cc|c} 2 & 3 & 7 \\ -1 & -2 & -5 \end{array}\right] \text{rref} \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 3 \end{array}\right] \begin{matrix} =A \\ =B \end{matrix}$$

$$\vec{X}_p = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$\boxed{\vec{X} = c_1\begin{bmatrix} -3 \\ 1 \end{bmatrix}e^{t} + c_2\begin{bmatrix} -1 \\ 1 \end{bmatrix}e^{-t} + \begin{bmatrix} -1 \\ 3 \end{bmatrix}}$$

#2. Use the method of undermined coefficients to solve the system:

$$\vec{X}' = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}\vec{X} + \begin{bmatrix} -2t^2 \\ t+5 \end{bmatrix}$$

$$\begin{vmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{vmatrix} = 0$$

$(1-\lambda)(1-\lambda) - 9 = 0$

$\lambda^2 - 2\lambda - 8 = 0$

$(\lambda - 4)(\lambda + 2) = 0$

$\lambda = 4,\ \lambda = -2$

$\underline{\lambda = -2}$

$$\left[\begin{array}{cc|c} 1+2 & 3 & 0 \\ 3 & 1+2 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 3 & 3 & 0 \\ 3 & 3 & 0 \end{array}\right]$$

rref

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$K_1 + K_2 = 0 \to K_1 = -K_2$

$(-K_2, K_2)\ (-1, 1)$

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$\underline{\lambda = 4}$

$$\left[\begin{array}{cc|c} 1-4 & 3 & 0 \\ 3 & 1-4 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} -3 & 3 & 0 \\ 3 & -3 & 0 \end{array}\right]$$

rref

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$K_1 - K_2 = 0 \to K_1 = K_2$

$(K_2, K_2)\ (1, 1)$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\vec{X_c} = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t}$$

for $\begin{bmatrix} -2t^2 \\ t+5 \end{bmatrix}$ $\quad \vec{X_p} = \begin{bmatrix} At^2 + Bt + C \\ Dt^2 + Et + F \end{bmatrix} \quad \vec{X}' = \begin{bmatrix} 2At + B \\ 2Dt + E \end{bmatrix}$

$$\begin{bmatrix} 2At + B \\ 2Dt + E \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} At^2 + Bt + C \\ Dt^2 + Et + F \end{bmatrix} + \begin{bmatrix} -2t^2 \\ t+5 \end{bmatrix}$$

$$\begin{cases} At^2 + Bt + C + 3Dt^2 + 3Et + 3F - 2t^2 = 2At + B \\ 3At^2 + 3Bt + 3C + Dt^2 + Et + F + t + 5 = 2Dt + E \end{cases}$$

$(A + 3D - 2)t^2 + (B + 3E)t + (C + 3F) = (0)t^2 + (2A)t + (B)$

$$\begin{cases} (A + 3D - 2)t^2 + (B + 3E)t + (C+3F) \\ (3A + D)t^2 + (3B + E + 1)t + (3C + F + 5) = (0)t^2 + (2D)t + (E) \end{cases}$$

$$\begin{cases} A + 3D - 2 = 0 \\ B + 3E = 2A \\ C + 3F = B \\ 3A + D = 0 \\ 3B + E + 1 = 2D \\ 3C + F + 5 = E \end{cases}$$

$$\left[\begin{array}{l} 1A + 0B + 0C + 3D + 0E + 0F = 2 \\ 1A + 0B + 0C + 3D + 0E + 0F = 0 \\ -2A + 1B + 0C + 0D + 3E + 0F = 0 \\ 0A - 1B + 1C + 0D + 0E + 3F = 0 \\ 3A + 0B + 0C + 1D + 0E + 0F = 0 \\ 0A + 3B + 0C - 2D + 1E + 0F = -1 \\ 0A + 0B + 3C + 0D - 1E + 1F = -5 \end{array}\right]$$

rref

$A = -1/4$

$B = 1/4$

$C = -2$

$D = 3/4$

$E = -1/4$

$F = 3/4$

$$\boxed{\vec{X} = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} + \begin{bmatrix} -\frac{1}{4}t^2 + \frac{1}{4}t - 2 \\ \frac{3}{4}t^2 - \frac{1}{4}t + \frac{3}{4} \end{bmatrix}}$$

#3. Use the method of undermined coefficients to solve the system:

$$\vec{X}' = \begin{bmatrix} 4 & 1/3 \\ 9 & 6 \end{bmatrix} \vec{X} + \begin{bmatrix} -3 \\ 10 \end{bmatrix} e^t$$

$$\begin{vmatrix} 4-\lambda & 1/3 \\ 9 & 6-\lambda \end{vmatrix} = 0 \qquad (4-\lambda)(6-\lambda) - 3 = 0$$

$$\lambda^2 - 10\lambda + 21 = 0$$

$$(\lambda-3)(\lambda-7) = 0$$

$$\lambda = 3, \ \lambda = 7$$

$\lambda = 3$

$$\left[\begin{array}{cc|c} 4-3 & 1/3 & 0 \\ 9 & 6-3 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 1/3 & 0 \\ 9 & 3 & 0 \end{array}\right]$$

rref

$$\left[\begin{array}{cc|c} 1 & 1/3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$k_1 + 1/3 k_2 = 0 \rightarrow k_1 = -1/3 k_2$

$(-1/3 k_2, k_2)$ $(-1, 3)$

$$\begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$\lambda = 7$

$$\left[\begin{array}{cc|c} 4-7 & 1/3 & 0 \\ 9 & 6-7 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} -3 & 1/3 & 0 \\ 9 & -1 & 0 \end{array}\right]$$

rref

$$\left[\begin{array}{cc|c} 1 & -1/9 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$k_1 - 1/9 k_2 = 0$

$k_1 = 1/9 k_2$

$(1/9 k_2, k_2)$

$(1, 9)$ $\begin{bmatrix} 1 \\ 9 \end{bmatrix}$

$$\vec{X_c} = C_1 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{3t} + C_2 \begin{bmatrix} 1 \\ 9 \end{bmatrix} e^{7t}$$

for $\begin{bmatrix} -3e^t \\ 10e^t \end{bmatrix}$ $\vec{X_p} = \begin{bmatrix} Ae^t \\ Be^t \end{bmatrix}$ $X' = \begin{bmatrix} Ae^t \\ Be^t \end{bmatrix}$

$$\begin{bmatrix} Ae^t \\ Be^t \end{bmatrix} = \begin{bmatrix} 4 & 1/3 \\ 9 & 6 \end{bmatrix} \begin{bmatrix} Ae^t \\ Be^t \end{bmatrix} + \begin{bmatrix} -3e^t \\ 10e^t \end{bmatrix}$$

$$\begin{cases} 4Ae^t + \frac{1}{3}Be^t - 3e^t = Ae^t \\ 9Ae^t + 6Be^t + 10e^t = Be^t \end{cases}$$

$$\begin{cases} (4A + \frac{1}{3}B - 3)e^t = (A)e^t \\ (9A + 6B + 10)e^t = (B)e^t \end{cases}$$

$$\begin{cases} 4A + \frac{1}{3}B - 3 = A \\ 9A + 6B + 10 = B \end{cases}$$

$$\begin{bmatrix} 3A + \frac{1}{3}B = 3 \\ 9A + 5B = -10 \end{bmatrix} \text{rref} \quad A = 55/36 \quad B = -19/4$$

$$\vec{X_p} = \begin{bmatrix} 55/36\, e^t \\ -19/4\, e^t \end{bmatrix}$$

$$\boxed{\vec{X} = C_1 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{3t} + C_2 \begin{bmatrix} 1 \\ 9 \end{bmatrix} e^{7t} + \begin{bmatrix} 55/36 \\ -19/4 \end{bmatrix} e^t}$$

#4. Use the method of undermined coefficients to solve the system:

$$\vec{X}' = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 5 \end{bmatrix} \vec{X} + \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} e^{4t}$$

$$\begin{array}{c} + \\ - \\ + \end{array}\begin{vmatrix} 1-\lambda & 1 & 1 \\ 0 & 2-\lambda & 3 \\ 0 & 0 & 5-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\left[(2-\lambda)(5-\lambda) - 0\right] - 0(\sim) + 0(\sim)$$

$$(1-\lambda)(2-\lambda)(5-\lambda) = 0$$

$$\lambda = 1,\ \lambda = 2,\ \lambda = 5$$

$\lambda = 1$

$$\left[\begin{array}{ccc|c} 1-1 & 1 & 1 & 0 \\ 0 & 2-1 & 3 & 0 \\ 0 & 0 & 5-1 & 0 \end{array}\right] \quad \left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 4 & 0 \end{array}\right]$$

rref

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$K_2 = 0,\ K_3 = 0$

$(K_1, 0, 0)$ $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$(1, 0, 0)$

$\lambda = 2$

$$\left[\begin{array}{ccc|c} 1-2 & 1 & 1 & 0 \\ 0 & 2-2 & 3 & 0 \\ 0 & 0 & 5-2 & 0 \end{array}\right] \quad \left[\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 3 & 0 \end{array}\right]$$

rref

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$K_1 - K_2 = 0 \to K_1 = K_2$

$K_3 = 0$

$(K_2, K_2, 0)$ $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$(1, 1, 0)$

$\lambda = 5$

$$\left[\begin{array}{ccc|c} 1-5 & 1 & 1 & 0 \\ 0 & 2-5 & 3 & 0 \\ 0 & 0 & 5-5 & 0 \end{array}\right] \quad \left[\begin{array}{ccc|c} -4 & 1 & 1 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

rref

$$\left[\begin{array}{ccc|c} 1 & 0 & -1/2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$K_1 - \frac{1}{2}K_3 = 0 \to K_1 = \frac{1}{2}K_3$

$K_2 - K_3 = 0 \quad K_2 = K_3$

$(\frac{1}{2}K_3, K_3, K_3)$ $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

$(1, 2, 2)$

$$\vec{X_c} = C_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}e^{t} + C_2\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}e^{2t} + C_3\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}e^{5t}$$

for $\begin{bmatrix} e^{4t} \\ -e^{4t} \\ 2e^{4t} \end{bmatrix}$ $\quad \vec{X_p} = \begin{bmatrix} Ae^{4t} \\ Be^{4t} \\ Ce^{4t} \end{bmatrix}$ $\quad \vec{X}' = \begin{bmatrix} 4Ae^{4t} \\ 4Be^{4t} \\ 4Ce^{4t} \end{bmatrix}$

$$\begin{bmatrix} 4Ae^{4t} \\ 4Be^{4t} \\ 4Ce^{4t} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} Ae^{4t} \\ Be^{4t} \\ Ce^{4t} \end{bmatrix} + \begin{bmatrix} e^{4t} \\ -e^{4t} \\ 2e^{4t} \end{bmatrix}$$

$$\begin{cases} Ae^{4t} + Be^{4t} + Ce^{4t} + e^{4t} = 4Ae^{4t} \\ 2Be^{4t} + 3Ce^{4t} - e^{4t} = 4Be^{4t} \\ 5Ce^{4t} + 2e^{4t} = 4Ce^{4t} \end{cases}$$

$$\begin{cases} A + B + C + 1 = 4A \\ 2B + 3C - 1 = 4B \\ 5C + 2 = 4C \end{cases}$$

$$\begin{bmatrix} -3A + 1B + 1C = -1 \\ 0A - 2B + 3C = 1 \\ 0A + 0B + C = -2 \end{bmatrix}$$

rref

$A = -3/2$

$B = -7/2$

$C = -2$

$$\boxed{\vec{X} = C_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}e^{t} + C_2\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}e^{2t} + C_3\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}e^{5t} + \begin{bmatrix} -3/2 \\ -7/2 \\ -2 \end{bmatrix}e^{4t}}$$

#5. Use the method of undermined coefficients to solve the initial-value problem:

$$\vec{X}' = \begin{bmatrix} -1 & -2 \\ 3 & 4 \end{bmatrix}\vec{X} + \begin{bmatrix} 3 \\ 3 \end{bmatrix}, \qquad \vec{X}(0) = \begin{bmatrix} -4 \\ 5 \end{bmatrix}$$

$$\begin{vmatrix} -1-\lambda & -2 \\ 3 & 4-\lambda \end{vmatrix} = 0 \qquad (-1-\lambda)(4-\lambda)+6=0$$

$$\lambda^2-3\lambda+2=0$$

$$(\lambda-1)(\lambda-2)=0$$

$$\lambda=1,\ \lambda=2$$

$\lambda=1$

$$\left[\begin{array}{cc|c} -1-1 & -2 & 0 \\ 3 & 4-1 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} -2 & -2 & 0 \\ 3 & 3 & 0 \end{array}\right]$$

rref

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$K_1+K_2=0 \rightarrow K_1=-K_2$

$(-K_2, K_2)$ $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$(-1, 1)$

$\lambda=2$

$$\left[\begin{array}{cc|c} -1-2 & -2 & 0 \\ 3 & 4-2 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} -3 & -2 & 0 \\ 3 & 2 & 0 \end{array}\right]$$

rref

$$\left[\begin{array}{cc|c} 1 & 2/3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$K_1+2/3K_2=0,\ K_1=-2/3K_2$

$(-2/3K_2, K_2)$ $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$

$(-2, 3)$

$$\vec{X}_c = C_1\begin{bmatrix} -1 \\ 1 \end{bmatrix}e^{t} + C_2\begin{bmatrix} -2 \\ 3 \end{bmatrix}e^{2t}$$

for $\begin{bmatrix} 3 \\ 3 \end{bmatrix}$ $\vec{X}_p = \begin{bmatrix} A \\ B \end{bmatrix}$ $\vec{X}'=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} + \begin{bmatrix} 3 \\ 3 \end{bmatrix} \quad \begin{cases} -A-2B+3=0 \\ 3A+4B+3=0 \end{cases} \quad \begin{bmatrix} -A-2B=-3 \\ 3A+4B=-3 \end{bmatrix} \text{rref} \quad \begin{matrix} A=-9 \\ B=6 \end{matrix} \quad \vec{X}_p=\begin{bmatrix} -9 \\ 6 \end{bmatrix}$$

$$\vec{X} = C_1\begin{bmatrix} -1 \\ 1 \end{bmatrix}e^{t} + C_2\begin{bmatrix} -2 \\ 3 \end{bmatrix}e^{2t} + \begin{bmatrix} -9 \\ 6 \end{bmatrix}$$

now $\vec{X}(0)=\begin{bmatrix} -4 \\ 5 \end{bmatrix}$

$x=-C_1e^{t}-2C_2e^{2t}-9$

$-4=-C_1(1)-2C_2(1)-9$

$-C_1-2C_2=5$

$y=C_1e^{t}+3C_2e^{2t}+6$

$5=C_1(1)+3C_2(1)+6$

$C_1+3C_2=-1$

$$\left[\begin{array}{cc|c} -1 & -2 & 5 \\ 1 & 3 & -1 \end{array}\right] \text{rref} \left[\begin{array}{cc|c} 1 & 0 & -13 \\ 0 & 1 & 4 \end{array}\right] \begin{matrix} =C_1 \\ =C_2 \end{matrix}$$

$$\boxed{\vec{X} = -13\begin{bmatrix} -1 \\ 1 \end{bmatrix}e^{t} + 4\begin{bmatrix} -2 \\ 3 \end{bmatrix}e^{2t} + \begin{bmatrix} -9 \\ 6 \end{bmatrix} \quad \text{or} \quad \vec{X} = \begin{bmatrix} 13e^{t}-8e^{2t}-9 \\ -13e^{t}+12e^{2t}+6 \end{bmatrix}}$$

## 8.3 day 2

#1. Use variation of parameters to solve the system:

$$\vec{X}' = \begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix} \vec{X} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t$$

$\begin{vmatrix} 0-\lambda & 2 \\ -1 & 3-\lambda \end{vmatrix} = 0$ $\quad -\lambda(3-\lambda)+2=0$ $\quad \lambda^2-3\lambda+2=0$ $\quad (\lambda-1)(\lambda-2)=0$ $\quad \underline{\lambda=1 \quad \lambda=2}$

$X_c = c_1\begin{bmatrix}2\\1\end{bmatrix}e^t + c_2\begin{bmatrix}1\\1\end{bmatrix}e^{2t}$

$\underline{\lambda=1}$

$\left[\begin{array}{cc|c} 0-1 & 2 & 0 \\ -1 & 3-1 & 0 \end{array}\right]$

$\left[\begin{array}{cc|c} -1 & 2 & 0 \\ -1 & 2 & 0 \end{array}\right]$ rref $\left[\begin{array}{cc|c} -1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $\quad \begin{bmatrix}2\\1\end{bmatrix}$

$-k_1+2k_2=0 \quad k_1=2k_2 \quad (2k_2, k_2)$

$(2,1)$

$\underline{\lambda=2}$

$\left[\begin{array}{cc|c} 0-2 & 2 & 0 \\ -1 & 3-2 & 0 \end{array}\right]$

$\left[\begin{array}{cc|c} -2 & 2 & 0 \\ -1 & 1 & 0 \end{array}\right]$ rref $\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$k_1-k_2=0, \; k_1=k_2 \quad (1,1) \quad \begin{bmatrix}1\\1\end{bmatrix}$

$\vec{\Phi} = \begin{bmatrix} 2e^t & e^{2t} \\ e^t & e^{2t} \end{bmatrix} \quad \begin{vmatrix} 2e^t & e^{2t} \\ e^t & e^{2t} \end{vmatrix} = 2e^{3t}-e^{3t} = e^{3t}$

$\Phi^{-1} = \frac{1}{e^{3t}}\begin{bmatrix} e^{2t} & -e^{2t} \\ -e^t & 2e^t \end{bmatrix} = \begin{bmatrix} e^{-t} & -e^{-t} \\ -e^{-2t} & 2e^{-2t} \end{bmatrix}$

$\vec{X}_p = \Phi \int \Phi^{-1} F\,dt = \begin{bmatrix} 2e^t & e^{2t} \\ e^t & e^{2t} \end{bmatrix} \int \begin{bmatrix} e^{-t} & -e^{-t} \\ -e^{-2t} & 2e^{-2t} \end{bmatrix}\begin{bmatrix} e^t \\ -e^t \end{bmatrix} dt$

$\int \begin{bmatrix} 1+1 \\ -e^{-t}-2e^{-t} \end{bmatrix} dt$

$\int \begin{bmatrix} 2 \\ -3e^{-t} \end{bmatrix} dt$

$X_p = \begin{bmatrix} 2e^t & e^{2t} \\ e^t & e^{2t} \end{bmatrix}\begin{bmatrix} 2t \\ 3e^{-t} \end{bmatrix} = \begin{bmatrix} 4te^t+3e^t \\ 2te^t+3e^t \end{bmatrix}$

$$\boxed{\vec{X} = c_1\begin{bmatrix}2\\1\end{bmatrix}e^t + c_2\begin{bmatrix}1\\1\end{bmatrix}e^{2t} + \begin{bmatrix}4\\2\end{bmatrix}te^t + \begin{bmatrix}3\\3\end{bmatrix}e^t}$$

#2. Use variation of parameters to solve the system:

$$\vec{X}' = \begin{bmatrix} 1 & 8 \\ 1 & -1 \end{bmatrix} \vec{X} + \begin{bmatrix} 12 \\ 12 \end{bmatrix} t$$

$\begin{vmatrix} 1-\lambda & 8 \\ 1 & -1-\lambda \end{vmatrix} = 0$ $\quad (1-\lambda)(-1-\lambda) - 8 = 0$ $\quad \lambda^2 - 9 = 0$ $\quad (\lambda-3)(\lambda+3) = 0$ $\quad \underline{\lambda = 3, \lambda = -3}$

$\underline{\lambda = 3}$

$\left[\begin{array}{cc|c} 1-3 & 8 & 0 \\ 1 & -1-3 & 0 \end{array}\right] \left[\begin{array}{cc|c} -2 & 8 & 0 \\ 1 & -4 & 0 \end{array}\right]$

rref

$\left[\begin{array}{cc|c} 1 & -4 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $\quad k_1 - 4k_2 = 0$, $k_1 = 4k_2$

$(4k_2, k_2)\ (4,1)\ \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

$\underline{\lambda = -3}$

$\left[\begin{array}{cc|c} 1+3 & 8 & 0 \\ 1 & -1+3 & 0 \end{array}\right] \left[\begin{array}{cc|c} 4 & 8 & 0 \\ 1 & 2 & 0 \end{array}\right]$

rref

$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $\quad k_1 + 2k_2 = 0$, $k_1 = -2k_2$

$(-2k_2, k_2)\ (-2,1)\ \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$$\vec{X}_c = C_1 \begin{bmatrix} 4 \\ 1 \end{bmatrix} e^{3t} + C_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{-3t}$$

$\Phi = \begin{bmatrix} 4e^{3t} & -2e^{-3t} \\ e^{3t} & e^{-3t} \end{bmatrix}$ $\quad \begin{vmatrix} 4e^{3t} & -2e^{-3t} \\ e^{3t} & e^{-3t} \end{vmatrix} = 4 + 2 = 6$

$\Phi^{-1} = \frac{1}{6} \begin{bmatrix} e^{-3t} & 2e^{-3t} \\ -e^{3t} & 4e^{3t} \end{bmatrix} = \begin{bmatrix} \frac{1}{6}e^{-3t} & \frac{1}{3}e^{-3t} \\ -\frac{1}{6}e^{3t} & \frac{2}{3}e^{3t} \end{bmatrix}$

$\vec{X}_p = \Phi \int \Phi^{-1} F\, dt = \begin{bmatrix} 4e^{3t} & -2e^{-3t} \\ e^{3t} & e^{-3t} \end{bmatrix} \int \begin{bmatrix} 1/6\, e^{-3t} & 1/3\, e^{-3t} \\ -1/6\, e^{3t} & 2/3\, e^{3t} \end{bmatrix} \begin{bmatrix} 12t \\ 12t \end{bmatrix} dt$

$\int \begin{bmatrix} 2te^{-3t} + 4te^{-3t} \\ -2te^{3t} + 8te^{3t} \end{bmatrix} dt$

$\begin{bmatrix} -\frac{2}{3}te^{-3t} - \frac{2}{9}e^{-3t} - \frac{4}{3}te^{-3t} - \frac{4}{9}e^{-3t} \\ -\frac{2}{3}te^{3t} + \frac{2}{9}e^{3t} + \frac{8}{3}te^{3t} - \frac{8}{9}e^{3t} \end{bmatrix}$

$\int te^{-3t}\, dt \quad u = t \quad dv = e^{-3t} dt$
$du = dt \quad v = -\frac{1}{3}e^{-3t}$
$-\frac{1}{3}te^{-3t} + \frac{1}{3}\int e^{-3t} dt$
$-\frac{1}{3}te^{-3t} - \frac{1}{9}e^{-3t}$

$\int te^{3t}\, dt \quad u = t \quad dv = e^{3t} dt$
$du = dt \quad v = \frac{1}{3}e^{3t}$
$\frac{1}{3}te^{3t} - \frac{1}{3}\int e^{3t} dt$
$\frac{1}{3}te^{3t} - \frac{1}{9}e^{3t}$

$\vec{X}_p = \begin{bmatrix} 4e^{3t} & -2e^{-3t} \\ e^{3t} & e^{-3t} \end{bmatrix} \begin{bmatrix} -2te^{-3t} - \frac{2}{3}e^{-3t} \\ 2te^{3t} - \frac{2}{3}e^{3t} \end{bmatrix}$

$\vec{X}_p = \begin{bmatrix} -8t - \frac{8}{3} - 4t + 4/3 \\ -2t - \frac{2}{3} + 2t - 2/3 \end{bmatrix} = \begin{bmatrix} -12t - 4/3 \\ -4/3 \end{bmatrix}$

$$\boxed{\vec{X} = C\begin{bmatrix} 4 \\ 1 \end{bmatrix} e^{3t} + C_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{-3t} + \begin{bmatrix} -12 \\ 0 \end{bmatrix} t + \begin{bmatrix} -4/3 \\ -4/3 \end{bmatrix}}$$

#3. Use variation of parameters to solve the system:

$$\vec{X}' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\vec{X} + \begin{bmatrix} \sec t \\ 0 \end{bmatrix}$$

$\begin{vmatrix} 0-\lambda & -1 \\ 1 & 0-\lambda \end{vmatrix} = 0 \qquad (-\lambda)(-\lambda)+1=0 \qquad \lambda^2+1=0 \qquad \lambda = 0 \pm i$

$\lambda = 0+i$

$\left[\begin{array}{cc|c} -i & -1 & 0 \\ 1 & -i & 0 \end{array}\right] \quad \begin{bmatrix} 1 \\ -i \end{bmatrix} \quad \vec{B_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \vec{B_2} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

$\vec{X_c} = c_1\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\cos t - \begin{bmatrix} 0 \\ -1 \end{bmatrix}\sin t\right)\cancel{e^{0t}} + c_2\left(\begin{bmatrix} 0 \\ -1 \end{bmatrix}\cos t + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\sin t\right)\cancel{e^{0t}}$

$-ik_1 - k_2 = 0$

$k_2 = -ik_1 \quad (k_1, -ik_1)$

$\Phi = \begin{bmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{bmatrix} \qquad \begin{vmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{vmatrix} = -\cos^2 t - \sin^2 t = -1$

$\Phi^{-1} = \frac{1}{-1}\begin{bmatrix} -\cos t & -\sin t \\ -\sin t & \cos t \end{bmatrix} = \begin{bmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{bmatrix}$

$\vec{X_p} = \Phi \int \Phi^{-1} F\,dt = \begin{bmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{bmatrix} \int \begin{bmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{bmatrix}\begin{bmatrix} \sec t \\ 0 \end{bmatrix} dt$

$\int \begin{bmatrix} \cos t \sec t \\ \sin t \sec t \end{bmatrix} dt$

$\int \cos t \sec t\, dt$

$\int 1\, dt$

$t$

$\int \sin t \sec t\, dt$

$\int \frac{\sin t}{\cos t}\, dt \qquad u = \cos t \quad du = -\sin t\, dt$

$-\int \frac{1}{u}\, du = -\ln|\cos t|$

$\begin{bmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{bmatrix}\begin{bmatrix} t \\ -\ln|\cos t| \end{bmatrix}$

$\vec{X_p} = \begin{bmatrix} t\cos t - \sin t \ln|\cos t| \\ t\sin t + \cos t \ln|\cos t| \end{bmatrix}$

$$\boxed{\vec{X} = c_1\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\cos t - \begin{bmatrix} 0 \\ -1 \end{bmatrix}\sin t\right) + c_2\left(\begin{bmatrix} 0 \\ -1 \end{bmatrix}\cos t + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\sin t\right) + \begin{bmatrix} t\cos t - \sin t \ln|\cos t| \\ t\sin t + \cos t \ln|\cos t| \end{bmatrix}}$$

#4. Use variation of parameters to solve the system:

$$\vec{X}' = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \vec{X} + \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} e^t$$

$$\begin{vmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} = 0 \quad (1-\lambda)(1-\lambda)+1=0 \quad \lambda^2 - 2\lambda + 2 = 0$$

$$\lambda = \frac{2 \pm \sqrt{4-4(2)}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$$

$\lambda = 1+i$

$$\left[\begin{array}{cc|c} 1-(1+i) & -1 & 0 \\ 1 & 1-(1+i) & 0 \end{array}\right]$$

$$\left(\begin{array}{cc|c} -i & -1 & 0 \\ 1 & -i & 0 \end{array}\right) \quad \begin{array}{l} k_1 - ik_2 = 0 \\ k_1 = ik_2 \\ (ik_2, k_2) \end{array} \quad \begin{bmatrix} i \\ 1 \end{bmatrix}$$

$$\vec{B_1} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\vec{X_c} = C_1\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\cos t - \begin{bmatrix} 1 \\ 0 \end{bmatrix}\sin t\right)e^t + C_2\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\cos t + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\sin t\right)e^t$$

$$\vec{X_c} = C_1\begin{bmatrix} -e^t\sin t \\ e^t\cos t \end{bmatrix} + C_2\begin{bmatrix} e^t\cos t \\ e^t\sin t \end{bmatrix}$$

$$\Phi = \begin{bmatrix} -e^t\sin t & e^t\cos t \\ e^t\cos t & e^t\sin t \end{bmatrix} \quad \begin{vmatrix} -e^t\sin t & e^t\cos t \\ e^t\cos t & e^t\sin t \end{vmatrix} = -e^{2t}\sin^2 t - e^{2t}\cos^2 t = -e^{2t}(\sin^2 t + \cos^2 t) = -e^{2t}$$

$$\Phi^{-1} = \frac{1}{-e^{2t}}\begin{bmatrix} e^t\sin t & -e^t\cos t \\ -e^t\cos t & -e^t\sin t \end{bmatrix} = \begin{bmatrix} -e^{-t}\sin t & e^{-t}\cos t \\ e^{-t}\cos t & e^{-t}\sin t \end{bmatrix}$$

$$\vec{X_p} = \Phi \int \Phi^{-1} F\,dt = \begin{bmatrix} -e^t\sin t & e^t\cos t \\ e^t\cos t & e^t\sin t \end{bmatrix} \int \begin{bmatrix} -e^{-t}\sin t & e^{-t}\cos t \\ e^{-t}\cos t & e^{-t}\sin t \end{bmatrix}\begin{bmatrix} e^t\cos t \\ e^t\sin t \end{bmatrix} dt$$

$$\int \begin{bmatrix} -\sin t\cos t + \sin t\cos t \\ \cos^2 t + \sin^2 t \end{bmatrix} dt$$

$$\int \begin{bmatrix} 0 \\ 1 \end{bmatrix} dt$$

$$\vec{X_p} = \begin{bmatrix} -e^t\sin t & e^t\cos t \\ e^t\cos t & e^t\sin t \end{bmatrix}\begin{bmatrix} 0 \\ t \end{bmatrix}$$

$$\vec{X_p} = \begin{bmatrix} te^t\cos t \\ te^t\sin t \end{bmatrix}$$

$$\boxed{\begin{array}{c} \vec{X} = C_1\begin{bmatrix} -e^t\sin t \\ e^t\cos t \end{bmatrix} + C_2\begin{bmatrix} e^t\cos t \\ e^t\sin t \end{bmatrix} + \begin{bmatrix} te^t\cos t \\ te^t\sin t \end{bmatrix} \\ \text{—or—} \\ \vec{X} = C_1\begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}e^t + C_2\begin{bmatrix} \cos t \\ \sin t \end{bmatrix}e^t + \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}te^t \end{array}}$$

**8.3 day 3**

#1. Solve the initial-value problem:

$\vec{X}' = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}\vec{X} + \begin{bmatrix} 4e^{2t} \\ 4e^{4t} \end{bmatrix}, \quad \vec{X}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\begin{vmatrix} 3-\lambda & -1 \\ -1 & 3-\lambda \end{vmatrix} = 0 \quad (3-\lambda)(3-\lambda) - 1 = 0 \quad \lambda^2 - 6\lambda + 8 = 0 \quad (\lambda-2)(\lambda-4) = 0 \quad \lambda = 2, \lambda = 4$

$\lambda = 4$

$\left[\begin{array}{cc|c} 3-4 & -1 & 0 \\ -1 & 3-4 & 0 \end{array}\right] \left[\begin{array}{cc|c} -1 & -1 & 0 \\ -1 & -1 & 0 \end{array}\right]$

rref $\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $k_1 + k_2 = 0$, $k_1 = -k_2$ $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\lambda = 2$

$\left[\begin{array}{cc|c} 3-2 & -1 & 0 \\ -1 & 3-2 & 0 \end{array}\right] \left[\begin{array}{cc|c} 1 & -1 & 0 \\ -1 & 1 & 0 \end{array}\right]$

rref $\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $k_1 - k_2 = 0$, $k_1 = k_2$ $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\vec{X_c} = c_1\begin{bmatrix} -1 \\ 1 \end{bmatrix}e^{4t} + c_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{2t}$

$\Phi = \begin{bmatrix} -e^{4t} & e^{2t} \\ e^{4t} & e^{2t} \end{bmatrix} \quad \begin{vmatrix} -e^{4t} & e^{2t} \\ e^{4t} & e^{2t} \end{vmatrix} = -e^{6t} - e^{6t} = -2e^{6t}$

$\Phi^{-1} = \frac{1}{-2e^{6t}}\begin{bmatrix} e^{2t} & -e^{2t} \\ -e^{4t} & -e^{4t} \end{bmatrix} = \begin{bmatrix} -1/2\, e^{-4t} & 1/2\, e^{-4t} \\ 1/2\, e^{-2t} & 1/2\, e^{-2t} \end{bmatrix}$

$\vec{X_p} = \Phi\int \Phi^{-1}F\,dt = \begin{bmatrix} -e^{4t} & e^{2t} \\ e^{4t} & e^{2t} \end{bmatrix}\int \begin{bmatrix} -1/2\, e^{-4t} & 1/2\, e^{-4t} \\ 1/2\, e^{-2t} & 1/2\, e^{-2t} \end{bmatrix}\begin{bmatrix} 4e^{2t} \\ 4e^{4t} \end{bmatrix}dt$

$\int \begin{bmatrix} -2e^{-2t} + 2 \\ 2 + 2e^{2t} \end{bmatrix} dt$

$\vec{X_p} = \begin{bmatrix} -e^{4t} & e^{2t} \\ e^{4t} & e^{2t} \end{bmatrix}\begin{bmatrix} e^{-2t} + 2t \\ 2t + e^{2t} \end{bmatrix}$

$\vec{X_p} = \begin{bmatrix} -e^{2t} - 2te^{4t} + 2te^{2t} + e^{4t} \\ e^{2t} + 2te^{4t} + 2te^{2t} + e^{4t} \end{bmatrix}$ now, use $\vec{X}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\vec{X} = c_1\begin{bmatrix} -1 \\ 1 \end{bmatrix}e^{4t} + c_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{2t} + \begin{bmatrix} -e^{2t} - 2te^{4t} + 2te^{2t} + e^{4t} \\ e^{2t} + 2te^{4t} + 2te^{2t} + e^{4t} \end{bmatrix}$

$x = -c_1e^{4t} + c_2e^{3t} - e^{2t} - 2te^{4t} + 2te^{2t} + e^{4t}$

$1 = -c_1(1) + c_2(1) - (1) - 2(0)(1) + 2(0)(1) + (1)$

$1 = -c_1 + c_2$

$\begin{cases} -c_1 + c_2 = 1 \\ c_1 + c_2 = -1 \end{cases}$

$2c_2 = 0$

$c_2 = 0$, $c_1 + (0) = -1$, $c_1 = -1$

$y = c_1e^{4t} + c_2e^{2t} + e^{2t} + 2te^{4t} + 2te^{2t} + e^{4t}$

$1 = c_1(1) + c_2(1) + (1) + 2(0)(1) + 2(0)(1) + (1)$

$1 = c_1 + c_2 + 2$

$\vec{X} = (-1)\begin{bmatrix} -1 \\ 1 \end{bmatrix}e^{4t} + (0)\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{2t} + \begin{bmatrix} -e^{2t} - 2te^{4t} + 2te^{2t} + e^{4t} \\ e^{2t} + 2te^{4t} + 2te^{2t} + e^{4t} \end{bmatrix}$

$\boxed{\vec{X} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}e^{4t} + \begin{bmatrix} -1 \\ 1 \end{bmatrix}e^{2t} + \begin{bmatrix} -2 \\ 2 \end{bmatrix}te^{4t} + \begin{bmatrix} 2 \\ 2 \end{bmatrix}te^{2t} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{4t}} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}e^{4t} + \begin{bmatrix} -1 \\ 1 \end{bmatrix}e^{2t} + \begin{bmatrix} -2 \\ 2 \end{bmatrix}te^{4t} + \begin{bmatrix} 2 \\ 2 \end{bmatrix}te^{2t}$

**8.3 day 4**

#1. Rewrite the following 3rd-order differential equation as a system of 3 1st-order differential equations (and also write the initial conditions in matrix form):

$y''' - 2y'' + 3y' = 9 + e^x$

$y(0) = 3, \quad y'(0) = 0, \quad y''(0) = -2$

(do not solve the system)

$$\begin{cases} y' = 0y + 1y' + 0y'' + 0 \\ y'' = 0y + 0y' + 1y'' + 0 \\ y''' = 0y - 3y' + 2y'' + 9 + e^x \end{cases}$$

$$\boxed{\vec{x}\,' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & 2 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ 0 \\ 9+e^x \end{bmatrix} \quad \text{where } \vec{x} = \begin{bmatrix} y \\ y' \\ y'' \end{bmatrix}}$$

$$\boxed{\text{w/ } \vec{x}(0) = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}}$$